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# On a Tauberian theorem for the weighted mean method of summability

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## ABSTRACT

We investigate conditions needed for a weighted mean summable series to be convergent by using Kloosterman's method. The results of this paper generalize the well known results of Landau and Hardy.

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**Keywords:** Summability by weighted means; Tauberian conditions.

## INTRODUCTION

Let $(a_n)$ be a sequence of real numbers. Throughout this paper, $N_0$ will denote the set of all nonnegative integers and the symbols $a_n = o(1)$, $a_n = O(1)$ and $a_n = O_L(1)$ mean respectively that $a_n \to 0$ as $n \to \infty$, $(a_n)$ is bounded and $(a_n)$ is bounded below for large enough $n$. Assume that $p = (p_n)$ is sequence of nonnegative numbers with $p_0 > 0$, such that

$$P_n := \sum_{k=0}^{n} p_k \to \infty, n \to \infty. \tag{1}$$

For given sequences $(a_n)$ and $(p_n)$ let

$$s_n = a_0 + a_1 + \ldots + a_n,$$

$$t_n = p_0 s_0 + p_1 s_1 + \ldots + p_n s_n.$$

The weighted mean of the sequence of the partial sums $(s_n)$ is defined by

$$\sigma_n(s) := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k$$

for all $n \in \mathbb{N}_0$.

The sequence $(s_n)$ is said to be summable by the weighted mean method determined by the sequence $p$; in short, $(\overline{N}, p)$ summable to a finite number $L$ if

$$\lim_{n\to\infty} \sigma_n(s) = L. \tag{2}$$

The $(\overline{N}, p)$ summability method is regular if and only if $P_n \to \infty$ as $n \to \infty$.

If the limit

$$\lim_{n\to\infty} s_n = L \tag{3}$$

exists, then (2) also exists. However, the converse part is not always true. Note that (2) may imply (3) under certain conditions. Such conditions are known as Tauberian conditions in the literature. Any theorem which states that convergence of sequences follows from $(\overline{N}, p)$ summability method and some Tauberian condition is said to be a Tauberian theorem.

If $p_n = 1$ for all $n \in N_0$, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

Tauberian theorems for $(\overline{N}, p)$ summability method were studied by a number of authors such as Ananda-Rau (1930); Tietz (1990); Tietz & Zeller (1998); Móricz & Rhoades (1995) and Móricz & Stadtmüller (2001).

Recently, Çanak & Totur (2011,2013) and Totur & Çanak (2012) have obtained Tauberian conditions for $(\overline{N}, p)$ summability method in terms of the weighted general control modulo analogous to the one defined by Dik (2001).

In Kloosterman (1940), Kloosterman has used an elegant approach and obtained simplified proofs of some well known Tauberian theorems as well as generalizations of these theorems. Inspired by his approach, we prove that if $\sum_{n=0}^{\infty} a_n$ is $(\overline{N}, p)$ summable to $L$, then one-sided boundedness of $(a_n)$ with some conditions imposed on $(a_n)$ and $(p_n)$ yields convergence of the series $\sum_{n=0}^{\infty} a_n$ to $L$.

## AUXILARY RESULTS

We need the following lemmas for the proof of our theorem.

**Lemma 1** For all $h > 0$, we have

$$t_{n+h} = t_n + (p_{n+1} + p_{n+2} + \ldots + p_{n+h})s_n + \left(p_{n+1} + 2p_{n+2} + \ldots + hp_{n+h}\right)a_\xi$$

where $a_\xi$ is a number such that

$$\min_{n<k\le n+h} a_k \le a_\xi \le \max_{n<k\le n+h} a_k.$$

**Proof.** By the definition of $t_n$ and $s_n$, we have

$$t_{n+h} = \sum_{k=0}^{n+h} p_k s_k$$

$$= \sum_{k=0}^{n} p_k s_k + \sum_{k=n+1}^{n+h} p_k s_k$$

$$= t_n + \left( p_{n+1}s_{n+1} + p_{n+2}s_{n+2} + \ldots + p_{n+h}s_{n+h} \right)$$

$$= t_n + \big( p_{n+1}s_n + p_{n+1}a_{n+1} + p_{n+2}s_n + p_{n+2}(a_{n+1} + a_{n+2}) + \ldots + p_{n+h}s_n + p_{n+h}(a_{n+1} + a_{n+2} + \ldots + a_{n+h}) \big)$$

$$= t_n + (p_{n+1} + p_{n+2} + \ldots + p_{n+h})s_n + \big( (p_{n+1} + p_{n+2} + \ldots + p_{n+h})a_{n+1} + (p_{n+2} + p_{n+3} + \ldots + p_{n+h})a_{n+2} + \ldots + p_{n+h}a_{n+h} \big).$$

Now,

$$(p_{n+1} + p_{n+2} + \ldots + p_{n+h})a_{n+1} + (p_{n+2} + p_{n+3} + \ldots + p_{n+h})a_{n+2} + \ldots + p_{n+h}a_{n+h}$$

$$\leq \big( (p_{n+1} + p_{n+2} + \ldots + p_{n+h}) + (p_{n+2} + p_{n+3} + \ldots + p_{n+h}) + \ldots + p_{n+h} \big) \max_{n<k\leq n+h} a_k$$

$$= (p_{n+1} + 2p_{n+2} + \ldots + hp_{n+h}) \max_{n<k\leq n+h} a_k,$$

and similarly

$$(p_{n+1} + 2p_{n+2} + \ldots + hp_{n+h}) \min_{n<k\leq n+h} a_k$$

$$\leq (p_{n+1} + p_{n+2} + \ldots + p_{n+h})a_{n+1} + (p_{n+2} + p_{n+3} + \ldots + p_{n+h})a_{n+2} + \ldots + p_{n+h}a_{n+h}.$$

So, the given lemma is valid.

**Lemma 2** For all $k = -i < 0$, we have

$$t_n = t_{n-i} + (p_{n-i+1} + p_{n-i+2} + \ldots + p_n)s_n - \big( (i-1)p_{n-i+1} + (i-2)p_{n-i+2} + \ldots + p_{n-1} \big) a_\gamma)$$

where $a_\gamma$ is a number such that

$$\min_{n-i+1<k\leq n} a_k \leq a_\gamma \leq \max_{n-i+1<k\leq n} a_k.$$

**Proof.** By the definition of $t_n$ and $s_n$, we have

$$
\begin{aligned}
t_n &= \sum_{k=0}^{n} p_k s_k \\
&= \sum_{k=0}^{n-i} p_k s_k + \sum_{k=n-i+1}^{n} p_k s_k \\
&= t_{n-i} + \left( p_{n-i+1} s_{n-i+1} + p_{n-i+2} s_{n-i+2} + \ldots + p_n s_n \right) \\
&= t_{n-i} + \Big( p_{n-i+1} s_n - p_{n-i+1} (a_{n-i+2} + a_{n-i+3} + \ldots + a_n) \\
&\qquad + p_{n-i+2} s_n - p_{n-i+2} (a_{n-i+3} + a_{n-i+4} + \ldots + a_n) + \ldots + p_n s_n \Big) \\
&= t_{n-i} + (p_{n-i+1} + p_{n-i+2} + \ldots + p_n) s_n - \Big( p_{n-i+1} a_{n-i+2} + (p_{n-i+1} + p_{n-i+2}) a_{n-i+3} \\
&\qquad + (p_{n-i+1} + p_{n-i+2} + \ldots + p_{n-1}) a_n \Big).
\end{aligned}
$$

Now,

$$
\begin{aligned}
& p_{n-i+1} a_{n-i+2} + (p_{n-i+1} + p_{n-i+2}) a_{n-i+3} + \ldots + (p_{n-i+1} + p_{n-i+2} + \ldots + p_{n-1}) a_n \\
&\le \left( p_{n-i+1} + (p_{n-i+1} + p_{n-i+2}) + \ldots + (p_{n-i+1} + p_{n-i+2} + \ldots + p_{n-1}) \right) \max_{n-i+1<k\le n} a_k \\
&\qquad \le \left( (i-1) p_{n-i+1} + (i-2) p_{n-i+2} + \ldots + p_{n-1} \right) \max_{n-i+1<k\le n} a_k
\end{aligned}
$$

and in a similar way

$$
\begin{aligned}
& \left( (i-1) p_{n-i+1} + (i-2) p_{n-i+2} + \ldots + p_{n-1} \right) \min_{n-i+1<k\le n} a_k \\
&\le p_{n-i+1} a_{n-i+2} + (p_{n-i+1} + p_{n-i+2}) a_{n-i+3} + \ldots + (p_{n-i+1} + p_{n-i+2} + \ldots + p_{n-1}) a_n .
\end{aligned}
$$

So, the given lemma is valid.

The next lemma which restricts $p = (p_n)$ were given by Móricz & Rhoades (2004).

**Lemma 3** If $(P_n)$ is a nondecreasing sequence of positive numbers, then the conditions

$$\liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} > 1 \text{ for every } \lambda > 1 \tag{4}$$

and

$$\liminf_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} > 1 \text{ for every } 0<\lambda<1 \tag{5}$$

are equivalent.

It is clear that condition (4) implies (1).

## MAIN RESULT

**Theorem 1** Let $\sum_{k=0}^{\infty} a_k = L(\overline{N}, p)$. If the conditions (4),

$$n\frac{p_n}{P_n} = O(1) \tag{6}$$

and

$$a_n\frac{P_n}{p_n} = O_L(1) \tag{7}$$

are satisfied, then $\sum_{k=0}^{\infty} a_k = L$.

**Proof.** We may assume that $\sum_{k=0}^{\infty} a_k = 0(\overline{N}, p)$ since, if $\sum_{k=0}^{\infty} a_k = L(\overline{N}, p)$, then we can replace $a_0$ by $a_0 - L$. This implies that $s_n$ and $\sigma_n$ will be decreased by $L$. Also by the definition of $\sigma_n$ we have $t_n = P_n\sigma_n$. Now, if we rewrite the equation in Lemma 1 in a form that isolates $s_n$, we get

$$s_n = \frac{t_{n+h} - t_n}{p_{n+1} + p_{n+2} + \ldots + p_{n+h}} - \frac{p_{n+1} + 2p_{n+2} + \ldots + hp_{n+h}}{p_{n+1} + p_{n+2} + \ldots + p_{n+h}} a_\xi,$$

where $n < \xi \le n+h$.

By (7), we get

$$s_n \le \frac{P_{n+h}\sigma_{n+h} - P_n\sigma_n}{P_{n+h} - P_n} + \frac{h(P_{n+h} - P_n)}{P_{n+h} - P_n} C\frac{p_\xi}{P_\xi}$$

$$= \frac{(P_{n+h} - P_n)\sigma_{n+h} + P_n(\sigma_{n+h} - \sigma_n)}{P_{n+h} - P_n} + hC\frac{p_\xi}{P_\xi}$$

$$= \sigma_{n+h} + \frac{P_n}{P_{n+h} - P_n}(\sigma_{n+h} - \sigma_n) + hC\frac{p_\xi}{P_\xi},$$

for some $C$.

Since this inequality holds for all $h > 0$, we choose $h = [\lambda n] - n$, for $\lambda > 1$ which gives the following inequality

$$s_n \le \sigma_{[\lambda n]} + \frac{P_n}{P_{[\lambda n]} - P_n}(\sigma_{[\lambda n]} - \sigma_n) + ([\lambda n] - n)C\frac{p_\xi}{P_\xi}. \tag{8}$$

Then we have

$$s_n \le \sigma_{[\lambda n]} + \frac{P_n}{P_{[\lambda n]} - P_n}(\sigma_{[\lambda n]} - \sigma_n) + 2C(\lambda - 1)\frac{n}{\xi}(\xi\frac{p_\xi}{P_\xi}) \tag{9}$$

for sufficiently large $n$. Since $\frac{n}{\xi} < 1$ and $\xi\frac{p_\xi}{P_\xi} = O(1)$, we get

$$s_n \le \sigma_{[\lambda n]} + \frac{P_n}{P_{[\lambda n]} - P_n}(\sigma_{[\lambda n]} - \sigma_n) + C_1(\lambda - 1) \tag{10}$$

for some $C_1$.

Taking the limit of both sides of the inequality (10) as $n \to \infty$, we have

$$\lim s_n \le \lim \sigma_{[\lambda n]} + \lim \frac{P_n}{P_{[\lambda n]} - P_n} \lim(\sigma_{[\lambda n]} - \sigma_n) + C_1(\lambda - 1). \tag{11}$$

By (4), we get

$$\limsup \frac{P_n}{P_{[\lambda n]} - P_n} = \left\{\liminf \frac{P_{[\lambda n]}}{P_n} - 1\right\}^{-1} < \infty \tag{12}$$

By taking into account (12) and the assumed summability $(\overline{N}, p)$ of $(s_n)$, we conclude from the inequality (11) that

$$\lim s_n \le C_1(\lambda - 1). \tag{13}$$

Taking the limit of both sides of the inequality (13) as $\lambda \to 1^+$, we have

$$\lim s_n \le 0. \tag{14}$$

Now we go through a similar process to obtain a lower bound for the limit of $s_n$. Rewriting the equation in Lemma 2 in a suitable form to group all the $s_n$ terms together, we have

$$s_n = \frac{t_n - t_{n-i}}{p_{n-i+1} + p_{n-i+2} + \ldots + p_n} + \frac{\left((i-1)p_{n-i+1} + (i-2)p_{n-i+2} + \ldots + p_{n-1}\right)}{p_{n-i+1} + p_{n-i+2} + \ldots + p_n} a_\gamma,$$

where $n - i + 1 < \gamma \le n$.

By (7), we get

$$s_n \geq \frac{P_n\sigma_n - P_{n-i}\sigma_{n-i}}{P_n - P_{n-i}} + \frac{(i-1)(P_n - P_{n-i})}{P_n - P_{n-i}}\left(-C\frac{p_\gamma}{P_\gamma}\right)$$

$$= \frac{(P_n - P_{n-i})\sigma_n + P_{n-i}(\sigma_n - \sigma_{n-i})}{P_n - P_{n-i}} - (i-1)C\frac{p_\gamma}{P_\gamma}$$

$$= \sigma_n + \frac{P_{n-i}}{P_n - P_{n-i}}(\sigma_n - \sigma_{n-i}) - (i-1)C\frac{p_\gamma}{P_\gamma},$$

for some $C$.

Since $i > 0$ is arbitrary, we can cleverly take $i = n - [\lambda n]$, for $0 < \lambda < 1$ which gives the following inequality

$$s_n \geq \sigma_n + \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}}(\sigma_n - \sigma_{[\lambda_n]}) - (n - [\lambda n] - 1)C\frac{p_\gamma}{P_\gamma}. \tag{15}$$

Then we have

$$s_n \geq \sigma_n + \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}}(\sigma_n - \sigma_{[\lambda_n]}) - 2C(1-\lambda)\frac{n}{\gamma}(\gamma\frac{p_\gamma}{P_\gamma}). \tag{16}$$

for sufficiently large $n$. Since $\frac{n}{\gamma} \geq 1$ and $\gamma\frac{p_\gamma}{P_\gamma} = O(1)$, we get

$$s_n \geq \sigma_n + \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}}(\sigma_n - \sigma_{[\lambda_n]}) - 2C_2(1-\lambda) \tag{17}$$

for some $C_2$.

Taking the limit of both sides of the inequality (17) as $n \to \infty$, we have

$$\lim s_n \geq \lim \sigma_n + \lim \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}} \lim(\sigma_n - \sigma_{[\lambda_n]}) - C_2(1-\lambda). \tag{18}$$

By (4), we get

$$\limsup \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}} = \left\{\liminf \frac{P_n}{P_{[\lambda n]}} - 1\right\}^{-1} < \infty \tag{19}$$

By taking into account (19) and the assumed summability $(\overline{N}, p)$ of $(s_n)$, we conclude from the inequality (18) that

$$\lim s_n \geq -C_2(1-\lambda). \tag{20}$$

Taking the limit of both sides of the inequality (20) as $\lambda \to 1^-$, we have

$$\lim s_n \geq 0. \tag{21}$$

Combining (14) and (21) yields convergence of $(s_n)$ to 0. This completes the proof.

Our main result includes the following classical Tauberian theorems given for the Cesàro summability method.

**Corollary 1 (Landau (1910))** If $\sum_{n=0}^{\infty} a_n = L(C)$ and $na_n = O_L(1)$, then $\sum_{n=0}^{\infty} a_n = L$.

**Corollary 2 (Hardy (1910))** If $\sum_{n=0}^{\infty} a_n = L(C)$ and $na_n = O(1)$, then $\sum_{n=0}^{\infty} a_n = L$.

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# حول نظرية توبريان لطريقة الوسط الموزون للجمعية

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## خلاصة

نبحث في هذه الدراسة عن الشروط التي تحتاجها متسلسلة وسط موزون قابلة للجمع حتى تكون متقاربة وذلك بإستخدام طريقة كلوسترمان. ونتائج بحثنا هذا تعمم النتائج المعروفة التي جاء بها لاندو وهاردي.