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An infinite dimensional fixed point theorem on function spaces of ordered metric spaces

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ABSTRACT

In this study, we extend the notion of multidimensional fixed point and coincidence point theorem to infinite dimensional product spaces. We also prove some theorems, which generalizes some results that are known in this field.

Keywords: Coincidence point; fixed point; metric space; mixed monotony property; function space.

INTRODUCTION

The concept of coupled fixed point was introduced in Guo & Lakshmikantham (1987). Then Gnana-Bhaskar & Lakshmikantham (2006) introduced a mixed monotony property for partially ordered metric spaces in 2006 and used it on the theory of coupled fixed points of contractive operators to prove some coupled fixed point theorems in partially ordered metric spaces. Then Lakshmikantham & Ćirić (2009) defined the $g-$ mixed monotony property and proved coupled coincidence point theorems for partially ordered metric spaces. Berinde & Borcut (2011) defined the notion of triple fixed point. Recently Roldán *et al.* (2012) obtained some existence and uniqueness theorems for nonlinear mappings with finite number of arguments.

Many infinite dimensional structures are involved in the study of fixed point theory. For example, Fisher (1982) demonstrated some fixed point results concerning possibly infinite bounded subset of a complete metric space X and Achari (1986) verified some common fixed point theorems for a family of multifunctions on a non-empty complete metric space.

In this study we consider multivalued fixed points of infinite dimensional functions. Thus we further generalize some former results on coupled fixed point theory. Also we apply our conclusion to a class of functional equations to show the existence and uniqueness of solutions.

PRELIMINARIES

Let (X,d) be a complete metric space, Y be a nonempty set and X^Y denote the set of all functions $f:Y\to X$. If either X is bounded or Y is finite, then the function $d^Y: X^Y\times X^Y\to\mathbb{R}$ with

$$d^Y(h,k)=\sup_{y\in Y}\left\{d\left(h(y),k(y)\right)\right\} \tag{1}$$

defines a metric on X^Y. It is known that if (X,d) is complete, then (X^Y,d^Y) is also a complete metric space.

If X is unbounded and Y is infinite, then for any different $y_1, y_2, \ldots\in Y$ and all $n\in\mathbb{N}$ we can pick $h_n, k_n\in Y$ such that $d(h_n,k_n)>n$ for all n. Then (X^Y,d^Y) will not be a metric space, since for any $h,k\in X^Y$ such that $h(y_n)=h_n$ and $k(y_n)=k_n$ for all n, $d^Y(h,k)=\infty$. To avoid this situation, if necessary, one can consider bounded metrics equivalent to d on X such as $\min\{d(x,y),1\}$ or $\frac{d(x,y)}{1+d(x,y)}$, which imposes a strict handicap on Lipschitz-like inequality to satisfy.

Definition 1. (Roldán *et al*., 2012). Let d and $\leq$ be a metric and a partial order on a nonempty set X, respectively, Then $(X,d,\leq)$ is referred as an ordered metric space.

Definition 2. (Roldán *et al*., 2012). Given an ordered metric space $(X,d,\leq)$ and a mapping $g:X\to X$. X is said to have sequential $g-$monotony property provided that, if (x_n) is non-decreasing sequence and $x_n\to x$, as $n\to\infty$, then $gx_n\leq gx$ for all n, and if (x_n) is non-increasing sequence and $x_n\to x$, as $n\to\infty$, then $gx\leq gx_n$ for all n. X is said to be have sequential monotony property, if it has sequential $g-$monotony property, where g is the identity map I_X.

MAIN RESULTS

Definition 3. Let $F:X^Y\to X$, $g:X\to X$ be functions. F and g are said to be commuting if $gF(h)=F(gh)$ for all $h\in X^Y$

In Definition 3, gh denotes the composition of g and h, and $F(h)$ denotes the value of h under F. This convention will be thoroughly used for complicated expressions to be easily readable.

Moreover, we assume that a partition $\{A,B\}$ of Y with possibly empty sets A and B is given and the sets

$$\begin{aligned}\Omega_{A,B}&=\left\{\tau:Y\to Y\mid\tau(A)\subseteq A\text{ and }\tau(B)\subseteq B\right\}\\ \Omega'_{A,B}&=\left\{\tau:Y\to Y\mid\tau(A)\subseteq B\text{ and }\tau(B)\subseteq A\right\}\end{aligned} \tag{2}$$

are readily defined.

The relation $\leq$ given on X, can be extended to X^Y by

$$h \leq k \Leftrightarrow h(y) \leq k(y) \text{ for all } y \in Y \tag{3}$$

be useful to define for each $y \in Y$ a partial order $\leq_y$ on X such that for all $x_1, x_2 \in X$

$$x_1 \leq_y x_2 \Leftrightarrow x_1 \leq x_2 \tag{4}$$

if $y \in A$, and

$$x_1 \leq_y x_2 \Leftrightarrow x_2 \leq x_1 \tag{5}$$

if $y \in B$. By these relations $\leq_y$, we can define a partial order $\leq_Y$ on X^Y such that for all $h, k \in X^Y$

$$h \leq_Y k \Leftrightarrow h(y) \leq_y k(y) \quad \text{for all} \quad y \in Y. \tag{6}$$

Definition 4. Let $F : X^Y \to X$, $g : X \to X$ be functions. If the implication

$$gh(y) \leq gk(y) \Rightarrow F(h) \leq_y F(k) \tag{7}$$

is satisfied for all $y \in Y$ and for all $h, k \in X^Y$ such that $k|_{Y\setminus\{y\}} = h|_{Y\setminus\{y\}}$, then F is called finitely mixed $g-$monotone.

Finitely mixed $g-$monotony property for F means that, $gh(y) \leq gk(y)$ implies $F(h) \leq F(k)$ for all $y \in A$ and all $h, k \in X^Y$ such that $h|_{Y\setminus\{y\}} = k|_{Y\setminus\{y\}}$, and $gh(y) \leq gk(y)$ implies $F(h) \geq F(k)$ for all $y \in B$ and all $h, k \in X^Y$ such that $h|_{Y\setminus\{y\}} = k|_{Y\setminus\{y\}}$. These can be unified under one expression: $gh(y) \leq_y gk(y)$ implies $F(h) \leq F(k)$ for all $y \in Y$ and all $h, k \in X^Y$ such that $h|_{Y\setminus\{y\}} = k|_{Y\setminus\{y\}}$. Moreover, for $z \neq y$, $gh(z) = gk(z)$ and so $gh(z) \leq_z gk(z)$. Thus F has finitely mixed $g-$monotony property iff

$$gh \leq_Y gk \Rightarrow F(h) \leq F(k) \tag{8}$$

for all $y \in Y$ and all $h, k \in X^Y$ such that $h|_{Y\setminus\{y\}} = k|_{Y\setminus\{y\}}$.

Definition 5. Let $F : X^Y \to X$, $g : X \to X$. If

$$gh \leq_Y gk \Rightarrow F(h) \leq F(k) \tag{9}$$

for $h, k \in X^Y$, then F is said to be mixed $g-$monotone.

Example 6. Let $X = Y = [-1,1]$ and consider the partition $\{A, B\}$ of Y, where $A = [-1,0]$ and $B = (0,1]$. Define functions $g : X \to X$, $g(x) = -x$ and $F : X^Y \to X$, $F(h) = \sup\{yh(y) : y \in Y\}$. Then F has mixed $g-$monotony property. We can observe for each $h, k \in X^Y$ that $gh \leq_Y gk$ implies $yh(y) \leq yk(y)$, which in turn gives $F(h) \leq F(k)$.

Clearly the mixed $g-$monotony property implies the finitely mixed $g-$monotony property. Moreover, in the case that Y is finite, these two properties are equivalent. Indeed, if $Y = \{y_1, \ldots, y_n\}$ and $gh \leq_Y gk$, for each $0 \leq i \leq n$ a function $h_i : Y \to X$ such that

$$h_i(y_j) = \begin{cases} h(y_j), & \text{if } i < j \\ k(y_j), & \text{if } i \geq j \end{cases} \tag{10}$$

gives $h_0 = h$, $h_n = k$ and $h_{i-1}|_{Y\setminus\{y_i\}} = h_i|_{Y\setminus\{y_i\}}$ for all i, $1 \leq i \leq n$. So we can apply finitely mixed $g-$monotony property n times to get $F(h) \leq F(k)$, since

$$F(h) = F(h_0) \leq F(h_1) \leq \cdots \leq F(h_n) = F(k). \tag{11}$$

Now we give an example of a function, which has finitely mixed $g-$monotony property, but does not have mixed $g-$monotony property.

Example 7. Let $X = [0,1]$, $Y = \mathbb{N}$, $A = \mathbb{N}$, $B = \varnothing$ and $g = I_X$. Let C_h denote the set of cluster points of the sequence $(h(\mathrm{n}))_{n\in\mathbb{N}}$, which is always nonempty by the Bolzano-Weierstrass theorem. Define the function $F : X^Y \to X$ as $F(h) = \sup h(Y) - \sup C_h$. Note that, if $h|_{Y\setminus\{y\}} = k|_{Y\setminus\{y\}}$, then the sequences $(h(n))$ and $(k(n))$ can differ in only one term, and so $C_h = C_k$. In this case, $gh \leq_Y gk$ implies $F(h) \leq F(k)$ and thus F has finitely mixed $g-$monotony property. However F does not have mixed $g-$monotony property. Observe that $gh \leq_Y gk$ for the functions $h(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$ and $k(n) = 1$ for all $n \in \mathbb{N}$, while $F(h) = 1 - 0 = 1$ and $F(k) = 1 - 1 = 0$.

Definition 8. Let $\sigma : Y \to Y^Y$, $\tau : Y \to Y$ and $\Phi = (\sigma, \tau)$. Assume that σ_y denote the function $\sigma(y) \in Y^Y$ for each $y \in Y$. If

$$F(h\sigma_y) = gh\tau(y) \tag{12}$$

for all $h \in X^Y$ and all $y \in Y$, h is called a $\Phi-$coincidence point of F and g. In particular, in the case that $g = I_X$, i.e. $F(h\sigma_y) = h\tau(y)$, h is called a $\Phi-$fixed point of F.

Example 9. Let $X = Y = \mathbb{N}$. Let $\sigma : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$, $\sigma(m) = \sigma_m : \mathbb{N} \to \mathbb{N}$, $\sigma_m(n) = m + n$, $\tau : \mathbb{N} \to \mathbb{N}$, $\tau(n) = n + 1$, $g : X \to X$, $g(n) = n + 1$, $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, $F(h) = \min\{n + h(n) : n \in \mathbb{N}\}$, for all $m, n \in \mathbb{N}$ and $h \in \mathbb{N}^{\mathbb{N}}$. Then $h : \mathbb{N} \to \mathbb{N}$, $h(y) = y^2 + 1$ is a $\Phi-$coincidence point of F and g, since $F(h\sigma_y) = y^2 + 2y + 3 = gh\tau(y)$ for all $y \in \mathbb{N}$.

Example 10. Let $(Y,\cdot)$ be any abelian group and $P(Y)$ denote the family of nonempty subsets of Y. We consider functions from Y to $P(Y)$, which also have been studied for their own sake in the view of the fixed point theory, e.g. Fisher (1982), Achari (1986). Let $F:P(Y)^Y \to P(Y)$ be defined as $F(h)=\{x: x\in h(x)\}$ for all maps $h:Y\to P(Y)$. Then for any subgroup of Y, the corresponding canonical map $q:Y\to P(Y)$ is a $\Phi-$ fixed point of F, where $\Phi=(\sigma,\tau)$, $\sigma:Y\to Y^Y$, $\sigma(\mathrm{y})=\sigma_y:Y\to Y$, $\sigma_y(z)=yz^2$ for all $y,z\in Y$ and $\tau:Y\to Y$, $\tau(y)=y^{-1}$, since $F(q\sigma_y)=\{x:x\in q(yx^2)\}=q(y^{-1})=q\tau(y)$.

Theorem 11. Let F and g be commuting. If $h\in X^Y$ is a $\Phi-$coincidence point of F and g, then gh is also a $\Phi-$coincidence point of F and g.

Proof: Since F and g are commuting, $gF(h)=F(gh)$ for all $h\in X^Y$ and since h is a $\Phi-$coincidence point of F and g, we have $F(h\sigma_y)=gh\tau(y)$ for each $y\in Y$. Thus

$$F(gh\sigma_y)=gF(h\sigma_y)=ggh\tau(y) \tag{13}$$

and also gh is a $\Phi-$coincidence point.

Theorem 12. Given an ordered metric space $(X,d,\leq)$. Let $F:X^Y\to X$ be a function with the mixed g-monotony property, where $g:X\to X$ be a continuous map such that F and g be commuting and $F(X^Y)\subseteq g(X)$. Let $\sigma:Y\to Y^Y$, $\sigma(y)=\sigma_y$, $\sigma(A)\subseteq\Omega_{A,B}$ and $\sigma(B)\subseteq\Omega'_{A,B}$. Assume that $\tau\in\Omega_{A,B}$ be a bijection and $\Phi=(\sigma,\tau)$. Suppose that there exists a constant $\lambda\in[0,1)$ such that

$$gh\leq_Y gk \Rightarrow d\left(F(h),F(\mathrm{k})\right)\leq \lambda d^Y(gh,gk) \tag{14}$$

for all $h,k\in X^Y$, and also suppose that there exists a point $h_0\in X^Y$, which satisfies $gh_0\tau(y)\leq_y F(h_0\sigma_y)$ for all $y\in Y$. If X has sequential g-monotony property or F is continuous, then F and g have some $\Phi-$coincidence point.

Proof: Since $F(X^Y)\subseteq g(X)$, there exists a point $x_y\in X$ for each $y\in Y$, such that $g(x_y)=F(h_0\sigma_y)$. By choice axiom, which is not needed in the case that g is injective, there exists a function $\rho:Y\to X$, for which $g\rho(y)=F(h_0\sigma_y)$ for all $y\in Y$. We now define $h_1\in X^Y$ such that $h_1(y)=\rho\tau^{-1}(y)$ for all $y\in Y$, so we have $gh_1\tau(y)=g\rho(y)=F(h_0\sigma_y)$. We can similarly define a function $h_2\in X^Y$ such that $gh_2\tau(y)=F(h_1\sigma_y)$ for all $y\in Y$. Continuing this process, one can obtain a sequence (h_n) on X^Y such that $gh_{n+1}\tau(y)=F(h_n\sigma_y)$ for all $n\in\mathbb{N}$.

Now we show that $gh_{n-1}\leq_Y gh_n$ for each positive integer n.

We may easily point out that $gh_0\tau(y)\leq_y F(h_0\sigma_y)=gh_1\tau(y)$ for all $y\in Y$ by the hypothesis of the theorem and the definition of h_1. This implies that $gh_0(y)\leq_y gh_1(y)$ for all $y\in Y$, since τ is surjective and since $\tau(A)\subseteq A$ and $\tau(B)\subseteq B$ by $\tau\in\Omega_{A,B}$, hence $gh_{n-1}\leq_Y gh_n$ for $n=1$.

Now assume as induction hypothesis that $gh_{n-1} \leq_Y gh_n$, i.e. $gh_{n-1}(z) \leq_z gh_n(z)$ for all $z \in Y$. We want to show that $gh_n \leq_Y gh_{n+1}$.

If $y \in A$, then $\sigma(A) \subseteq \Omega_{A,B}$ and so $\sigma_y(z) \in A$ for $z \in A$ and $\sigma_y(z) \in B$ for $z \in B$. Thus $\leq_{\sigma_y(z)}$ is identical to $\leq_z$ for all z. For $\sigma_y(z) \in Y$ we can write $gh_{n-1}\sigma_y(z) \leq_{\sigma_y(z)} gh_n\sigma_y(z)$, which gives that $gh_{n-1}\sigma_y(z) \leq_z gh_n\sigma_y(z)$ for all $z \in Y$. Hence $gh_{n-1}\sigma_y \leq_Y gh_n\sigma_y$. Since F has the mixed $g-$monotony property, thus $F(h_{n-1}\sigma_y) \leq F(h_n\sigma_y)$.

If $y \in B$, then $\sigma_y(z) \in B$ for $z \in A$ and $\sigma_y(z) \in B$ for $z \in A$. Thus $\leq_{\sigma_y(z)}$ is identical to $\geq_z$ for all $z \in Y$, where $\geq_z$ denotes the inverse of $\leq_z$. We can write $gh_{n-1}\sigma_y(z) \leq_{\sigma_y(z)} gh_n\sigma_y(z)$ for $\sigma_y(z) \in Y$, so $gh_{n-1}\sigma_y(z) \geq_z gh_n\sigma_y(z)$ for all $z \in Y$. Hence $gh_n\sigma_y \leq_Y gh_{n-1}\sigma_y$ and $F(h_n\sigma_y) \leq F(h_{n-1}\sigma_y)$ by mixed $g-$monotony property.

Using two inequalities for $y \in A$ and for $y \in B$, it can be written as $F(h_{n-1}\sigma_y) \leq_y F(h_n\sigma_y)$ for all $y \in Y$. So we obtain $gh_n\tau(y) \leq_y gh_{n+1}\tau(y)$, since by the definition of the sequence (h_n) we have $gh_n\tau(y) = F(h_{n-1}\sigma_y)$ and $gh_{n+1}\tau(y) = F(h_n\sigma_y)$. But $gh_n(y) \leq_y gh_{n+1}(y)$ for all $y \in Y$ since $\tau \in \Omega_{A,B}$ is bijective and $\leq_y$ is identical to $\leq_{\tau(y)}$, and now we have $gh_n \leq_Y gh_{n+1}$.

We have shown that $gh_{n-1} \leq_Y gh_n$ for all positive integers n. Additionally we get $gh_{n-1}\sigma_y \leq_Y gh_n\sigma_y$ or $gh_n\sigma_y \leq_Y gh_{n-1}\sigma_y$, when $y \in A$ or $y \in B$ respectively. In both cases we can compare the functions $gh_{n-1}\sigma_y$ and $gh_n\sigma_y$ under the relation $\leq_Y$. For each $y \in Y$ we may write

$$d\left(F(h_{n-1}\sigma_y), F(h_n\sigma_y)\right) \leq \lambda d^Y(gh_{n-1}\sigma_y, gh_n\sigma_y) \tag{15}$$

since we know by the hypothesis of the theorem that $d\left(F(h), F(k)\right) \leq \lambda d^Y(gh, gk)$ for all $h, k \in X^Y$ such that $gh \leq_Y gk$ (or $gk \leq_Y gh$, since metric function is symmetric). Thus

$$\begin{aligned} d\left(gh_n\tau(y), gh_{n+1}\tau(y)\right) &= d\left(F(h_{n-1}\sigma_y), F(h_n\sigma_y)\right) \leq \lambda d^Y(gh_{n-1}\sigma_y, gh_n\sigma_y) \\ &= \lambda \sup_{z \in Y} d\left(gh_{n-1}\sigma_y(z), gh_n\sigma_y(z)\right) = \lambda \sup_{z \in \sigma_y(Y)} d\left(gh_{n-1}(z), gh_n(z)\right) \\ &\leq \lambda \sup_{z \in Y} d\left(gh_{n-1}(z), gh_n(z)\right) = \lambda d^Y(gh_{n-1}, gh_n) \end{aligned}$$

for all positive integers n and $y \in Y$. So

$$\sup_{y \in Y} d\left(gh_n\tau(y), gh_{n+1}\tau(\mathrm{y})\right) \leq \lambda \sup_{y \in Y} d^Y(gh_{n-1}, gh_n) \tag{16}$$

and since τ is a bijection

$$\sup_{y \in Y} d\left(gh_n(y), gh_{n+1}(y)\right) \leq \lambda d^Y(gh_{n-1}, gh_n) \tag{17}$$

Now,

$$d^{Y}(gh_{n},gh_{n+1})\leq\lambda d^{Y}(gh_{n-1},gh_{n}) \tag{18}$$

for all positive integers n. Thus we have the inequalities

$$d^{Y}(gh_{n},gh_{n+1})\leq\lambda d^{Y}(gh_{n-1},gh_{n})\leq\lambda^{2}d^{Y}(gh_{n-2},gh_{n-1})\leq\ldots\leq\lambda^{n}d^{Y}(gh_{0},gh_{1}) \tag{19}$$

for all integers $n\geq 0$, and by triangle inequality

$$\begin{aligned} d^{Y}(gh_{n},gh_{n+p}) &\leq d^{Y}(gh_{n},gh_{n+1})+d^{Y}(gh_{n+1},gh_{n+2})+\cdots+d^{Y}(gh_{n+p-1},gh_{n+p}) \\ &\leq\lambda^{n}d^{Y}(gh_{0},gh_{1})+\lambda^{n+1}d^{Y}(gh_{0},gh_{1})+\cdots+\lambda^{n+p-1}d^{Y}(gh_{0},gh_{1}) \\ &=\lambda^{n}(1+\lambda+\cdots+\lambda^{p-1})\cdot d^{Y}(gh_{0},gh_{1})=\lambda^{n}\tfrac{1-\lambda^{p}}{1-\lambda}\cdot d^{Y}(gh_{0},gh_{1})\leq\tfrac{\lambda^{n}}{1-\lambda}d^{Y}(gh_{0},gh_{1}) \end{aligned}$$

for all integers $p\geq 1$ and $n\geq 0$.

Now we are able to show that (gh_{n}) is a Cauchy sequence. Let $\varepsilon>0$. Choose an integer n_{0} such that $\tfrac{\lambda^{n_0}}{1-\lambda}d^{Y}(gh_{0},gh_{1})<\varepsilon$. Since

$$d^{Y}(gh_{n},gh_{n+p})\leq\tfrac{\lambda^{n}}{1-\lambda}d^{Y}(gh_{0},gh_{1})\leq\tfrac{\lambda^{n_0}}{1-\lambda}d^{Y}(gh_{0},gh_{1})<\varepsilon \tag{20}$$

for $n\geq n_{0}$ and $p\geq 1$, (gh_{n}) is a Cauchy sequence. The completeness of (X,d) implies that of (X^{Y},d^{Y}), and there exists a $h\in X^{Y}$ such that $gh_{n}\to h$, as $n\to\infty$. Then, for all $y\in Y$, $gh_{n}(y)\to h(y)$, as $n\to\infty$ on (X,d), since $d\big(gh_{n}(y),h(y)\big)\leq d^{Y}(gh_{n},h)$ by the definition of d^{Y}.

On the other hand if $\upsilon:Y\to Y$ is any function, then the inequality

$$d^{Y}(h\upsilon,k\upsilon)=\sup_{y\in Y}d\big(h\upsilon(y),k\upsilon(y)\big)=\sup_{y\in\upsilon(Y)}d\big(h(y),k(y)\big)\leq\sup_{y\in Y}d\big(h(\mathrm{y}),k(y)\big)=d^{Y}\big(h,k\big) \tag{21}$$

is true for any $h,k\in X^{Y}$. Since in particular $d^{Y}\big(gh_{n},gh_{n+p}\big)\geq d^{Y}\big(gh_{n}\upsilon,gh_{n+p}\upsilon\big)$, $\big(gh_{n}\upsilon\big)$ is also a Cauchy sequence on X^{Y}. Say $gh_{n}\upsilon\to k\in X^{Y}$, as $n\to\infty$. Again,

$$d\big(gh_{n}\upsilon(y),h\upsilon(y)\big)\leq d^{Y}(gh_{n}\upsilon,h\upsilon) \tag{22}$$

and $gh_{n}\upsilon(y)\to k(y)$, as $n\to\infty$ for all $y\in Y$. Since

$$\begin{aligned} d\big(k(y),h\upsilon(y)\big) &\leq d\big(k(y),gh_{n}\upsilon(y)\big)+d\big(gh_{n}\upsilon(y),h\upsilon(y)\big) \\ &\leq d^{Y}(k,gh_{n}\upsilon)+d^{Y}(gh_{n}\upsilon,h\upsilon) \\ &\leq d^{Y}(k,gh_{n}\upsilon)+d^{Y}(gh_{n},h)<\varepsilon \end{aligned}$$

for all $y\in Y$ and $\varepsilon>0$, $k(y)=h\upsilon(y)$ for all $y\in Y$, so $k=h\upsilon$. Hence $gh_{n}\upsilon\to h\upsilon$, as $n\to\infty$.

Now for $\upsilon = \tau$ we have $gh_n\tau(y) \to h\tau(y)$, as $n \to \infty$, or equivalently $gh_{n+1}\tau(y) \to h\tau(y)$, as $n \to \infty$ and since g is continuous $ggh_{n+1}\tau(y) \to gh\tau(y)$, as $n \to \infty$. As F and g are commuting

$$ggh_{n+1}\tau\left(y\right) = gF(h_n\sigma_y) = F(gh_n\sigma_y) \tag{23}$$

and also for any $y \in Y$ and for $\upsilon = \sigma_y$, we can write $gh_n\sigma_y \to h\sigma_y$, as $n \to \infty$.

Now, if F is continuous, then

$$F(h\sigma_y) = F(\lim gh_n\sigma_y) = \lim F(gh_n\sigma_y) = \lim ggh_{n+1}\tau(y) = gh\tau(y). \tag{24}$$

Thus $F(h\sigma_y) = gh\tau(y)$ for all $y \in Y$. So h be a $\Phi-$coincidence point for F and g.

We complete the proof by considering the case, where X has sequential g-monotony property.

We know that $gh_n \leq_Y gh_{n+1}$, and so $gh_n(y) \leq_y gh_{n+1}(y)$ for all n and $y \in Y$.

If $y \in A$, then $z \in A \Leftrightarrow \sigma_y(z) \in A$. So the inequality $gh_n\sigma_y(z) \leq_{\sigma_y(z)} gh_{n+1}\sigma_y(z)$ can be written as $gh_n\sigma_y(z) \leq_z gh_{n+1}\sigma_y(z)$. If $y \in B$, then $z \in A \Leftrightarrow \sigma_y(z) \in B$ and so $gh_n\sigma_y(z) \leq_{\sigma_y(z)} gh_{n+1}\sigma_y(z)$ becomes $gh_{n+1}\sigma_y(z) \leq_z gh_n\sigma_y(z)$. Then we deduce that

$$\begin{aligned} gh_n\sigma_y(z) &\leq gh_{n+1}\sigma_y(z), \text{ if either } y, z \in A \text{ or } y, z \in B \\ gh_n\sigma_y(z) &\geq gh_{n+1}\sigma_y(z), \text{ if either } y \in A,\ z \in B \text{ or } y \in B,\ z \in A \end{aligned} \tag{25}$$

From the fact that $gh_n \to h$, as $n \to \infty$, $gh_n\sigma_y(z) \to h\sigma_y(z)$, as $n \to \infty$ for all $y, z \in Y$ and by sequential $g-$monotony property

$$\begin{aligned} ggh_n\sigma_y(z) &\leq gh\sigma_y(z), \text{ if either } y, z \in A \text{ or } y, z \in B \\ ggh_n\sigma_y(z) &\geq gh\sigma_y(z), \text{ if either } y \in A,\ z \in B \text{ or } y \in B,\ z \in A \end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$. This means that, $ggh_n\sigma_y \leq_Y gh\sigma_y$, while $y \in A$ and $gh\sigma_y \leq_Y ggh_n\sigma_y$, while $y \in B$. In both cases $ggh_n\sigma_y$ and $gh\sigma_y$ are comparable under $\leq_Y$. Then

$$d\left(F(gh_n\sigma_y), F(h\sigma_y)\right) \leq \lambda d^Y(ggh_n\sigma_y, gh\sigma_y) \tag{27}$$

since F has mixed $g-$ monotony property. Now $gh_n\sigma_y \to h\sigma_y$, as $n \to \infty$ and g is continuous, so $ggh_n\sigma_y \to gh\sigma_y$, as $n \to \infty$, and $d\left(F(gh_n\sigma_y), F(h\sigma_y)\right) \to 0$, as $n \to \infty$. This yields again

$$F(h\sigma_y) = \lim F(gh_n\sigma_y) = \lim gF(h_n\sigma_y) = \lim ggh_{n+1}\tau(y) = g \lim gh_{n+1}\tau(\mathrm{y}) = gh\tau(y), \tag{28}$$

which completes the proof.

Theorem 12 does not guarantee uniqueness. For example, if F and g are constant mappings with the same constant value, then the hypothesis of the theorem is satisfied, but each point of X^Y is a $\Phi-$coincidence point.

Theorem 13. Under the hypothesis of Theorem 12, assume that for each $h,k\in X^Y$ being $\Phi-$coincidence point of F and g, there is a $q\in X^Y$ such that gq is comparable with both gh and gk under the relation $\leq_Y$. Then there is a unique point $s\in X^Y$ satisfying $gs=s$ and being a $\Phi-$coincidence point of F and g.

Proof: Let h and k be two $\Phi-$coincidence points of F and g. Assume that there exists a $q\in X^Y$ such that gq is comparable with both gh and gk under the relation $\leq_Y$.

Say $q_0:=q$ and by the similar way used in the proof of Theorem 12 we obtain a sequence $(q_1,q_2,\ldots)$ on X^Y such that $gq_{n+1}\tau(y)=F(q_n\sigma_y)$ for all positive integers n. Since gq is comparable with gh, $gh\leq_Y gq_0$ or $gq_0\leq_Y gh$ and considering the case $gq_0\leq_Y gh$,

$$\begin{aligned}
gq_0\leq_Y gh &\Rightarrow gq_0(z)\leq_z gh(z), \text{ for all } z\in Y\\
&\Rightarrow gq_0\sigma_y(z)\leq_{\sigma_y(z)} gh\sigma_y(z), \text{ for all } y,z\in Y\\
&\Rightarrow \begin{cases} gq_0\sigma_y(z)\leq_z gh\sigma_y(z), & \text{if } y\in A\\ gh\sigma_y(z)\leq_z gq_0\sigma_y(z), & \text{if } y\in B\end{cases}, \text{ for all } y,z\in Y\\
&\Rightarrow \begin{cases} gq_0\sigma_y\leq_Y gh\sigma_y, & \text{if } y\in A\\ gh\sigma_y\leq_Y gq_0\sigma_y, & \text{if } y\in B\end{cases}, \text{ for all } y\in Y\\
&\Rightarrow \begin{cases} F(q_0\sigma_y)\leq F(h\sigma_y), & \text{if } y\in A\\ F(h\sigma_y)\leq F(q_0\sigma_y), & \text{if } y\in B\end{cases}, \text{ for all } y\in Y\\
&\Rightarrow F(q_0\sigma_y)\leq_y F(h\sigma_y), \text{ for all } y\in Y\\
&\Rightarrow gq_1\tau(y)\leq_y gh\tau(y), \text{ for all } y\in Y\\
&\Rightarrow gq_1(y)\leq_y gh(y), \text{ for all } y\in Y\\
&\Rightarrow gq_1\leq_Y gh
\end{aligned}$$

by the mixed $g-$monotony property. Continuing this process yields $gq_n\leq_Y gh$ for all $n\in\mathbb{N}$. Moreover, we see that

$$gq_n\sigma_y\leq_Y gh\sigma_y \quad\text{or}\quad gh\sigma_y\leq_Y gq_n\sigma_y \tag{29}$$

for all positive integers n and $y\in Y$. On the other hand if $gh\leq_Y gq_0$, similarly $gh\leq_Y gq_n$ for all $n\in\mathbb{N}$ and

$$\text{either } gq_n\sigma_y \leq_Y gh\sigma_y \text{ or } gh\sigma_y \leq_Y gq_n\sigma_y \tag{30}$$

for all $y \in A$. Thus, in all cases $gq_n\sigma_y$ and $gh\sigma_y$ are comparable under the relation $\leq_Y$, and by hypothesis of existence theorem

$$d\left(gq_{n+1}\tau(y), gh\tau(y)\right) = d\left(F(q_n\sigma_y), F(h\sigma_y)\right) \leq \lambda d^Y(gq_n\sigma_y, gh\sigma_y) \leq \lambda d^Y(gq_n, gh). \tag{31}$$

Since τ is bijection, supremum on the left side yields

$$d^Y(gq_{n+1}, gh) \leq \lambda d^Y(gq_n, gh). \tag{32}$$

Additionally since $\lambda \in [0,1)$, this means that $d^Y(gq_n, gh) \to 0$, as $n \to \infty$. Hence $gq_n \to gh$, as $n \to \infty$. It can be similarly shown that $gq_n \to gk$, as $n \to \infty$.

Consequently $gh = gk$ for any two $\Phi-$coincidence points h and k of F and g. If h is a $\Phi-$coincidence point of F and g, then gh is so, thus $gh = ggh$. So $gs = s$ for at least one $\Phi-$coincidence point of F and g. On the other hand, let $s_1, s_2 \in X^Y$ be $\Phi-$coincidence points of F and g such that $gs_1 = s_1$ and $gs_2 = s_2$. Then $gs_1 = gs_2$ and so $s_1 = s_2$.

Let $q \in X^Y$. We can define a function $q^* \in X^Y$ such that $q^*(y) = F(q\sigma_{\tau^{-1}(y)})$. Then, Theorem 13 holds also under the following alternative hypothesis: For each $h, k \in X^Y$ being $\Phi-$coincidence point of F and g, there exists $q \in X^Y$ such that q^* is comparable with both gh and gk under the relation $\leq_Y$. For $q_0 := q$, we can begin a similar proof by $gq_1\tau(y) = F(q_0\sigma_y) = F(q\sigma_y) = q^*\tau(y)$ is comparable with $gh\tau(y)$ under $\leq$.

Corollary 14. In addition to the hypothesis of Theorem 13, if $gh = gk$ implies $h = k$ for all $\Phi-$coincidence points h and k of F and g, then there is exactly one $\Phi-$coincidence point of F and g.

Corollary 15. Besides the hypothesis of Theorem 13, assume also that $s\sigma_y$ is comparable to $s\sigma_z$ under $\leq_Y$ for all $y, z \in Y$. Then $s \in X^Y$ is a constant function.

Proof: Say $M = \sup_{y,z \in Y} d\left(s(y), s(z)\right)$. For any bijection τ, $M = \sup_{y,z \in Y} d\left(s\tau(y), s\tau(z)\right)$. Let $s\sigma_y$ and $s\sigma_z$ be comparable, so either $s\sigma_y \leq_Y s\sigma_z$ or $s\sigma_z \leq_Y s\sigma_y$, and since $gs = s$, either $gs\sigma_y \leq_Y gs\sigma_z$ or $gs\sigma_z \leq_Y gs\sigma_y$. By the hypothesis of the existence theorem

$$\begin{aligned} d\left(s\tau(y), s\tau(z)\right) &= d\left(gs\tau(y), gs\tau(z)\right) = d\left(F(s\sigma_y), F(s\sigma_z)\right) \\ &\leq \lambda d^Y(gs\sigma_y, gs\sigma_z) = \lambda d^Y(s\sigma_y, s\sigma_z) = \lambda \sup_{w \in Y} d\left(s\sigma_y(w), s\sigma_z(w)\right) \\ &\leq \lambda \sup_{y,z \in Y} d\left(s(y), s(z)\right) \leq \lambda d^Y(gs\sigma_y, gs\sigma_z) = \lambda d^Y(s\sigma_y, s\sigma_z) \end{aligned}$$

which yields by taking supremum that

$$\sup_{y,z\in Y} d\left(s(y),s(z)\right)=\sup_{y,z\in Y} d\left(s\tau(y),s\tau(z)\right)$$
$$\leq \lambda \sup_{y,z\in Y} d\left(s(y),s(z)\right).$$

Then, since $0\leq \lambda <1$, $s(y)=s(z)$ for all $y,z\in Y$.

Considering the case $g=I_X$, the facts obtained about $\Phi-$coincidence points can be restated for $\Phi-$fixed points.

Corollary 16. Given an ordered metric space $(X,d,\leq)$ and a non-decreasing function $F:(X^Y,\leq_Y)\to(X,\leq)$. Let $\Phi=(\sigma,\tau)$, where $\tau\in\Omega_{A,B}$ is a bijection, $\sigma:Y\to Y^Y$, $\sigma(y)=\sigma_y$, $\sigma(A)\subseteq\Omega_{A,B}$ and $\sigma(B)\subseteq\Omega'_{A,B}$. Suppose that there exists a constant $\lambda\in[0,1)$ such that

$$h\leq_Y k\Rightarrow d\left(F(h),F(k)\right)\leq \lambda d^Y(h,k) \tag{33}$$

for all $h,k\in X^Y$, and there exists a point $h_0\in X^Y$ such that $h_0\tau(y)\leq_y F(h_0\sigma_y)$ for all $y\in Y$. If F is continuous or X has sequential monotony property, then F has at least one $\Phi-$fixed point.

If, in addition, there exists $q\in X^Y$ such that q is comparable with both h and k under the relation $\leq_Y$ for each $\Phi-$fixed point $h,k\in X^Y$ of F, then there exists a unique $\Phi-$fixed point s of F.

Moreover, also if $s\sigma_y$ is comparable with $s\sigma_z$ under $\leq_Y$ for all $y,z\in Y$, then s is a constant function.

Example 17. Let $X=[0,1]$ given with usual metric and $Y=\mathbb{N}$. Then X^Y corresponds to the set of all sequences on $[0,1]$. Assume that $A=\mathbb{N}$, $B=\varnothing$ and $\tau=\sigma_y=I_Y$ for all $y\in Y$. For any constant c, $0\leq c\leq 1$, define the function $F:X^Y\to X$ by $F(h)=\Sigma_{n=1}^{\infty}\frac{h(n)+c}{3^n}$. Then,

$$d\left(F(h),F(k)\right)=\left|\Sigma_{n=1}^{\infty}\frac{h(n)+c}{3^n}-\Sigma_{n=1}^{\infty}\frac{k(n)+c}{3^n}\right|\leq \sup_{n\in\mathbb{N}}\left|h(n)-k(n)\right|\cdot\Sigma_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}\cdot d^Y(h,k) \tag{34}$$

for all $h,k\in X^Y$. Also it is clear that F is a non-decreasing function and the space $[0,1]$ is sequential monotone. Consider $h_0\in X^Y$ as the constant function $h_0:\mathbb{N}\to[0,1]$, $h_0(n)=0$. So $h_0\tau(y)=0\leq_n F(h_0\sigma_y)$ for all $n\in\mathbb{N}$, since $n\in A=\mathbb{N}$ so that $\leq_n=\leq$. Hence F has at least one $\Phi-$fixed point. Indeed for the constant function $s:\mathbb{N}\to[0,1]$, $s(n)=c$ for all $n\in\mathbb{N}$, then

$$F(s\sigma_y)=F(s)=\Sigma_{n=1}^{\infty}\frac{s(n)+c}{3^n}=2c\Sigma_{n=1}^{\infty}\frac{1}{3^n}=c=s\tau(n) \tag{35}$$

This $\Phi-$ fixed point is unique, since the function q, defined as $q(n)=\min\{h(n),k(n)\}$ is comparable with both h and k under the relation $\leq_{\mathbb{N}}$, and by the fact that $s\sigma_y = s\sigma_z$ for all $y,z\in\mathbb{N}$ since $\sigma_y=\sigma_z=I_{\mathbb{N}}$, we again see that s is a constant function.

APPLICATION

As stated in Bellman & Lee (1978), many functional equations arising in dynamic programming have the form

$$g(p)=\max_{q} G\left(p,q,g\left(T(p,q)\right)\right) \tag{36}$$

where p and q are state and decision vectors, g is the optimal return function and T is transformation of the process. Here we give an existence and uniqueness theorem, as an application of Corollary 16, for a special case of these equations, in which the function G is independent of state, i.e. constant in the first argument.

Theorem 18. Let U and V be Banach spaces, $Y\subseteq U$ and $Z\subseteq V$. Also let $X\subseteq \mathbb{R}$ be a bounded subset and $T:Y\times Z\to Y$, and $H:Z\times X\to X$ be functions. Suppose that the following conditions hold:

i) There exists a bijection $f:Y\to Z$ such that $T\left(y_1,f(y_2)\right)=T\left(y_2,f(y_1)\right)$ for all $y_1,y_2\in Y$.

ii) There exists a $\lambda\in\mathbb{R}$ such that $0\leq\frac{H(z,b)-H(z,a)}{b-a}<\lambda<1$ for all $z\in Z$ and $a,b\in X$, $a\neq b$.

Then the functional equation

$$h(y)=\sup_{z\in Z} H\left(z,h\left(T(y,z)\right)\right) \tag{37}$$

has a unique solution.

Proof: Let d denote the standard metric on the bounded set $X\subseteq\mathbb{R}$, ordered with the usual order $\leq$. Define $F:X^Y\to X$ as $F(h)=\sup\limits_{z\in Z} H\left(z,h\left(f^{-1}(z)\right)\right)$. Say $A=Y$ and $B=\varnothing$. So $\leq_y$ is identical to $\leq$ for all $y\in Y$. For all $y\in Y$, we define $\sigma_y:Y\to Y$ with $\sigma_y(y')=T\left(y,f(y')\right)$ for all $y'\in Y$. By the selection of the sets A and B, it is clear that $\sigma\left(A\right)\subseteq\Omega_{A,B}$ and $\sigma\left(B\right)\subseteq\Omega'_{A,B}$.

Let $h,k\in X^Y$ such that $h\leq_Y k$. Since $0\leq\frac{H(z,b)-H(z,a)}{b-a}$ for all $z\in Z$, $a,b\in X$, H is non-decreasing in second argument, so that

$$F\left(h\right)=\sup_{z\in Z} H\left(z,h\left(f^{-1}(z)\right)\right)\leq\sup_{z\in Z} H\left(z,k\left(f^{-1}(z)\right)\right)\leq F(k). \tag{38}$$

Hence $F:(X^Y,\leq_Y)\to(X,\leq)$ is non-decreasing, and since

$$H\left(z,h\left(k^{-1}(z)\right)\right)-H\left(z,h\left(f^{-1}(z)\right)\right)\leq \lambda\left[k\left(f^{-1}(z)\right)-h\left(f^{-1}(z)\right)\right]\leq \lambda\sup_{y\in Y}\{k(y)-h(y)\} \quad (39)$$

for all $z\in Z$ from (ii), we have

$$d\left(F(h),F(k)\right)=\sup_{z\in Z}H\left(z,k\left(f^{-1}(z)\right)\right)-\sup_{z\in Z}H\left(z,h\left(f^{-1}(z)\right)\right)$$
$$\leq \lambda\sup_{y\in Y}\{k(y)-h(y)\}=\lambda d^{Y}(h,k).$$

Define $h_0:Y\to X$ as the function with the constant value $\inf X$. Then $h_0(y)\leq F(h_0\sigma_y)$ for all $y\in Y$. Finally, X has the sequential monotony property, since it is endowed with the standard metric.

Thus all hypotheses related to existence in Corollary 16 are satisfied for $\tau = I_Y$, and F has a $\Phi-$fixed point h, where $\Phi=(\sigma,\tau)$. Now we have

$$F(h\sigma_y)=\sup_{z\in Z}H\left(z,h\sigma_y\left(f^{-1}(z)\right)\right)=\sup_{z\in Z}H\left(z,hT\left(y,f\left(f^{-1}(z)\right)\right)\right)=h\tau(y). \quad (40)$$

So there exists a function $h:Y\to X$ such that $h(y)=\sup_{z\in Z}H\left(z,h\left(T(y,z)\right)\right)$. In addition, this function is unique by Corollary 16, since for any pair $h,k\in X^Y$, and the function q defined as $q(y):=\max\{h(y),k(y)\}$ is comparable with both h and k under the relation $\leq_Y$.

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حول نظرية نقطة صامدة لا منتهية البعدية على فضاءات مترية مرتبة

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خلاصة

نقوم في هذه الدارسة بتوسيع نظرية النقطة الصامدة متعددة البعدية و نقطة التطابق على فضاءات الجداء لا منتهية البعدية. نقوم كذلك بتعميم بعض النتائج المعروفة في هذا المجال.