***Kuwait J. Sci.* 42 (3)** pp. 105-119, 2015

An improvement of the douglas scheme for the Black-Scholes equation

FARES AL-AZEMI

Department of Mathematics, Kuwait University, Safat, 13060, Kuwait.

fares@sci.kuniv.edu.kw

ABSTRACT

A well-known finite difference scheme for the valuation of options from the Black-Sholes equation is the Crank-Nicolson scheme. However, in the case of non-smooth payoffs, the Crank-Nicolson scheme is known to produce unwanted oscillations for the computed solution. As an alternative, Douglas scheme is generally recommended for better resolution of option price because it has fourth order accuracy in asset derivative. However, as noted by Shaw in his book, both these methods show "potentially nasty behavior when applied to simple option pricing". We note that both the Crank-Nicolson scheme and the Douglas scheme use a trapezoidal formula for time integration which is known to produce unwanted oscillations in the computed solution. This works since the trapezoidal formula is only A-stable and not L-stable. Chawla and Evans proposed a new L-stable Simpson rule. We investigate the application of this L-stable third order rule for the time integration in the Black-Sholes equation after it has been semi-discretized in the asset derivative by Numerov discretisation. By numerical experimentation with real option valuation problems, we compare the performance of this new improved version of Douglas with both Crank-Nicolson and Douglas schemes. We also study the performance of this scheme for the valuation of the Greeks.

Key words: Black-Scholes; crank-nicolson; douglas scheme; option pricing.

MS Classification (2000): Primary: 74H15; Secondary: 91G20

INTRODUCTION

Let $V(S,t)$ denote the value of an option contingent on an asset with value S at time t, volatility of the underlying σ, exercise price E, expiration time T, and risk-free interest rate r. Then the celebrated Black-Scholes equationin Black & Scholes (1973) satisfied by the valuation of an option is described by

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0,\ 0<S<\infty,\ 0\leq t\leq T. \quad (1)$$

This is a backward parabolic equation and, for a complete mathematical specification, we need two boundary conditions

$$V(0,t) = V_0(t), \;\; V(\infty,t) = V_\infty(t), \tag{2}$$

and a final condition on the payoff

$$V(S,T) = V_T(S) \tag{3}$$

for given $V_0(t), V_\infty(t)$ and .$V_T(S)$.

Brennan & Schwartz (1978) were the first to describe the application of finite difference methods to option pricing. Geskec & Shastri (1985) give a comparison of the efficiency of various finite difference and other numerical methods for option pricing.

A well-known finite difference scheme for the valuation of an option from the Black-Sholes equation is Crank & Nicolson (1947); Shaw (1998) Chapters 13, 14, and 15, pp. 258-305 and Wilmott *et al.* 1995. However for non-smooth payoff, the Crank-Nicolson scheme is known to produce unwanted oscillations in the computed solution. As an alternative, Douglas scheme is generally recommended for better resolution of option price because it has fourth order accuracy in asset derivative. As noted by Shaw in his book Shaw (1998), both these methods show "potentially nasty behavior when applied to simple option pricing". We note that both the Crank-Nicolson scheme and the Douglas scheme use a trapezoidal formula for the time integration. This is known to produce unwanted oscillations in the computed solution since the trapezoidal formula is only A-stable and not L-stable. As an alternative, three-time level version of the Douglas scheme is generally recommended; see Shaw (1998), and Smith (1985). The Crank-Nicolson scheme has a local truncation error of order $O(h^2) + O(k^2)$, while the Douglas scheme has a local truncation error of order $O(h^4) + O(k^2)$. However, if a larger time step is used, there is no significant improvement by using Douglas over Crank-Nicolson scheme. We also note that the three-time level Douglas is only second order in time and not self starting.

Chawla & Evans (2005) proposed a new L-stable Simpson rule. We investigate the application of this L-stable third order rule for the time integration of the Black-Sholes equation after it has been semi-discretised in the asset derivative by Numerov discretisation (Numerov, 1924). By numerical experimentation with real option valuation problems, we compare the performance of this new improved version of Douglas with both Crank-Nicolson and Douglas schemes. We also study the performance of this scheme for the valuation of the Greeks. We assume no dividends are paid during the life of the option.

To develop finite difference methods for the numerical solution of the Black-

Scholes equation, we first need transform it into a standard forward diffusion problem as in the following section.

THE LOG-TRANSFORMATION

With the transformations

$$S = Ee^{x}, \;\; t = T - \frac{2\tau}{\sigma^2} \tag{4}$$

$$V(s,t) = Ee^{\frac{-1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x,\tau), \tag{5}$$

where $k = \frac{2r}{\sigma^2}$, the Black-Scholes equation is transformed into the following standard forward diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < \tau < \frac{1}{2}\tau^2 T \tag{6}$$

With suitably selected $x_{-\infty}$ to represent $-\infty$ and x_∞ to represent ∞, the boundary conditions are transformed into

$$u(x_{-\infty},\tau) = u_{-\infty}(\tau) = E^{-1}\exp\left\{\left[\frac{1}{2}(k-1)x_{-\infty}\right] + \left[\frac{1}{4}(k+1)^2\tau\right]\right\} V_0\left(T - \frac{2\tau}{\sigma^2}\right) \tag{7}$$

$$u(x_{\infty},\tau) = u_{\infty}(\tau) = E^{-1}\exp\left\{\left[\frac{1}{2}(k-1)x_{\infty}\right] + \left[\frac{1}{4}(k+1)^2\tau\right]\right\} V_\infty\left(T - \frac{2\tau}{\sigma^2}\right) \tag{8}$$

The final condition is transformed into the initial condition

$$u(x,0) = u_0(x), \tag{9}$$

where

$$u_0(x) = E^{-1}e^{\frac{1}{2}(k-1)x}V_T(Ee^{x}). \tag{10}$$

It is worth noting the well known property of the diffusion equation that in the case of double infinite domain the prescription of specific conditions at $\pm\infty$ is irrelevant since these cannot affect the solution in the finite part of the domain.

SECOND ORDER CENTRAL DIFFERENCE SEMI-DISCRETISATION

For suitable positive integers N^+, N^- and step h, define the spatial grid $x_i = (i - N^- - 1)h, i = 1, \cdots, N$, where we have set $N = N^- + N^+ + 1$, and $X_{-\infty} = -(N^- + 1)h, X_\infty = (N^+ + 1)h$.

For a positive integer M, define the temporal grid $\tau_j = jk, j = 0,1,\cdots$, where $k = \frac{1}{2}\sigma^2\frac{T}{M}$. In the following, we set $\rho = \frac{k}{h^2}$, and $u_i(\tau) = u(x_i,\tau), u_{ij} = u_i(\tau_j)$, etc.

Now, a central difference discretisation of the transformed Black-Scholes differential equation in (6) is given by

$$\frac{\partial}{\partial\tau}u_i(\tau) = \frac{1}{h^2}[u_{i+1}(\tau)2u_i(\tau) + u_{i-1}(\tau)], i = 1, \dots, N. \tag{11}$$

Consider the matrices

$$\mathbf{u}(\tau) = \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \cdot \\ \vdots \\ u_{N-1}(\tau) \\ u_N(\tau) \end{bmatrix}, \mathrm{C}(\tau) = \begin{bmatrix} u_{-\infty}(\tau) \\ 0 \\ \cdot \\ \vdots \\ 0 \\ u_\infty(\tau) \end{bmatrix}, \mathrm{J} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Then the discretisation equations (11), together with the boundary conditions (7)–(8) can be written in the matrix form

$$\frac{\partial}{\partial\tau}\mathbf{u}(\tau) = \frac{1}{h^2}[\mathbf{C}(\tau) - \mathbf{J}\mathbf{u}(\tau)]. \tag{12}$$

We first note that the Crank-Nicolson scheme results from time integration of (12) by the classical trapezoidal formula. This is given by

$$\left[I + \frac{1}{2}\rho\mathbf{J}\right]\mathbf{u}_{j+1} = \left[i - \frac{1}{2}\rho\mathbf{J}\right]u_j + \frac{1}{2}\rho\left(\mathbf{C}_j + \mathbf{C}_{j+1}\right) \tag{13}$$

where ***I*** denotes the identity matrix. This is the Crank-Nicolson (C-N) scheme for the transformed Black-Scholes equation.

FOURTH ORDER NUMEROV SEMI-DISCRETISATION

Consider the special second ordinary differential equation

$$y^n = f(x,y),\ a < x < b. \tag{14}$$

For a positive integer $N > 1$, let $h = \frac{b-a}{N+1}, x_i = a + ih,\ i = O(1)N + 1$ and set

$$y_i = y(x_i),\quad f_i = f(x_i, y_i). \tag{15}$$

A well known three-point fourth order discretisation of the differential equation in (14) is due to Numerov (1924) and is given by the following

$$-y_{i-1} + 2y_i - y_{i+1} + \frac{h^2}{12}[f_{i-1} + 10f_i + f_{i+1}] = t_i(h),\quad i = 1, \dots, N, \tag{16}$$

with the local truncation error given by

$$t_i(h) = \frac{h^6}{240} y_i^6 + O(h^8). \quad (17)$$

We now describe a fourth order special discretisation of (6) by the Numerov method. This method (16) can be written as

$$-u_{i-1} + 2u_i - u_{i+1} + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}[u_{i-1} + 10u_i + u_{i+1}] = 0, \;\; i = 1, \dots, N. \quad (18)$$

For the differential equation in (6) this gives

$$\frac{\partial}{\partial \tau}[u_{i-1} + 10u_i + u_{i+1}] = \frac{-12}{h^2}[-u_{i-1} + 2u_i - u_{i+1}], \;\; i = 1, \dots, N. \quad (19)$$

In order to write this system in matrix form, it is useful to consider

$$\mathbf{B} = \begin{bmatrix} 10 & 1 & & & \\ 1 & 10 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & 10 & 1 \\ & & & 1 & 10 \end{bmatrix}$$

Then the semi-discretisation (19) for $i = 1, \dots, N$ together with the boundary conditions can be written in matrix form as

$$\frac{\partial}{\partial \tau}[\mathbf{Bu}(t) + \mathbf{C}(t)] = \frac{12}{h^2}[\mathbf{C}(t) - \mathbf{Ju}(t)]. \quad (20)$$

Employing the trapezoidal formula for the time integration of (20) we obtain

$$(\mathbf{B} + 6\rho\mathbf{J})\mathbf{u}_{j+1} = (\mathbf{B} - 6\rho\mathbf{J})\mathbf{u}_j + (6\rho + 1)\mathbf{C}_j + (6\rho - 1)\mathbf{C}_{J+1} \quad (21)$$

This is the Douglas method for the diffusion equation (6). In comparison with the Crank-Nicolson method, the Douglas method is fourth order in asset and second order in time; its local truncation error is of order $O(h^4) + O(k^2)$. To improve accuracy of the Douglas method we need to consider an excessively small time step, case in which both Crank-Nicolson and Douglas methods provide a stable solution.

Smith (1985) gives an example for the diffusion equation wherein Douglas method has better accuracy than the Crank-Nicolson method in the third or fourth decimal place for sufficiently small time step.

Unconditional stability of Douglas scheme

The Douglas method is unconditionally stable as shown below.

For homogeneous boundary conditions, the Douglas method (21) can be written as

$$\mathbf{u}_{J+1} = \mathbf{Q}^D \mathbf{u}_j, \tag{22}$$

where the amplification matrix is given by

$$\mathbf{Q}^{\mathrm{D}} = (\mathbf{B} + 6\rho\mathbf{J})^{-1}(\mathbf{B} - 6\rho\mathbf{J}) \tag{23}$$

The eigenvalues of **J** are known to be all positive (see Smith (1985)), and are given by

$$\lambda_s(\mathbf{J}) = 4\sin^2(\theta_s) \tag{24}$$

where

$$\theta_s = \frac{s\pi}{2(N+1)}, \quad s = 1. \dots, N. \tag{25}$$

Since $\mathbf{B} = 12I - \mathbf{J}$ we can write

$$\mathbf{Q}^D = [12I + 6(\rho - 1)\mathbf{J}]^{-1}[12I - (6\rho + 1)\mathbf{J}]. \tag{26}$$

The eigenvalues of $\mathbf{Q}^D$ are now given by(see Smith (1985))

$$\lambda_s(\mathbf{Q}^D) = \frac{12-(6\rho+1)\lambda_s}{12+(6\rho-1)\lambda_s}, \quad s = 1, \dots, N. \tag{27}$$

It is easy to check that

$$\left|\lambda_s(\mathbf{Q}^D)\right| \leq 1, \ \ s = 1, \dots, N, \forall\rho > 0, \tag{28}$$

and hence the Douglas method is unconditionally stable. However, it is worthy to note that for large we have $\lambda_s(\mathbf{Q}^D) \sim -1$. This is due to the A-stability of the trapezoidal rule employed for time integration in the Douglas method and helps explain unwanted oscillations in the computed solution by the Douglas method, especially for large time steps.

IMPROVED DOUGLAS SCHEME

We now present the improved Douglas method by employing an L-stable third order time integration scheme for the time integration of (20).

For the numerical integration of the first order initial value problem

$$y' = f(t, y), \ \ y(t_0) = \eta \tag{29}$$

an optimal two-step fourth order is the Simpson rule

$$y_{j+2} = y_j + \frac{k}{3}\left(f_j + 4f_{j+1} + f_{j+2}\right) \tag{30}$$

Here, for a positive time step $t_j = t_0 + jk, j = 0,1,2, \ldots$ we have set $y_j = y(t_j)$ and $f_j = f(t_j, y_j)$.

However, it is known that Simpson rule is unconditionally unstable; see, e.g., Lambert (1991) and Chawla & Evans (2005) who gave a new third order L-stable version of Simpson rule. This rule, compressed to a single interval, is described as follows. For half-step points, we set $t_{j+1/2} = t_j + k/2$, $y_{j+\frac{1}{2}} = y(t_{j+1/2})$, etc.

$$\bar{y}_{j+1/2} = \frac{1}{4}\left(y_j + 3y_{j+1}\right) - \frac{k}{4} f_{j+1}, \quad \bar{f}_{j+\frac{1}{2}} = f(t_{j+1/2}, \bar{y}_{j+1/2}). \tag{31}$$

$$y_{j+1} = y_j + \frac{k}{6}\left(f_j + 4\bar{f}_{j+1/2} + f_{j+1}\right). \tag{32}$$

As in Chawla & Evans (2005), we call this version of Simpson rule by LSIMP3. Now, employing LSIMP3 for the time, we have

$$\mathbf{B}\bar{\mathbf{u}}_{j+1/2} = \frac{1}{4}\mathbf{B}\mathbf{u}_{\mathrm{j}} + \frac{3}{4}(\mathbf{B} + \rho\mathbf{J})\mathbf{u}_{j+1} + \frac{1}{4}\mathbf{C}_j + \frac{3}{4}(1 - 4\rho)\mathbf{C}_{j+1} - \mathbf{C}_{j+1/2} \tag{33}$$

and

$$(\mathbf{B} + 2\rho\mathbf{J})\mathbf{u}_{j+1} = (\mathbf{B} - 2\rho\mathbf{J})\mathbf{u}_{\mathrm{j}} + (2\rho + 1)\mathbf{C}_j + (2\rho - 1)\mathbf{C}_{j+1} + 8\rho\mathbf{C}_{j+1/2} - 8\rho\mathbf{J}\bar{\mathbf{u}}_{j+1/2} \tag{34}$$

Since the matrices **B** and **J** commute, multiplying equation (34) by **B** and substituting for $\mathbf{B}\bar{\mathbf{u}}_{j+1/2}$ from (33), and noting that $\mathbf{B} + \mathbf{J} = 12I$, we obtain

$$\left[\mathbf{B}^2 + 8\rho\mathbf{J}\mathbf{B} + 24\rho^2\mathbf{J}^2\right]\mathbf{u}_{j+1} = \left[\mathbf{B}^2 - 4\rho\mathbf{J}\mathbf{B}\right]\mathbf{u}_j + [(2\rho + 1)\mathbf{B} - 2\rho\mathbf{J}]\mathbf{C}_j + [(2\rho - 1)\mathbf{B} + 6\rho(4\rho - 1)\mathbf{J}]\mathbf{C}_{j+1} + 96\rho\mathbf{C}_{j+1/2} \tag{35}$$

This is the improved Douglas method. The improved Douglas method is fourth order in asset and third order in time, and it can be shown to be unconditionally stable. Unconditional stability of the improved Douglas method means that we can afford to use relatively large time steps for integration, and since it is based on an L-stable version of Simpson rule, with a larger time step we can expect better stability than with the classical Douglas method. These ideas are illustrated in the section on numerical experiments.

Unconditional stability of improved Douglas scheme

We next discuss unconditional stability of the improved Douglas method. For homogeneous boundary conditions, the improved Douglas (**ID**) method (35) can be written as

$$\mathbf{u}_{J+1} = \mathbf{Q}^{ID}\mathbf{u}_j \tag{36}$$

with

$$\mathbf{Q}^{ID} = \left[\mathbf{B}^2 + 8\rho\mathbf{JB} + 24\rho^2\,\mathbf{J}^2\right]^{-1}\left[\mathbf{B}^2 - 4\rho\mathbf{JB}\right]. \tag{37}$$

As in section (5.1), since $\mathbf{B} = 12I - \mathbf{J}$, we can write

$$\mathbf{Q}^{ID} = \left[144I + 24(4\rho - 1)\mathbf{J} + (24\rho^2 - 8\rho + 1)\mathbf{J}^2\right]^{-1}[144I - 24(2\rho + 1)\mathbf{J} + (4\rho + 1)\mathbf{J}^2] \tag{38}$$

and the eigenvalues of are given by

$$\lambda_s\left(\mathbf{Q}^{ID}\right) = \frac{\mathbf{Num}(\lambda_s)}{\mathbf{Dennom}(\lambda_s)}, s = 1, \dots, N. \tag{39}$$

where

$$\mathbf{Num}(\lambda) = 144 - 24(2\rho + 1)\lambda + (4\rho + 1)\lambda^2. \tag{40}$$

$$\mathbf{Denom}(\lambda) = 144 + 24(4\rho - 1)\lambda + (24\rho^2 - 8\rho + 1)\lambda^2. \tag{41}$$

Since

$$\mathbf{Denom}(\lambda) - \mathbf{Num}(\lambda) = 12\rho\lambda[12 + (2\rho - 1)\lambda] > 0, \tag{42}$$

$$\mathbf{Denom}(\lambda) + \mathbf{Num}(\lambda) = 288 + 144\rho\lambda + 2(12\rho^2 - 2\rho + 1)\lambda^2 > 0 \tag{43}$$

it follows that

$$\left|\lambda_s\left(\mathbf{Q}^{ID}\right)\right| \leq 1, \ \ s = 1,\dots,N, \ \ \forall\rho > 0 \tag{44}$$

Thus the improved Douglas method is unconditionally stable. However, for the improved Douglas method that for large ρ, we have $\lambda_s(\mathbf{Q}^{ID}) \sim 0$.

This is due to the L-stability of the LSIMP3 rule employed for time integration in the improved Douglas method and helps explain more stable approximations obtained by the improved Douglas method in comparison with the Douglas method for large time steps.

THE GREEKS

The delta of an option is defined by

$$\Delta = \frac{\partial V}{\partial S} \tag{45}$$

For a European call and put, the delta is given respectively by

$$\Delta_c = N(d_1),\ \Delta_p = N(d_1) - 1, \tag{46}$$

where d_1 is as given by

$$d_1 = \frac{\ln\frac{S_0}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

For numerical computation of the delta from the values of the option at the interior points we obtain

$$\Delta_i \approx \frac{V_{i+1}-V_{i-1}}{S_{i+1}-S_{i-1}} \tag{47}$$

The gamma for a call is given by

$$\Gamma_c = \frac{1}{\sqrt{2\pi(T-t)}}\frac{1}{\sigma S}e^{-\frac{1}{2}d_1^2} \tag{48}$$

For numerical computation of the gamma, at the interior points we use the approximation

$$\Gamma_i = \frac{\Delta_{i+1}-\Delta_{i-1}}{S_{i+1}-S_{i-1}} \tag{49}$$

The theta for a call is given from the Black-Scholes equation by

$$\theta_c = -\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 S^2 \Gamma + erS\Delta = rC. \tag{50}$$

Substituting for C, Δ and Γ, then the theta is given by

$$\theta_c = \frac{\sigma S}{2\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}d_1^2} + rEe^{-r(T-t)}N(d_2). \tag{51}$$

For numerical computation, having computed Δ and Γ, we compute theta using the Black-Scholes equation

$$\theta_i = \frac{1}{2}\sigma^2 S_i^2 \Gamma_i + rS_i\Delta_i - rC_i. \tag{52}$$

The computation of these Greeks is illustrated in the following section.

NUMERICAL EXPERIMENTS

In this section we present some comparisons and show the advantage of using the improved Douglas method over the usual Douglas method. For this purpose, we consider the valuation of some European options and the calculation of the Greeks. For all the following computations we take $\mathrm{r} = 0.1, \sigma = 0.45, \mathrm{E} = 10, \mathrm{T} = 0.33, \mathrm{h} = 0.05$ and $\mathrm{M} = 1$.

Problem 1. We consider the valuation of a European put $V(S,t) = P(S,t)$ with payoff

$$V_T(S) = \mathbf{max}(E - S, 0), \tag{53}$$

and boundary conditions

$$P(S,t) = 0,\ \ P(0,t) = Ee^{-r(T-t)},\ S \to \infty \tag{54}$$

In transformed coordinates, the payoff is

$$u_0(x) = \left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right), \tag{55}$$

and the boundary conditions become

$$u_{-\infty}(\tau) = e^{\frac{1}{2}(k-1)x_{-\infty}+\frac{1}{4}(k-1)^2\tau},\ u(x_\infty, \tau) = 0. \tag{56}$$

We note here that having calculated the value of a put, then the value of a call C(S,t) can be calculated using the put-call parity

$$S + P - C\ =\ Ee^{-r(T-t)}. \tag{57}$$

The exact values of European call and put are given (Wilmott *et al.,* 1995) by

$$C(S,t)\ =\ SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{58}$$

$$P(S,t)\ =\ Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{59}$$

where

$$d_1 = \frac{\ln\frac{S}{E}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\ d_2 = \frac{\ln\frac{S}{E}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{60}$$

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{\left(-\frac{1}{2}s^2\right)}ds. \tag{61}$$

We computed the value of the put by the Crank-Nicolson method, the Douglas method and by the improved Douglas method. The computed values are displayed in Figure 1. The improved accuracy and stability of the improved Douglas method, especially close to the exercise price, is clear from Figure 1.

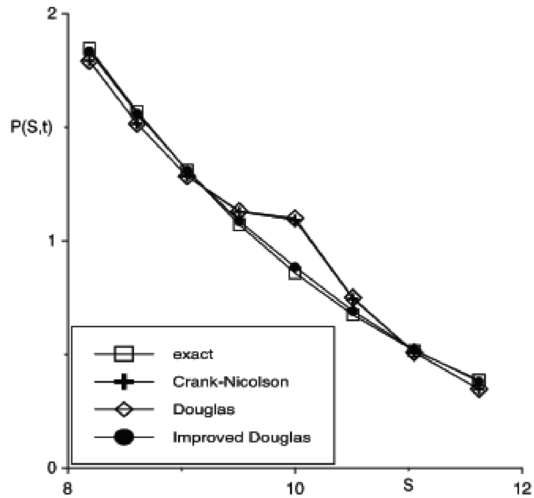

Fig. 1. Problem 1 European put

Problem 2. Consider the valuation of a cash-or-nothing call with payoff, where is the Heaviside unit step function. At expiration, if , then the payoff is; otherwise the payoff is zero. The boundary conditions are

$$CON(0,t) = 0, \quad CON(S,t) = B, \quad S \to \infty \tag{62}$$

In terms of the transformed coordinates, the payoff is

$$\mathbf{u}_0(x) = \frac{\mathbf{B}}{\mathbf{E}} e^{\frac{1}{2}(k-1)x} H(x), \tag{63}$$

and the boundary conditions are

$$\mathbf{u}_{-\infty}(\tau) = 0, \quad \mathbf{u}_{\infty}(\tau) = \frac{\mathbf{B}}{\mathbf{E}} e^{\frac{1}{2}(k-1)x_\infty + \frac{1}{4}(k+1)^2\tau} \tag{64}$$

The exact value of the cash-or-nothing call is given (Wilmott *et al.,* 1995) by

$$CON(S,t) = \mathbf{B}e^{-r}(T-t)N(d_2), \tag{65}$$

Where d_2 is as given above.

We computed the value of the CON by the Crank-Nicolson method, the Douglas method and by the improved Douglas method. The computed values are displayed in Figure 2. The superior accuracy and stability of the improved Douglas method close to the exercise is clear from Figure 2.

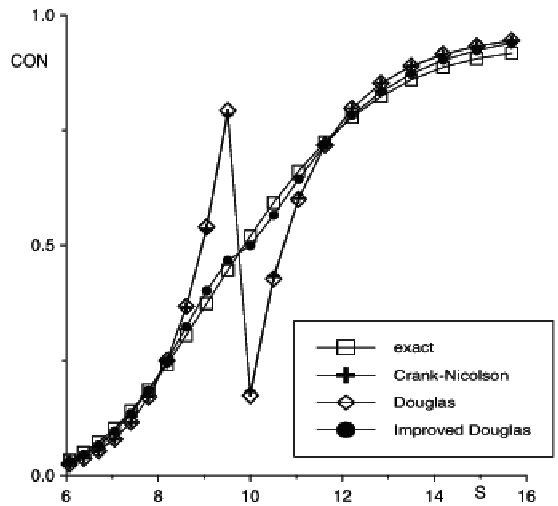

Fig. 2. Problem 2 Cash-or-nothing call

Problem 3. We consider the calculation of the delta for the European call. The delta computed by the Crank-Nicolson, the Douglas method and the improved Douglas method are shown versus the exact value in Figure 3. Superior approximation provided by the improved Douglas method is clear from Figure 3.

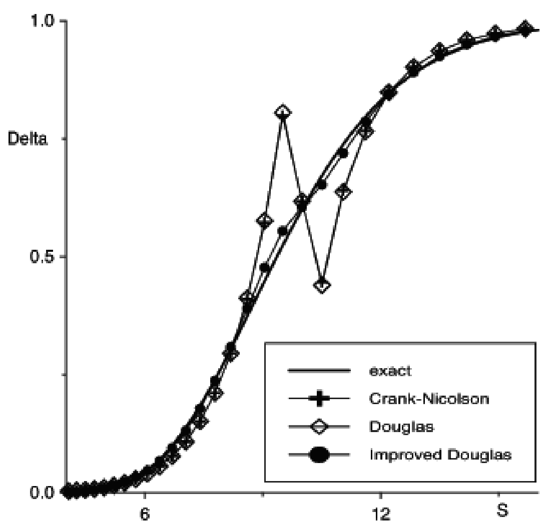

Fig. 3. Problem 3 Delta for call

Problem 4. We consider the calculation of the gamma for the European call. The gamma computed by the Crank-Nicolson, the Douglas method and the improved Douglas method are shown versus the exact value in Figure 4. Superior approximation provided by the improved Douglas method is clear from Figure 4.

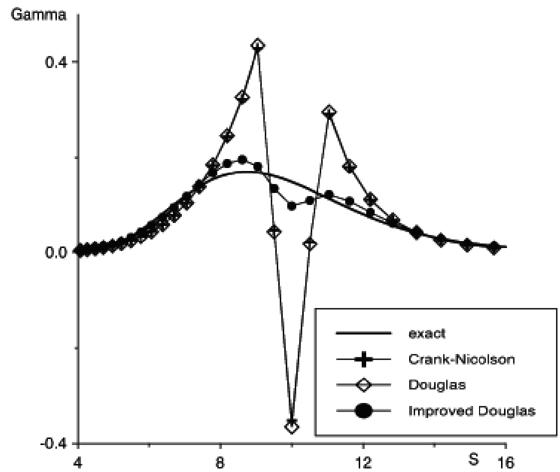

Fig. 4. Problem 4 Gamma for call

Problem 5. We consider the calculation of theta for the European call. The theta computed by the Crank-Nicolson, the Douglas method and the improved Douglas method are shown versus the exact value in Figure 5. Superior approximation provided by the improved Douglas method is clear from Figure 5.

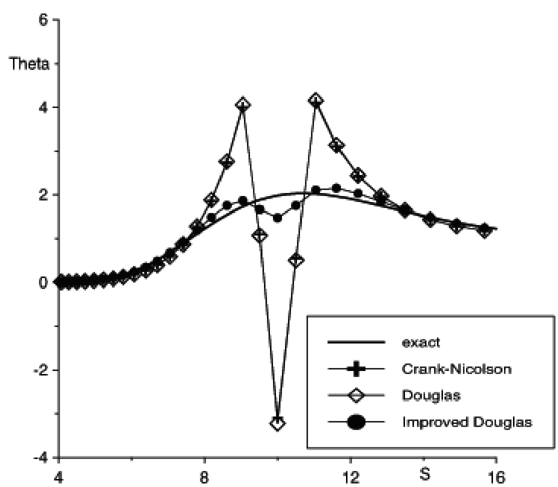

Fig. 5. Problem 5 Theta for a call

REFERENCES

Black, F. & Scholes, M. 1973. The pricing of options and corporate liabilities, The Journal of Political Economy, **81**, 637-659.

Brennan, M.J. & Schwartz, E. S. 1978. Finite difference methods and jump processes arising in the pricing of contingent claims, A synthesis, Journal of Financial and Quantitative Analysis., **13**, 461-474.

Chawla, M. M. & Evans, D. J. 2005. A new L-stable Simpson rule for the diffusion equation, International Journal of Computer Mathematics, **82** , 601-607.

Crank, J. & Nicolson, P. 1947. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, Mathematical Proceedings of the Cambridge Philosophical Society., **43**, 50-67.

Geske, R. & Shastri, K. 1985. Valuation by approximation: a comparison of alternative option valuation techniques, Journal of Financial and Quantitative Analysis., **20,** 45-71.

Lambert, J. D. 1991. Numerical methods for ordinary differential systems, John Wiley, New York.

Numerov, B. V. 1924. A method of extrapolation of perturbations, Monthly Notices of the Royal Astronomical Society **84**, 592-601.

Shaw, W. T. 1998. Modelling financial derivatives with mathematica, Cambridge University Press, Cambridge.

Smith, G. D. 1985. Numerical solution of partial differential equations, 3rd edition, Oxford University Press, Oxford.

Wilmott, P., Howison, S. & Dewynne, J. 1995. The mathematics of financial derivatives, A Student Introduction, Cambridge University Press, Cambridge

Submitted **:** 03/04/2014

Revised **:** 27/05/2014

Accepted **:** 27/05/2014

تحسين على مخطط دوغلاس لمعادلة بلاك – شولز

فارس العازمي

قسم الرياضيات – جامعة الكويت – الصفاة – 13060 – الكويت

خلاصة

يعتبر مخطط كرانك – نيكلسون مخططاً معروفاً من مخططات الفرق المنته لتقدير خيارات معادلة بلاك – شولز، لكن هذا المخطط ينتج تذبذبات غير مرغوبة للحل وذلك في حالة العوائد غير الملساء . وفي هذه الحالة يفضل مخطط دوغلاس كبديل أفضل. ولكن وكما لاحظ شو في كتابة فإن لكل من هذين المخططين سلوكاً كريهاً عندما يجري تطبيقهما في عمليات تسعير الأسهم البسيطة . نلاحظ أن كلا المخططين يستخدم صيغة شبة منحرفية لتكامل الزمن وهذه الصيغة هي التي تعطي التذبذبات غير المرغوبة . لذا فقد أقترح شاولاو إيفان طريقة سمبسون جدية و مستقرة. ونقوم نحن بدراسة تطبيق هذه الطريقة على معادلة بلاك – شولز بعد تقطيعها جزئياً و ذلك بإستخدام طريقة نوميروف. ونقوم بقارنة أداء بإستخدام طريقة نوميروف ونقوم بمقارنة أداء هذه الطريقة الجديدة بكل من مخطط كرانك – نيكلسون ومخطط دوغلاس وذلك بإستخدام تجارب عددية على مسائل تقدير حقيقية . كما نقوم بدارسة هذا المخطط لتقييم الغريكي.