Kuwait J. Sci. **42 (3)** pp. 10-27, 2015

Generalized Bertrand curves with timelike (1,3)-normal plane in Minkowski space-time

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ABSTRACT

In this paper, we reconsider the $(1,3)$-Bertrand curves with respect to the casual characters of $(1,3)$-normal plane which is a plane spanned by the principal normal and the second binormal vector fields of the given curve. Here, we restrict our investigation of $(1,3)$-Bertrand curves to the timelike $(1,3)$-normal plane in Minkowski space-time. We obtain the necessary and sufficient conditions for the curves with timelike $(1,3)$-normal plane to be $(1,3)$-Bertrand curves and we give the related examples for these curves.

Mathematics Subject Classification (2010): 53C50, 53C40.

Keywords: Bertrand curve; Minkowski space-time; Frenet planes.

INTRODUCTION

Much work has been done on the general theory of curves in an Euclidean space (or more generally in a Riemannian manifold). So now, we have extensive knowledge on its local geometry as well as its global geometry. Characterization of a regular curve is one of the important and interesting problems in the theory of curves in Euclidean space. There are two ways widely used to solve these problems: to figure out the relationship between the Frenet vectors of the curves (Kuhnel, 1999), and to determine the shape and size of a regular curve by using its curvatures k_1 (or $\varkappa$) and k_2 (or τ).

In 1845, Saint Venant (Venant, 1845) proposed the question whether the principal normal of a curve is the principal normal of another's on the surface generated by the principal normal of the given one. Bertrand answered this question in a paper published in 1850 (Bertrand, 1850). He proved that a necessary and sufficient condition for the existence of such a second curve is required in fact a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in R$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves (Kuhnel, 1999).

There are many important papers on Bertrand curves in Euclidean space (Bioche, 1889; Burke, 1960; Pears, 1935).

When we investigate the properties of Bertrand curves in Euclidean n-space, it is easy to see that either k_2 or k_3 is zero which means that Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ are degenerate curves (Pears, 1935). This result is restated by Matsuda and Yorozu (Matsuda & Yorozu, 2003). They proved that there was not any special Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (Balgetir *et al.,* 2004; Balgetir *et al.,* 2004/05; Ekmekci & İlarslan, 2001; Jin, 2008; Kahraman *et al.,* 2014; Whittemore, 1940; Gök *et al.,* 2014) as well as in Euclidean space. In addition, there are some other studies about Bertrand curves such as (Ersoy & İnalcik 2014; Lucas & Ortega- Yagues, 2012; Oztekin, 2009; Yilmaz & Bektaş, 2008).

Many researchers have dealt with $(1,3)$-type Bertrand curves in Minkowski space-time. However, they only considered the casual character of the curves. In this paper, we reconsider $(1,3)$-type Bertrand curves in Minkowski space-time with respect to the casual character of the plane spanned by the principal normal and the second binormal of the curve. For now, we look into the timelike case of the plane. The spacelike case of the plane will be considered in our next paper.

PRELIMINARIES

The Minkowski space-time $\mathbb{E}_1^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that a vector $v \in \mathbb{E}_1^4 \setminus \{0\}$ can be spacelike if $g(v,v) > 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$. In particular, the vector $v = 0$ is said to be a spacelike. The norm of a vector v is given by $\| v \| = \sqrt{| g(v,v) |}$. Two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (O'Neill, 1983).

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s),\alpha''(s))=1$ (Bonnor, 1969; Otsuki, 1961). On the other hand, a non-null curve α is parametrized by the arclength parameter s if $g(\alpha'(s),\alpha'(s))=\pm 1$.

Let $\{T,N,B_1,B_2\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^4$, consisting of the tangent, the principal normal, the first binormal and the second binormal vector field respectively.

If α is a spacelike or a timelike curve whose the Frenet frame $\{T,N,B_1,B_2\}$ contains only non-null vector fields, the Frenet equations are given by (İlarslan & Nesovic, 2009)

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{1}$$

where $g(T,T)=\varepsilon_1$, $g(N,N)=\varepsilon_2$, $g(B_1,B_1)=\varepsilon_3$, $g(B_2,B_2)=\varepsilon_4$, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4=-1$, $\varepsilon_i\in\{-1,1\}$, $i\in\{1,2,3,4\}$. In particular, the following conditions hold:

$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(N,B_2)=g(B_1,B_2)=0.$$

If α is a pseudo null curve, the Frenet formulas read (Bonnor, 1985; Otsuki, 1961)

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2}$$

where the first curvature $k_1(s)=0$, if α is straight line, or $k_1(s)=1$ in all other cases. Then, the following conditions are satisfied:

$$g(T,T)=g(B_1,B_1)=1,\quad g(N,N)=g(B_2,B_2)=0,$$

$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(B_1,B_2)=0,\quad g(N,B_2)=1.$$

ON (1,3)-BERTRAND CURVES WITH TIMELIKE PLANE $sp\{N,B_2\}$ IN $\mathbb{E}_1^4$

In this section, we discuss (1,3)-Bertrand curves according to their (1,3)-normal planes, which are planes spanned by the principal normal vectors and second binormal vectors of the curves. Here, we assume that the $(1,3)$-normal planes are timelike. As a result, we obtain the necessary and sufficient conditions for the curves to be $(1,3)$-Bertrand curves with spacelike $(1,3)$-normal plane.

Definition 1. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ and $\beta^* : I^* \subset \mathbb{R} \to \mathbb{E}_1^4$ be C^∞-special Frenet curves in Minkowski space-time $\mathbb{E}_1^4$ and $f : I \to I^*$ a regular C^∞-map such that each point $\beta(s)$ of β corresponds to the point $\beta^*(s^*) = \beta^*(f(s))$ of β^* for all $s \in I$. Here s and s^* are arc -length or pseudo arc parameters of β and β^*, respectively. If the Frenet $(1,3)$-normal plane at each point $\beta(s)$ of β coincides with the Frenet $(1,3)$-normal plane at each point $\beta^*(s^*) = \beta^*(f(s))$ of β^* for all s, then β is called a $(1,3)$-Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$ and β^* is called a $(1,3)$-Bertrand mate curve of β (Gök *et al*., 2014).

Let $\beta : I \to \mathbb{E}_1^4$ be (1,3)-Bertrand curve in $\mathbb{E}_1^4$ with the Frenet frame $\{T, N, B_1, B_2\}$ and the curvatures $\kappa_1, \kappa_2, \kappa_3$ and $\beta^* : I \to \mathbb{E}_1^4$ be a $(1,3)$-Bertrand mate curve of β with the Frenet frame $\{T^*, N^*, B_1^*, B_2^*\}$ and the curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$. We assume that the $(1,3)$-normal plane spanned by $\{N, B_2\}$ is a timeelike plane. Since $sp\{N, B_2\} = sp\{N^*, B_2^*\}$ is a timelike plane, we have the following four cases:

Case 1. β is a spacelike curve with non zero curvature functions κ_1, κ_2, κ_3 and spacelike vector B_1, and β^* is a spacelike curve with non zero curvature functions $\kappa_1^*, \kappa_2^*, \kappa_3^*$ and spacelike vector B_1^*;

Case 2. β is a spacelike curve with non zero curvature functions κ_1, κ_2, κ_3 and spacelike vector B_1 and β^* is a pseudo null curve with curvature functions $\kappa_1^* = 1$, $\kappa_2^* \neq 0, \kappa_3^*$;

Case 3. β is a pseudo null curve with curvature functions $\kappa_1 = 1$, $\kappa_2 \neq 0$, κ_3 and β^* is a spacelike curve with non zero curvature functions $\kappa_1^*, \kappa_2^*, \kappa_3^*$ and spacelike vector B_1^*;

Case 4. β is a pseudo null curve with curvature functions $\kappa_1 = 1$, $\kappa_2 \neq 0$, κ_3 and β^* is also a pseudo null curve with curvature functions $\kappa_1^* = 1$, $\kappa_2^* \neq 0, \kappa_3^*$.

In what follows, we consider these four cases separately.

Case 1. Let β be a spacelike curve with non zero curvature functions κ_1, κ_2, κ_3 and spacelike vector B_1, and β^* be a spacelike curve with non zero curvature functions $\kappa_1^*, \kappa_2^*, \kappa_3^*$ and spacelike vector B_1^*. In this case, we get the following theorem.

Theorem 1. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a spacelike curve parametrized by arc-lenght parameter s with non-zero curvature functions $\kappa_1, \kappa_2, \kappa_3$ and spacelike binormal vector B_1. Then the curve β is a $(1,3)$-Bertrand curve and its $(1,3)$-Bertrand mate curve is also spacelike curve with spacelike binormal vector and non-zero curvature functions if and only if there exist constant real numbers a, b, h, $\mu \neq \pm 1$ satisfying

$$a\kappa_2(s)-b\kappa_3(s)\neq 0, \tag{3}$$

$$1=a\kappa_1(s)+h(a\kappa_2(s)-b\kappa_3(s)), \tag{4}$$

$$-\mu\kappa_3(s)=h\kappa_1(s)-\kappa_2(s), \tag{5}$$

$$\kappa_1(s)\kappa_2(s)(h^2-1)+h(\kappa_1^2(s)-\kappa_2^2(s)+\kappa_3^2(s))\neq 0 \tag{6}$$

for all $s\in I$.

Proof. We assume that $\beta : I\subset\mathbb{R}\to\mathbb{E}_1^4$ is a spacelike $(1,3)$-Bertrand curve parametrized by arc-length s with non-zero curvature functions $\kappa_1,\kappa_2,\kappa_3$ and spacelike binormal vector B_1, and the curve $\beta^* : I^*\subset\mathbb{R}\to\mathbb{E}_1^4$ is the spacelike $(1,3)$-Bertrand mate curve of the curve β parametrized by arc-length parameter s^* with non-zero curvature functions $\kappa_1^*,\kappa_2^*,\kappa_3^*$ and spacelike binormal vector B_1^*. Then we can write the curve β^* as follows

$$\beta^*(s^*)=\beta^*(f(s))=\beta(s)+a(s)N(s)+b(s)B_2(s) \tag{7}$$

for all $s^*\in I^*$, $s\in I$ where $a(s)$ and $b(s)$ are $C^\infty-$functions on I. Differentiating (7) with respect to s and using the Frenet formulae (1), we get

$$T^*f'=(1-a\kappa_1)T+a'N+(a\kappa_2-b\kappa_3)B_1+b'B_2. \tag{8}$$

Multiplying the equation (8) with N and B_2 respectively, we have

$$a'=0 \text{ and } b'=0. \tag{9}$$

Substituting (9) in (8), we find

$$T^*f'=(1-a\kappa_1)T+(a\kappa_2-b\kappa_3)B_1. \tag{10}$$

Multiplying the equation (10) with itself, we obtain

$$(f')^2=(1-a\kappa_1)^2+(a\kappa_2-b\kappa_3)^2. \tag{11}$$

If we denote

$$\delta=\frac{1-a\kappa_1}{f'} \text{ and } \gamma=\frac{a\kappa_2-b\kappa_3}{f'}, \tag{12}$$

we get

$$T^*=\delta T+\gamma B_1. \tag{13}$$

Differentiating (13) with respect to s and using the Frenet formulae (1), we have

$$\varepsilon_2^* f' \kappa_1^* N^* = \delta' T + \varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)N + \gamma' B_1 - \varepsilon_2\gamma\kappa_3 B_2. \tag{14}$$

Multiplying the equation (14) with T and B_1 ,respectively, we get

$$\delta' = 0 \text{ and } \gamma' = 0. \tag{15}$$

Assume that $\gamma = 0$. From (13), $T^* = \delta T$. Then

$$T^* = \pm T. \tag{16}$$

Differentiating (16) with respect to s and using the Frenet formulae (1), we find

$$\varepsilon_2^* f' \kappa_1^* N^* = \varepsilon_2\kappa_1 N, \tag{17}$$

which implies that N is linearly dependent with N^*, which is a contradiction. Since $\gamma \neq 0$, from (12), we find (3)

$$a\kappa_2 - b\kappa_3 \neq 0.$$

From (12), we have (4)

$$1 = a\kappa_1 + h(a\kappa_2 - b\kappa_3) \tag{18}$$

where $h = \delta/\gamma$. Substituting (15) in (14), we get

$$\varepsilon_2^* f' \kappa_1^* N^* = \varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)N - \varepsilon_2\gamma\kappa_3 B_2. \tag{19}$$

By taking the scalar product of (19) with itself, we obtain

$$\varepsilon_2^*(f')^2(\kappa_1^*)^2 = \varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)^2 + \varepsilon_4\gamma^2\kappa_3^2. \tag{20}$$

Substituting (12) in (20), we find

$$\varepsilon_2^*(f')^2(\kappa_1^*)^2 = \frac{(a\kappa_2 - b\kappa_3)^2}{(f')^2}[\varepsilon_2(h\kappa_1 - \kappa_2)^2 + \varepsilon_4\kappa_3^2]. \tag{21}$$

Substituting (18) in (11), we have

$$(f')^2 = (a\kappa_2 - b\kappa_3)^2[h^2 + 1]. \tag{22}$$

Substituting (22) in (21), we get

$$\varepsilon_2^*(f')^2(\kappa_1^*)^2 = \frac{1}{h^2 + 1}[\varepsilon_2(h\kappa_1 - \kappa_2)^2 + \varepsilon_4\kappa_3^2]. \tag{23}$$

If we denote

$$\lambda_1 = \frac{\varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)}{\varepsilon_2^* f' \kappa_1^*} = \frac{\varepsilon_2(a\kappa_2 - b\kappa_3)}{\varepsilon_2^*(f')^2 \kappa_1^*}[h\kappa_1 - \kappa_2], \tag{24}$$

$$\lambda_2 = \frac{-\varepsilon_2\gamma\kappa_3}{\varepsilon_2^* f' \kappa_1^*} = \frac{\varepsilon_2(a\kappa_2 - b\kappa_3)}{\varepsilon_2^*(f')^2 \kappa_1^*}\kappa_3, \tag{25}$$

we get

$$N^* = \lambda_1 N + \lambda_2 B_2. \tag{26}$$

Differentiating (26) with respect to s and using the Frenet formulae (1), we find

$$-f'\kappa_1^* T^* + f'\kappa_2^* B_1^* = -\kappa_1\lambda_1 T + \lambda_1' N + (\lambda_1\kappa_2 + \lambda_2\kappa_3)B_1 + \lambda_2' B_2. \tag{27}$$

Multiplying the equation (27) with N and B_2 respectively, we obtain

$$\lambda_1' = 0 \text{ and } \lambda_2' = 0. \tag{28}$$

From (24) and (25), since $\lambda_2 \neq 0$, we have (5)

$$-\mu\kappa_3 = h\kappa_1 - \kappa_2 \tag{29}$$

where $\mu = \lambda_1/\lambda_2 \neq \pm 1$ from (21). Substituting (28) in (27), we find

$$-f'\kappa_1^* T^* + f'\kappa_2^* B_1^* = -\kappa_1\lambda_1 T + (\lambda_1\kappa_2 - \lambda_2\kappa_3)B_1. \tag{30}$$

From (10) and (30), we obtain

$$f'\kappa_2^* B_1^* = A(s)T + B(s)B_1 \tag{31}$$

where

$$A(s) = \frac{-\varepsilon_2(a\kappa_2 - b\kappa_3)}{\varepsilon_2^*(f')^2\kappa_1^*(h^2+1)}[\kappa_1\kappa_2(h^2-1) + h(-\kappa_2^2 + \kappa_3^2 + \kappa_1^2)] \tag{32}$$

and

$$B(s) = \frac{\varepsilon_2 h(a\kappa_2 - b\kappa_3)}{\varepsilon_2^*(f')^2\kappa_1^*(h^2+1)}[\kappa_1\kappa_2(h^2-1) + h(-\kappa_2^2 + \kappa_3^2 + \kappa_1^2)]. \tag{33}$$

Since $f'\kappa_2^* B_1^* \neq 0$, we get (6)

$$\kappa_1\kappa_2(h^2-1) + h(-\kappa_2^2 + \kappa_3^2 + \kappa_1^2) \neq 0.$$

Conversely, assume that β is a spacelike curve parametrized by arc-lenght parameter s with non-zero curvature functions $\kappa_1, \kappa_2, \kappa_3$ and spacelike binormal vector B_1 and the relations (3), (4), (5) and (6) hold for constant real numbers $a,\ b,\ h,\ \mu \neq \pm 1$. Then we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + aN(s) + bB_2(s). \tag{34}$$

Differentiating (34) with respect to s and using the Frenet formulae (1), we find

$$\frac{d\beta^*}{ds} = (1 - a\kappa_1)T + (a\kappa_2 - b\kappa_3)B_1. \tag{35}$$

From (35) and (4), we get

$$\frac{d\beta^*}{ds} = (a\kappa_2 - b\kappa_3)[hT + B_1]. \tag{36}$$

From (36), we have

$$f' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = m_1(a\kappa_2 - b\kappa_3)\sqrt{h^2+1} > 0 \tag{37}$$

where $m_1 = \pm 1$ such that $m_1(a\kappa_2 - b\kappa_3) > 0.$ By rewriting (36), we obtain

$$T^* = \frac{m_1}{\sqrt{h^2+1}}[hT + B_1] \tag{38}$$

which implies that $g(T^*, T^*) = 1.$ Differentiating (38) with respect to s and using the Frenet formulae (1), we find

$$\frac{dT^*}{ds^*} = \frac{m_1\varepsilon_2}{f'\sqrt{h^2+1}}[(h\kappa_1 - \kappa_2)N - \kappa_3 B_2]. \tag{39}$$

Using (39), we have

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{\sqrt{m_2[(h\kappa_1 - \kappa_2)^2 - \kappa_3^2]}}{f'\sqrt{h^2+1}} \tag{40}$$

where $m_2 = \mp 1$ such that $m_2[(h\kappa_1 - \kappa_2)^2 - \kappa_3^2] > 0.$ From (39) and (40), we have

$$N^* = \frac{\varepsilon_2^*}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{m_1\varepsilon_2^*\varepsilon_2}{\sqrt{m_2[(h\kappa_1 - \kappa_2)^2 - \kappa_3^2]}}[(h\kappa_1 - \kappa_2)N - \kappa_3 B_2] \tag{41}$$

which implies that $g(N^*, N^*) = \varepsilon_2 m_2 = \varepsilon_2^*$. If we denote

$$\lambda_3 = \frac{m_1\varepsilon_2^*\varepsilon_2(h\kappa_1 - \kappa_2)}{\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]}} \text{ and } \lambda_4 = \frac{-m_1\varepsilon_2^*\varepsilon_2\kappa_3}{\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]}}, \tag{42}$$

we obtain

$$N^* = \lambda_3 N + \lambda_4 B_2. \tag{43}$$

Differentiating (43) with respect to s and using the Frenet formulae (1), we find

$$f'\frac{dN^*}{ds^*} = -\lambda_3\kappa_1 T + \lambda_3' N + (\lambda_3\kappa_2 - \lambda_4\kappa_3)B_1 + \lambda_4' B_2. \tag{44}$$

Differentiating (5) with respect to s, we have

$$(h\kappa_1' - \kappa_2')\kappa_3 - (h\kappa_1 - \kappa_2)\kappa_3' = 0. \tag{45}$$

Differentiating (42) with respect to s and using (45), we get

$$\lambda_3' = 0 \text{ and } \lambda_4' = 0. \tag{46}$$

Substituting (42) and (46) in (44), we obtain

$$\frac{dN^*}{ds^*} = \frac{-m_1\varepsilon_2^*\varepsilon_2\kappa_1(h\kappa_1-\kappa_2)}{f'\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]}}T + \frac{m_1\varepsilon_2^*\varepsilon_2[\kappa_2(h\kappa_1-\kappa_2)+\kappa_3^2]}{f'\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]}}B_1. \tag{47}$$

From (38) and (40), we find

$$\kappa_1^* T^* = \frac{m_1\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]}}{f'(h^2+1)}[hT + B_1]. \tag{48}$$

From (47) and (48), we get

$$\frac{dN^*}{ds^*} + \kappa_1^* T^* = \frac{P(s)}{R(s)}[T - hB_1] \tag{49}$$

where

$$\begin{aligned} P(s) &= -m_1\varepsilon_2^*\varepsilon_2[\kappa_1\kappa_2(h^2-1) + h(\kappa_1^2 - \kappa_2^2 + \kappa_3^2)] \neq 0, \\ R(s) &= f'(h^2+1)\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]} \neq 0. \end{aligned} \tag{50}$$

From (49) we find κ_2^* as

$$\kappa_2^* = \frac{\left|\kappa_1\kappa_2(h^2-1) + h(\kappa_1^2 - \kappa_2^2 + \kappa_3^2)\right|}{f'\sqrt{h^2+1}\sqrt{m_2[(h\kappa_1-\kappa_2)^2-\kappa_3^2]}} > 0. \tag{51}$$

From (49) and (51), we obtain B_1^* as

$$B_1^* = \frac{1}{\kappa_2^*}[\frac{dN^*}{ds^*} + \kappa_1^* T^*] = \frac{-m_1 m_3 \varepsilon_2^* \varepsilon_2}{\sqrt{h^2+1}}[T - hB_1]. \tag{52}$$

where

$$m_3 = \frac{\left|\kappa_1\kappa_2(h^2-1) + h(\kappa_1^2 - \kappa_2^2 + \kappa_3^2)\right|}{\left(\kappa_1\kappa_2(h^2-1) + h(\kappa_1^2 - \kappa_2^2 + \kappa_3^2)\right)} = \pm 1.$$

From (52), we have $g(B_1^*, B_1^*) = 1$. Now, we can define a unit vector B_2^* as $B_2^* = -\gamma_1 N - \delta_1 B_2$, that is

$$B_2^* = \frac{m_1 \varepsilon_2^* \varepsilon_2 \kappa_3}{\sqrt{m_2[(h\kappa_1 - \kappa_2)^2 - \kappa_3^2]}} N - \frac{m_1 \varepsilon_2^* \varepsilon_2 (h\kappa_1 - \kappa_2)}{\sqrt{m_2[(h\kappa_1 - \kappa_2)^2 - \kappa_3^2]}} B_2. \tag{53}$$

which implies that $g(B_2^*, B_2^*) = m_2 \varepsilon_4 = \varepsilon_4^* = -\varepsilon_2^*$. Lastly, we can define κ_3^* as

$$\kappa_3^* = g(B_1^{*\prime}, B_2^*) = \frac{-m_3 \kappa_1 \kappa_3 \sqrt{h^2+1}}{f' \sqrt{m_2[(h\kappa_1 - \kappa_2)^2 - \kappa_3^2]}} \neq 0.$$

Consequently, we find that β^* is a spacelike curve with spacelike B_1^* and a $(1,3)$-Bertrand mate curve of the curve β since span $\{N^*, B_2^*\} =$ span $\{N, B_2\}$.

Case 2. Let β be a spacelike curve with non zero curvature functions κ_1, κ_2, κ_3 and spacelike vector B_1 and β^* be a pseudo null curve with curvature functions $\kappa_1^* = 1$, $\kappa_2^* \neq 0$, κ_3^*. In this case, we get the following theorem.

Theorem 2. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a spacelike curve parametrized by arc-lenght parameter s with non-zero curvature functions $\kappa_1, \kappa_2, \kappa_3$ and spacelike binormal vector B_1. Then the curve β is a $(1,3)$-Bertrand curve and its $(1,3)$-Bertrand mate curve is a pseudo null curve with non-zero second curvature function if and only if there exist constant real numbers a, b, h, $\mu_1 \neq 0, \mu_2 \neq 0$ satisfying

$$a\kappa_2(s) - b\kappa_3(s) \neq 0, \tag{54}$$

$$1 - a\kappa_1(s) = h(a\kappa_2(s) - b\kappa_3(s)), \tag{55}$$

$$h\kappa_1(s) - \kappa_2(s) = \mu_1(a\kappa_2(s) - b\kappa_3(s)), \tag{56}$$

$$\kappa_3(s) = \mu_2(a\kappa_2(s) - b\kappa_3(s)) \tag{57}$$

for all $s \in I$ where $\mu_1^2 = \mu_2^2$.

Proof. We assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a spacelike $(1,3)$ -Bertrand curve parametrized by arc-length s with non-zero curvature functions $\kappa_1, \kappa_2, \kappa_3$ and spacelike binormal vector B_1, and the curve $\beta^* : I^* \subset \mathbb{R} \to \mathbb{E}_1^4$ is a pseudo null $(1,3)$ -Bertrand mate curve of the curve β parametrized by pseudo arc parameter s^* with $\kappa_1^* = 1, \kappa_2^* \neq 0$ and κ_3^*. Then we can write the curve β^* as follows

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + a(s)N(s) + b(s)B_2(s) \tag{58}$$

for all $s^* \in I^*$, $s \in I$ where $a(s)$ and $b(s)$ are $C^\infty -$ functions on I. Differentiating (58) with respect to s and using the Frenet formulae (1) and (2), we get

$$T^* f' = (1 - a\kappa_1)T + a'N + (a\kappa_2 - b\kappa_3)B_1 + b'B_2. \tag{59}$$

Multiplying the equation (59) with N and B_2 respectively, we have

$$a' = 0 \text{ and } b' = 0. \tag{60}$$

Substituting (60) in (59), we find

$$T^* f' = (1 - a\kappa_1)T + (a\kappa_2 - b\kappa_3)B_1. \tag{61}$$

Multiplying the equation (61) with itself , we obtain

$$(f')^2 = (1 - a\kappa_1)^2 + (a\kappa_2 - b\kappa_3)^2. \tag{62}$$

If we denote

$$\delta = \frac{1 - a\kappa_1}{f'} \text{ and } \gamma = \frac{a\kappa_2 - b\kappa_3}{f'}, \tag{63}$$

we get

$$T^* = \delta T + \gamma B_1. \tag{64}$$

Differentiating (64) with respect to s and using the Frenet formulae (1) and (2), we have

$$\varepsilon_2^* f' \kappa_1^* N^* = \delta' T + \varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)N + \gamma' B_1 - \varepsilon_2 \gamma \kappa_3 B_2. \tag{65}$$

Multiplying the equation (65) with T and B_1, respectively, we get

$$\delta' = 0 \text{ and } \gamma' = 0. \tag{66}$$

Assume that $\gamma = 0$. From (64), we get that N is linearly dependent with N^*, which is a contradiction. Since $\gamma \neq 0$, from (63), we find (54)

$$a\kappa_2 - b\kappa_3 \neq 0.$$

From (63), we have (55)

$$1 - a\kappa_1 = h(a\kappa_2 - b\kappa_3) \tag{67}$$

where $h = \delta/\gamma$. Substituting (66) in (65), we get

$$\varepsilon_2^* f' \kappa_1^* N^* = \varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)N - \varepsilon_2\gamma\kappa_3 B_2. \tag{68}$$

By taking the scalar product of (68) with itself, we obtain

$$(\delta\kappa_1 - \gamma\kappa_2)^2 = \gamma^2\kappa_3^2. \tag{69}$$

Substituting (67) in (62), we have

$$(f')^2 = (a\kappa_2 - b\kappa_3)^2[h^2 + 1]. \tag{70}$$

If we denote

$$\lambda_1 = \frac{\varepsilon_2(\delta\kappa_1 - \gamma\kappa_2)}{\varepsilon_2^* f' \kappa_1^*} = \frac{\varepsilon_2(a\kappa_2 - b\kappa_3)}{(f')^2}[h\kappa_1 - \kappa_2], \tag{71}$$

$$\lambda_2 = \frac{-\varepsilon_2\gamma\kappa_3}{\varepsilon_2^* f' \kappa_1^*} = \frac{\varepsilon_2(a\kappa_2 - b\kappa_3)}{(f')^2}\kappa_3, \tag{72}$$

we get

$$N^* = \lambda_1 N + \lambda_2 B_2. \tag{73}$$

Differentiating (73) with respect to s and using the Frenet formulae (1) and (2), we find

$$f' \kappa_2^* B_1^* = -\kappa_1\lambda_1 T + \lambda_1' N + (\lambda_1\kappa_2 + \lambda_2\kappa_3)B_1 + \lambda_2' B_2. \tag{74}$$

Multiplying the equation (74) with N and B_2 respectively, we obtain

$$\lambda_1' = 0 \text{ and } \lambda_2' = 0. \tag{75}$$

By using (63), (69), (71), (72) and (75), we have

$$h\kappa_1 - \kappa_2 = \mu_1(a\kappa_2 - b\kappa_3),$$
$$\kappa_3 = \mu_2(a\kappa_2 - b\kappa_3)$$

where $\mu_1^2 = \mu_2^2,\ \mu_1, \mu_2 \in \mathbb{R}/\{0\}$.

Conversely, assume that β is a spacelike curve parametrized by arc-lenght parameter s with non-zero curvature functions $\kappa_1, \kappa_2, \kappa_3$ and spacelike binormal vector B_1, and the relations (54), (55), (56) and (57) hold for constant real numbers $a,\ b,\ h,\ \mu \neq \pm 1$. Then we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + aN(s) + bB_2(s). \tag{76}$$

Differentiating (76) with respect to s and using the Frenet formulae (1), we find

$$\frac{d\beta^*}{ds} = (1 - a\kappa_1)T + (a\kappa_2 - b\kappa_3)B_1. \tag{77}$$

From (77) and (55), we get

$$\frac{d\beta^*}{ds} = (a\kappa_2 - b\kappa_3)[hT + B_1]. \tag{78}$$

From (78), we have

$$f' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = m_1(a\kappa_2 - b\kappa_3)\sqrt{h^2+1} > 0 \tag{79}$$

where $m_1 = \pm 1$ such that $m_1(a\kappa_2 - b\kappa_3) > 0$. By rewriting (78), we obtain

$$T^* = \frac{m_1}{\sqrt{h^2+1}}[hT + B_1] \tag{80}$$

which implies that $g(T^*, T^*) = 1$. Differentiating (80) with respect to s and using the Frenet formulae (1), (56) and (57), we find

$$\frac{dT^*}{ds^*} = \frac{\varepsilon_2}{\sqrt{h^2+1}}[\mu_1 N - \mu_2 B_2] \tag{81}$$

from which we have $g\left(\frac{dT^*}{ds^*}, \frac{dT^*}{ds^*}\right) = 0$. Thus using (81), we have $\kappa_1^* = 1$ and

$$N^* = \frac{\varepsilon_2}{\sqrt{h^2+1}}[\mu_1 N - \mu_2 B_2]. \tag{82}$$

Differentiating (82) with respect to s and using the Frenet formulae (1), we obtain

$$\frac{dN^*}{ds^*} = \frac{\varepsilon_2}{f'(h^2+1)}[-\mu_1\kappa_1 T + (\mu_1\kappa_2 + \mu_2\kappa_3)B_1]. \tag{83}$$

Using (83), we get

$$\kappa_2^* = \left\| \frac{dN^*}{ds^*} \right\| = \frac{\sqrt{\mu_1^2\kappa_1^2 + (\mu_1\kappa_2 + \mu_2\kappa_3)^2}}{f'(h^2+1)} \neq 0. \tag{84}$$

From (83) and (84), we find

$$B_1^* = \frac{\varepsilon_2}{\sqrt{\mu_1^2\kappa_1^2 + (\mu_1\kappa_2 + \mu_2\kappa_3)^2}}[-\mu_1\kappa_1 T + (\mu_1\kappa_2 + \mu_2\kappa_3)B_1]. \tag{85}$$

Now we can define a null vector B_2^* such that $g(N^*, B_2^*) = 1$ as

$$B_2^* = \frac{h^2+1}{2}\left[\frac{1}{\mu_1}N + \frac{1}{\mu_2}B_2\right]. \tag{86}$$

Using (85) and (86), we have

$$\kappa_3^* = \frac{-\varepsilon_2(h^2+1)(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)}{2f'\sqrt{\mu_1^2\kappa_1^2 + (\mu_1\kappa_2 + \mu_2\kappa_3)^2}}.$$

Consequently, we find that β^* is a pseudo null curve with non-zero second curvature function, and a $(1,3)$-Bertrand mate curve of the curve β since span $\{N^*, B_2^*\}$ = span $\{N, B_2\}$.

Case 3. Let β be a pseudo null curve with curvature functions $\kappa_1 = 1$, $\kappa_2 \neq 0$, κ_3 and β^* be a spacelike curve with non zero curvature functions $\kappa_1^*, \kappa_2^*, \kappa_3^*$ and spacelike vector B_1^*. In this case, we get the following theorem.

Theorem 3. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a pseudo null curve parametrized by arc-length parameter s with the curvature functions $\kappa_1 = 1$, $\kappa_2 \neq 0$, κ_3. Then, the curve β is a $(1,3)$-Bertrand curve and its $(1,3)$-Bertrand curve is a spacelike curve with spacelike binormal vector if and only if there exist constant real numbers a, b, h, μ satisfying

$$a\kappa_2(s) - b\kappa_3(s) \neq 0, \tag{87}$$

$$1 = b + h(a\kappa_2(s) - b\kappa_3(s)), \tag{88}$$

$$-\mu\kappa_2(s) = h + \kappa_3(s), \tag{89}$$

$$\kappa_2(s) - 2\kappa_2(s)\kappa_3(s)h - \kappa_2(s)h^2 \neq 0 \tag{90}$$

for all $s \in I$.

Proof. The theorem can be proved by the similar technique in the first and second theorems. Therefore we omit the proof of the theorem.

Example 1. (The pseudo null curve equation given in İlarslan & Nesovic 2011) Let us consider the pseudo null curve with the equation

$$\beta(s)=\frac{3}{\sqrt{10}}\left(\frac{1}{9}\cosh(3s),\frac{1}{9}\sinh(3s),\sin(s),-\cos(s)\right).$$

The Frenet Frame of β is given by

$$T(s)=\frac{3}{\sqrt{10}}\left(\frac{1}{3}\sinh(3s),\frac{1}{3}\cosh(3s),\cos(s),\sin(s)\right),$$

$$N(s)=\frac{3}{\sqrt{10}}\left(\cosh(3s),\sinh(3s),-\sin(s),\cos(s)\right),$$

$$B_1(s)=\frac{1}{\sqrt{10}}\left(3\sinh(3s),3\cosh(3s),-\cos(s),-\sin(s)\right),$$

$$B_2(s)=\frac{5}{3\sqrt{10}}\left(-\cosh(3s),-\sinh(3s),-\sin(s),\cos(s)\right).$$

The curvatures of β are $k_1(s)=1$, $k_2(s)=3$, $k_3(s)=4/3$. Let us take $a=b=-9/16$, $h=-5/3$ and $\mu=1/9$ in Theorem 3. Then, it is obvious that the relations (87), (88), (89) and (90) are hold. Therefore the curve β is a $(1,3)$-Bertrand curve in E_1^4 and the $(1,3)$-Bertrand mate curve β^* of the curve β is a spacelike curve with spacelike B_1^* given as follows:

$$\beta^*(s)=\left(-\frac{1}{24}\sqrt{10}\cosh 3s,-\frac{1}{24}\sqrt{10}\sinh 3s,\frac{9}{16}\sqrt{10}\sin s,-\frac{9}{16}\sqrt{10}\cos s\right).$$

The Frenet Frame of β^* is given by

$$T^*(s)=\left(-\frac{2}{85}\sqrt{85}\sinh 3s,-\frac{2}{85}\sqrt{85}\cosh 3s,\frac{9}{85}\sqrt{85}\cos s,\frac{9}{85}\sqrt{85}\sin s\right),$$

$$N^*(s)=\left(-\frac{2}{5}\sqrt{5}\cosh 3s,-\frac{2}{5}\sqrt{5}\sinh 3s,-\frac{3}{5}\sqrt{5}\sin s,\frac{3}{5}\sqrt{5}\cos s\right),$$

$$B_1^*(s)=\left(-\frac{9}{85}\sqrt{85}\sinh 3s,-\frac{9}{85}\sqrt{85}\cosh 3s,-\frac{2}{85}\sqrt{85}\cos s,-\frac{2}{85}\sqrt{85}\sin s\right),$$

$$B_2^*(s)=\left(\frac{3}{5}\sqrt{5}\cosh 3s,\frac{3}{5}\sqrt{5}\sinh 3s,\frac{2}{5}\sqrt{5}\sin s,-\frac{2}{5}\sqrt{5}\cos s\right).$$

The curvatures of β^* are $k_1^*(s)=24\sqrt{2}/85$, $k_2^*(s)=96\sqrt{2}/85$, $k_3^*(s)=8\sqrt{2}/5$ and also the following equalities hold

$$N^*(s)=\frac{1}{6}\sqrt{2}N(s)+\frac{3}{2}\sqrt{2}B_2(s),$$

$$B_2^*(s)=\frac{1}{6}\sqrt{2}N(s)-\frac{3}{2}\sqrt{2}B_2(s).$$

Case 4. Let β be a pseudo null curve with curvature functions $\kappa_1 = 1$, $\kappa_2 \neq 0$, κ_3 and β^* be also a pseudo null curve with curvature functions $\kappa_1^* = 1$, $\kappa_2^* \neq 0, \kappa_3^*$. In this case, we get the following theorem.

Theorem 4. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a pseudo null curve parametrized by arc-length parameter s with the curvature functions $\kappa_1 = 1$, $\kappa_2 \neq 0$, κ_3. Then, the curve β is a $(1,3)$-Bertrand curve and its $(1,3)$-Bertrand curve is a pseudo null curve with non-zero second curvature function if and only if there exist constant real numbers $\lambda \neq 0$, $\gamma \neq 0, \mu \neq 0$ satisfying the followings

$$\lambda\kappa_2(s) - \mu\kappa_3(s) \neq 0, \tag{91}$$

$$1 = \mu + \gamma(\lambda\kappa_2(s) - \mu\kappa_3(s)), \tag{92}$$

$$\gamma + \kappa_3(s) = 0 \tag{93}$$

for all $s \in I$.

Proof. We omit the proof of the theorem since it can be seen in (Gök *et al.,* 2014).

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Submitted **:** 22/04/2014

Revised **:** 08/07/2014

Accepted **:** 14/07/2014

منحنيات برتراند معممة لها مستويات ناظمة مثل زمنية (1،3) في فضاء منكوفسكي الزمكاني

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خلاصة

نقوم في هذا البحث بإعادة دراسة منحينيات برتراند (1،3) بالنسبة إلى الميزات السببية للمستوى الناظم – (1،3) وهذا المستوى الناظم هو المستوى المولد بواسطة الناظم الأساسي و حقل المتجهات ثنائي النظامية الثاني للمنحنى المعطى. لكننا نقصر دراستنا على منحنيات برتراند للمستوى الناظم – (1،3) في زمكان منكوفسكي. ونحصل على الشروط اللازمة و الكافية لهذه المنحنيات ، كما نعطي عدداً من الأمثلة المتعلقة بالموضوع .