

Mixed finite element approximation for a contact problem in electro-elasticity

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ABSTRACT

The present paper is concerned with the frictionless contact problem between two electro-elastic bodies in a bidimensional context. We consider a mixed formulation in which the unknowns are the displacement field, the electric potential field and the contact pressure. We use the mixed finite element method to approximate the solutions. Error estimates are derived on the approximative solutions from which the convergence of the algorithm is deduced under suitable regularity conditions on the exact solution.

Keywords: Adhesion; electro-elastic materials; error estimates; frictionless contact; mixed formulation.

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INTRODUCTION

The piezoelectric effect was discovered in 1880 by Jacques and Pierre Curie; it consists the apparition of electric charges on the surfaces of some crystals after their deformation. The reverse effect was outlined in 1881; in the generation of stress and strain in crystals under the action of electric field on the boundary. A deformable material which undergoes piezoelectric effects is called a piezoelectric material. An elastic material with piezoelectric effect is called electro-elastic material, and the discipline dealing with the study of electro-elastic materials is the theory of electro-elasticity. Their fundamentals were studied by Voigt (1910) who provided the first mathematical model of a linear elastic material which takes into account the interaction between mechanical and electrical properties. General models for elastic

materials with piezoelectric effects can be found in Mindlin (1968), Mindlin (1972), Toupin (1963) and, more recently, in Batra & Yang. (1995). The importance of this paper is to make the coupling of the piezoelectric problem and a frictionless contact problem without adhesion.

In this paper we consider unilateral contact problems between two electro-elastic bodies. The contact is frictionless and it is modelled with the nonlinear elastic-piezoelectric constitutive law and Signorini's condition. The variational formulations are derived in a form of a coupled system for the displacement-electric potential fields and the contact pressure. An existence and uniqueness result is recalled. Further, a discrete scheme is introduced based on the mixed finite element method to approximate the displacements fields, the electric potentials fields, and the contact pressure. Under appropriate regularity assumptions on the exact solution, optimal order error estimates are derived.

The paper is structured as follows. First, we establish the continuous mixed variational formulation of the considered problem. Next, we define two mixed finite element methods using quadratic finite elements with multipliers which are continuous and piecewise of degree two on the contact part. The difference between both approaches is that the non-interpenetration conditions are either of linear type (i.e. hold at the discretization nodes of the method) or of quadratic type (i.e. hold everywhere on the contact part). The link of the mixed methods with the corresponding variational inequality formulation is given. Finally, is concerned with the convergence study of the methods for which we prove identical convergence rates under various regularity hypotheses.

PROBLEM STATEMENT AND VARIATIONAL FORMULATION

We consider the following physical setting. Let us consider two electro-elastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space \mathbb{R}^d ($d = 2, 3$). For $\ell = 1, 2$, the boundary $\partial\Omega^\ell$ of Ω^ℓ is assumed be "smooth", and is the union of three non-overlapping portions $\Gamma_1^\ell, \Gamma_2^\ell$ and Γ_3^ℓ , or into two disjoint parts Γ_a^ℓ and Γ_b^ℓ , such that $meas(\Gamma_1^\ell) > 0, meas(\Gamma_a^\ell) > 0$. The Ω^ℓ body is submitted to \mathbf{f}_0^ℓ forces and volume electric charges of density q_0^ℓ . The bodies are assumed to be clamped on Γ_1^ℓ . The surface tractions \mathbf{f}_2^ℓ act on Γ_2^ℓ . We also assume that the electrical potential vanishes on Γ_a^ℓ and a surface electric charge of density q_2^ℓ is prescribed on Γ_b^ℓ . The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$, (see Figure 1).

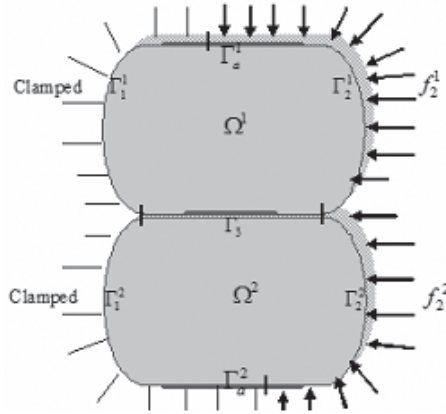


Fig.1. Frictionless contact problem between two electro-elastic bodies

With these assumptions. The classical formulation of the mechanical problem of piezoelectric material, frictionless contact between two deformable bodies is the following

Problem P.

For $\ell = 1, 2$, find a displacement field $\mathbf{u}^\ell : \Omega^\ell \rightarrow \mathbb{R}^d$, a stress field $\sigma^\ell : \Omega^\ell \rightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \rightarrow \mathbb{R}$ and an electric displacement field $\mathbf{D}^\ell : \Omega^\ell \rightarrow \mathbb{R}^d$ such that

$$\sigma^\ell = \mathcal{A}\varepsilon(\mathbf{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell \quad \text{in } \Omega^\ell, \quad (1)$$

$$\mathbf{D}^\ell = \mathcal{E}^\ell \varepsilon(\mathbf{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell, \quad (2)$$

$$\text{Div } \sigma^\ell + \mathbf{f}_0^\ell = \mathbf{0} \quad \text{in } \Omega^\ell, \quad (3)$$

$$\text{div } \mathbf{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell, \quad (4)$$

$$\mathbf{u}^\ell = \mathbf{0} \quad \text{on } \Gamma_2^\ell, \quad (5)$$

$$\sigma^\ell \nu^\ell = \mathbf{f}_2^\ell \quad \text{on } \Gamma_2^\ell, \quad (6)$$

$$\sigma_\nu^1 = \sigma_\nu^2 = -p_\nu([u_\nu]), \quad \text{on } \Gamma_3, \quad (7)$$

$$[u_\nu] \leq 0, \quad \sigma_\nu \leq 0, \quad [u_\nu] \sigma_\nu = 0 \quad \text{on } \Gamma_3, \quad (8)$$

$$\sigma_{\tau}^1 = \sigma_{\tau}^2 = \mathbf{0} \text{ on } \Gamma_3, \tag{9}$$

$$\varphi^\ell = \mathbf{0} \text{ on } \Gamma_a, \tag{10}$$

$$\mathbf{D}^\ell \cdot \mathbf{v}^\ell = q_2^\ell \text{ on } \Gamma_b. \tag{11}$$

First, equations (1) and (2) represent the electro-elastic constitutive law in which $\varepsilon(\mathbf{u}^\ell)$ denotes the linearized strain tensor, $E(\varphi^\ell) = -\nabla \varphi^\ell$ is the electric field, where φ^ℓ is the electric potential, \mathcal{A}^ℓ is a given nonlinear function, \mathcal{E}^ℓ represents the piezoelectric operator, $(\mathcal{E}^\ell)^*$ is its transpose, \mathcal{B}^ℓ denotes the electric permittivity operator, and $\mathbf{D}^\ell = (D_1^\ell, \dots, D_d^\ell)$ is the electric displacement vector. Details on the constitutive equations of the form (1) and (2) can be found, for instance, in Batra (1995), Bisenga *et al.*(2002). Next, (3) and (4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which “Div” and “div” denote the divergence operator for tensor and vector valued functions, respectively. Equations (5) and (6) represent the displacement and stress boundary conditions. Condition (7) represents the normal compliance conditions where p_ν is a given positive function which will be described below and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the jump of the displacement in normal direction: either contact (i.e. $[u_\nu] = 0$) or separation (i.e. $[u_\nu] < 0$) are allowed, in other words ($[u_\nu] \leq 0$) is the nonpenetration condition. Conditions (8) and (9) represent the Signorini contact condition without friction. Finally, (10) and (11) represent the electric boundary conditions, in which “.” denote the inner product on \mathbb{R}^d .

In order to proceed with the variational formulation, we need the following spaces:

$$H^\ell = \{\mathbf{v}^\ell = (v_i^\ell); v_i^\ell \in L^2(\Omega^\ell)\}, \quad \mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell); \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\},$$

$$H_1^\ell = \{\mathbf{v}^\ell = (v_i^\ell); v_i^\ell \in H^1(\Omega^\ell)\}, \quad \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell; \text{div } \boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces H^ℓ , \mathcal{H}^ℓ , H_1^ℓ and \mathcal{H}_1^ℓ are real Hilbert spaces endowed with the inner products given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell \cdot \mathbf{v}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell \cdot \boldsymbol{\tau}^\ell dx,$$

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell \cdot \mathbf{v}^\ell dx + \int_{\Omega^\ell} \nabla \mathbf{u}^\ell \cdot \nabla \mathbf{v}^\ell dx,$$

$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell \cdot \boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} \text{div } \boldsymbol{\sigma}^\ell \cdot \text{Div } \boldsymbol{\tau}^\ell dx$$

respectively. Here and below, we use the notation

$$\begin{aligned}\nabla \mathbf{u}^\ell &= (u_{i,j}^\ell), \quad \varepsilon(\mathbf{u}^\ell) = (\varepsilon_{ij}(\mathbf{u}^\ell)), \quad \varepsilon_{ij}(\mathbf{u}^\ell) = \frac{1}{2}(u_{i,j}^\ell + u_{j,i}^\ell), \quad \forall \mathbf{u}^\ell \in H_1^\ell, \\ \text{Div } \boldsymbol{\sigma}^\ell &= (\sigma_{ij,j}^\ell), \quad \forall \boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell.\end{aligned}$$

The associated norms on the spaces H^ℓ , H_1^ℓ , \mathcal{H}^ℓ and \mathcal{H}_1^ℓ are denoted by $\|\cdot\|_{H^\ell}$, $\|\cdot\|_{H_1^\ell}$, $\|\cdot\|_{\mathcal{H}^\ell}$, and $\|\cdot\|_{\mathcal{H}_1^\ell}$ respectively.

For every element $\mathbf{v}^\ell \in H_1^\ell$, we also use the notation \mathbf{v}^ℓ for the trace of \mathbf{v}^ℓ on Γ^ℓ and we denote by v_ν^ℓ and \mathbf{v}_τ^ℓ the normal and the tangential components of \mathbf{v}^ℓ on the boundary Γ^ℓ given by

$$v_\nu^\ell = \mathbf{v}^\ell \cdot \boldsymbol{\nu}^\ell, \quad \mathbf{v}_\tau^\ell = \mathbf{v}^\ell - v_\nu^\ell \boldsymbol{\nu}^\ell.$$

Let H'_{Γ^ℓ} be the dual of $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^d$ and denote by $(\cdot, \cdot)_{-\frac{1}{2}, \frac{1}{2}, \Gamma^\ell}$ the duality pairing between H'_{Γ^ℓ} and H_{Γ^ℓ} . For every element $\sigma^\ell \in \mathcal{H}_1^\ell$ let $\sigma^\ell \boldsymbol{\nu}^\ell$ be the element of H'_{Γ^ℓ} given by

$$(\sigma^\ell \boldsymbol{\nu}^\ell, \mathbf{v}^\ell)_{-\frac{1}{2}, \frac{1}{2}, \Gamma^\ell} = (\sigma^\ell, \varepsilon(\mathbf{v}^\ell))_{H^\ell} + (\text{Div } \sigma^\ell, \mathbf{v}^\ell)_{H^\ell} \quad \forall \mathbf{v}^\ell \in H_1^\ell.$$

Denote by σ_ν^ℓ and σ_τ^ℓ the normal and the tangential traces of $\sigma^\ell \in \mathcal{H}_1^\ell$, respectively. If σ^ℓ is continuously differentiable on $\Omega^\ell \cup \Gamma^\ell$, then

$$\begin{aligned}\sigma_\nu^\ell &= (\sigma^\ell \boldsymbol{\nu}^\ell) \cdot \boldsymbol{\nu}^\ell, \quad \sigma_\tau^\ell = \sigma^\ell \boldsymbol{\nu}^\ell - \sigma_\nu^\ell \boldsymbol{\nu}^\ell, \\ (\sigma^\ell \boldsymbol{\nu}^\ell, \mathbf{v}^\ell)_{-\frac{1}{2}, \frac{1}{2}, \Gamma^\ell} &= \int_{\Gamma^\ell} \sigma^\ell \boldsymbol{\nu}^\ell \cdot \mathbf{v}^\ell da\end{aligned}$$

for all $\mathbf{v}^\ell \in H_1^\ell$, where da is the surface measure element.

To obtain the variational formulation of the problem (1)-(11), we introduce for the displacement field the closed subspace of H_1^ℓ , defined by

$$V^\ell = \{\mathbf{v}^\ell \in H_1^\ell; \mathbf{v}^\ell = 0 \text{ on } \Gamma_1^\ell\}.$$

Next, we define the convex cone of Lagrange multipliers denoted by M and defined as follows:

$$M = \left\{ \mu \in H^{-\frac{1}{2}}(\Gamma_3); \int_{\Gamma_3} \mu \cdot \psi d\Gamma_3 \geq 0, \text{ for all } \psi \in H^{\frac{1}{2}}(\Gamma_3), \psi \geq 0 \right\}.$$

Since $meas(\Gamma_1^\ell) > 0$, the following Korn's inequality holds :

$$\|\varepsilon(\mathbf{v}^\ell)\|_{\mathcal{H}^\ell} \geq c_K \|\mathbf{v}^\ell\|_{H_1^\ell} \quad \forall \mathbf{v}^\ell \in V^\ell, \quad (12)$$

where c_K denotes a positive constant which may depends only on Ω^ℓ , Γ_1^ℓ (Neč as & Hlaváček, 1981). We endow V^ℓ with the inner product

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{V^\ell} = (\varepsilon(\mathbf{u}^\ell), \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \mathbf{u}^\ell, \mathbf{v}^\ell \in V^\ell, \quad (13)$$

and $\|\cdot\|_{V^\ell}$ is the associated norm. It follows from Korn's inequality (12) that the norms $\|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{V^\ell}$ are equivalent on V^ℓ . Then $(V^\ell, \|\cdot\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (13), there exists a constant $c_0 > 0$, depending only on Ω^ℓ , Γ_1^ℓ and Γ_3 such that

$$\|\mathbf{v}^\ell\|_{L^2(\Gamma_3)^d} \leq c_0 \|\mathbf{v}^\ell\|_{V^\ell} \quad \forall \mathbf{v}^\ell \in V^\ell.$$

We also introduce the spaces

$$\begin{aligned} W^\ell &= \{\psi^\ell \in H^1(\Omega^\ell); \psi^\ell = 0 \text{ on } \Gamma_a^\ell\}, \\ \mathcal{W}^\ell &= \{\mathbf{D}^\ell = (D_i^\ell); D_i^\ell \in L^2(\Omega^\ell), \operatorname{div} \mathbf{D}^\ell \in L^2(\Omega^\ell)\}, \end{aligned}$$

which are real Hilbert spaces with the inner products

$$\begin{aligned} (\varphi^\ell, \psi^\ell)_{W^\ell} &= \int_{\Omega^\ell} \nabla \varphi^\ell \cdot \nabla \psi^\ell dx, \\ (\mathbf{D}^\ell, \mathbf{E}^\ell)_{\mathcal{W}^\ell} &= \int_{\Omega^\ell} \mathbf{D}^\ell \cdot \mathbf{E}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \mathbf{D}^\ell \cdot \operatorname{div} \mathbf{E}^\ell dx. \end{aligned}$$

The associated norms will be denoted by $|\cdot|_{W^\ell}$ and $|\cdot|_{\mathcal{W}^\ell}$, respectively. Notice also that, since $meas(\Gamma_a^\ell) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla \psi^\ell\|_{L^2(\Omega^\ell)^d} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell,$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ , Γ_a^ℓ .

In order to simplify the notations, we define the product spaces

$$\begin{aligned} H_1 &= H_1^1 \times H_1^2, \quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \\ \mathbf{V} &= V^1 \times V^2, \quad \mathbf{W} = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{aligned}$$

The spaces \mathbf{V} , \mathcal{W} and \mathcal{W} are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot, \cdot)_{\mathbf{V}}$, $(\cdot, \cdot)_{\mathcal{W}}$, and $(\cdot, \cdot)_{\mathcal{W}}$. The associate norms will be denoted by $\|\cdot\|_{\mathbf{V}}$, $\|\cdot\|_{\mathcal{W}}$, and $\|\cdot\|_{\mathcal{W}}$, respectively. In the study of the Problem **P**, we consider the following assumptions:

We assume that the elasticity operator $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies:

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \quad \|\mathcal{A}^\ell(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\ell(\mathbf{x}, \boldsymbol{\xi}_2)\| \leq L_{\mathcal{A}^\ell} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\ \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega^\ell. \\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \quad (\mathcal{A}^\ell(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\ell(\mathbf{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathcal{A}^\ell} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\ \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega^\ell. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{A}^\ell(\mathbf{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}^\ell(\mathbf{x}, \mathbf{0}) \text{ belongs to } \mathcal{H}^\ell. \end{array} \right. \quad (14)$$

The piezoelectric tensor $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \rightarrow \mathbb{R}^d$ satisfies:

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{E}^\ell = (e_{ijk}^\ell), \quad e_{ijk}^\ell = e_{ikj}^\ell \in L^\infty(\Omega^\ell), \quad 1 \leq i, j, k \leq d. \\ \text{(b) } \mathcal{E}^\ell \sigma \cdot \tau = \sigma \cdot (\mathcal{E}^\ell)^* \tau \quad \forall \sigma, \tau \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega^\ell. \end{array} \right. \quad (15)$$

The permittivity operator $\mathcal{B}^\ell : \Omega^\ell \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ verifies:

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{B}^\ell(\mathbf{x}, \mathbf{E}) = (b_{ij}^\ell(\mathbf{x}) E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega^\ell. \\ \text{(b) } b_{ij}^\ell = b_{ji}^\ell, \quad b_{ij}^\ell \in L^\infty(\Omega^\ell), \quad 1 \leq i, j \leq d. \\ \text{(c) There exists } m_{\mathcal{B}^\ell} > 0 \text{ such that } \mathcal{B}^\ell \mathbf{E} \cdot \mathbf{E} \geq m_{\mathcal{B}^\ell} |\mathbf{E}|^2 \\ \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega^\ell. \end{array} \right. \quad (16)$$

The normal compliance functions $p_\nu : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ satisfies:

$$\left\{ \begin{array}{l} \text{(a) } \exists L_\nu > 0 \text{ such that } |p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2)| \leq L_\nu |r_1 - r_2| \\ \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \\ \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) The mapping } \mathbf{x} \mapsto p_\nu(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \forall r \in \mathbb{R}. \\ \text{(d) } p_\nu(\mathbf{x}, r) = 0, \text{ for all } r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array} \right. \quad (17)$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\mathbf{f}_0^\ell \in H^\ell, \quad \mathbf{f}_2^\ell \in L^2(\Gamma_2^\ell)^d, \quad q_0^\ell \in L^2(\Omega^\ell), \quad q_2^\ell \in L^2(\Gamma_b^\ell). \quad (18)$$

Using the Riesz's theorem, we define the linear mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) \in \mathbf{V}$ and \in as follows:

$$\begin{aligned} (\mathbf{f}, \mathbf{v})_{\mathbf{V}} &= \sum_{\ell=1}^2 \int_{\Omega^\ell} \mathbf{f}_0^\ell \cdot \mathbf{v}^\ell dx + \sum_{\ell=1}^2 \int_{\Gamma_2} \mathbf{f}_2^\ell \cdot \mathbf{v}^\ell da \quad \forall \mathbf{v} \in \mathbf{V}, \\ (q, \zeta)_W &= \sum_{\ell=1}^2 \int_{\Omega^\ell} q_0^\ell \zeta^\ell dx - \sum_{\ell=1}^2 \int_{\Gamma_b^\ell} q_2^\ell \zeta^\ell da \quad \forall \zeta \in W. \end{aligned}$$

For the Signorini Problem, we use the convex subset of admissible displacements given by

$$U_{ad} = \{ \mathbf{v} = (v^1, v^2) \in \mathbf{V}; [v_\nu] \leq 0 \text{ on } \Gamma_3 \}.$$

Let us denote by $j_{\nu c} : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ the normal compliance functional given by

$$j_{\nu c}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] da.$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical problem (1)--(11).

Problem P^V.

Find a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) \in \mathbf{V}$, an electric potential field $\phi = (\phi^1, \phi^2) \in W$, such that

$$\begin{aligned} \sum_{\ell=1}^2 (\mathcal{A}^\ell \varepsilon(\mathbf{u}^\ell), \varepsilon(\mathbf{v}^\ell - \mathbf{u}^\ell))_{\mathcal{H}^\ell} + \sum_{\ell=1}^2 ((\mathcal{E}^\ell)^* \nabla \phi^\ell, \varepsilon(\mathbf{v}^\ell - \mathbf{u}^\ell))_{\mathcal{H}^\ell} \\ + j_{\nu c}(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_{\mathbf{V}} \quad \forall \mathbf{v} \in U_{ad}, \end{aligned} \quad (19)$$

$$\sum_{\ell=1}^2 (\mathcal{B}^\ell \nabla \phi^\ell, \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^2 (\mathcal{E}^\ell \varepsilon(\mathbf{u}^\ell), \nabla \phi^\ell)_{H^\ell} = (q, \phi)_W \quad \forall \phi \in W. \quad (20)$$

Let us introduce the functional space $\mathcal{V} = \mathbf{V} \times W$, that is a Hilbert space endowed with the inner product

$$(\tilde{u}, \tilde{v})_{\mathcal{V}} = (u, v)_{\mathbf{V}} + (\varphi, \phi)_{\mathcal{W}}, \quad \tilde{u} = (u, \varphi), \quad \tilde{v} = (v, \phi) \in \mathcal{V},$$

the corresponding norm is denoted by $\|\cdot\|_{\mathcal{V}}$.

For any $\tilde{u} = (u, \varphi)$, $\tilde{v} = (v, \phi) \in \mathcal{V}$, and for any $\mu \in M$, we define

$$\begin{aligned} a(\tilde{u}, \tilde{v}) = & \sum_{\ell=1}^2 (\mathcal{A}^{\ell} \varepsilon(\mathbf{u}^{\ell}), \varepsilon(\mathbf{v}^{\ell}))_{\mathcal{H}^{\ell}} + \sum_{\ell=1}^2 ((\mathcal{E}^{\ell})^* \nabla \varphi^{\ell}, \varepsilon(\mathbf{v}^{\ell}))_{\mathcal{H}^{\ell}} \\ & + \sum_{\ell=1}^2 (\mathcal{B}^{\ell} \nabla \varphi^{\ell}, \nabla \phi^{\ell})_{H^{\ell}} - \sum_{\ell=1}^2 (\mathcal{E}^{\ell} \varepsilon(\mathbf{u}^{\ell}), \nabla \phi^{\ell})_{H^{\ell}}, \end{aligned} \quad (21)$$

$$b(\tilde{u}, \mu) = \int_{\Gamma_3} \mu [u_{\nu}] d\Gamma_3. \quad (22)$$

Using (19)-(22) and from (7), we get

$$a(\tilde{u}, \tilde{v}) + b(\tilde{v}, -\sigma_{\nu}) = (\tilde{\mathbf{f}}, \tilde{v})_{\mathcal{V}}, \quad \forall \tilde{v} \in \mathcal{V},$$

where $\tilde{\mathbf{f}} = (\mathbf{f}, q) \in \mathcal{V}$.

Using now (7) and (9), we deduce that

$$b(\tilde{u}, \sigma_{\nu}) = 0.$$

On the other hand, obviously $[u_{\nu}] \leq 0$ on Γ_3 . From $\mu \in M$ and (8), we deduce

$$b(\tilde{u}, \mu) \leq 0,$$

resulting in $b(\tilde{u}, \mu - \lambda) \leq 0$, for all $\mu \in M$, with $\lambda = -\sigma_{\nu}$. Thus, we can write the following mixed formulation of Problem **P**.

Problem \mathbf{P}_m^V .

Find $\tilde{u} \in \mathcal{V}$ and $\lambda \in M$ such that

$$a(\tilde{u}, \tilde{v}) + b(\tilde{v}, \lambda) = (\tilde{\mathbf{f}}, \tilde{v})_{\mathcal{V}}, \quad \forall \tilde{v} \in \mathcal{V}, \quad (23)$$

$$b(\tilde{v}, \mu - \lambda) \leq 0, \quad \forall \mu \in M. \quad (24)$$

The functional $\tilde{v} \mapsto (\tilde{\mathbf{f}}, \tilde{v})_{\mathcal{V}}$ is linear and continuous on \mathcal{V} . The bilinear form $a(\cdot, \cdot)$ is symmetric, continuous and V -elliptic.

The existence and uniqueness of solution to (23)--(24) have been stated in Haslinger *et al.* (1996). We recall this result in the following lemma.

Lemma 1 Problem \mathbf{P}_m^V admits a unique solution $(\tilde{u}, \lambda) \in \mathcal{V} \times M$, where $\lambda = -\sigma_{\nu}^1 = -\sigma_{\nu}^2$.

Remark 1 Let (\tilde{u}, λ) be the solution of problem \mathbf{P}_m^V .

Then (\tilde{u}, λ) is the saddle-point of the functional $\mathcal{L}(\cdot, \cdot)$ over $\mathcal{V} \times M$ where

$$\mathcal{L}(\tilde{v}, \mu) = \frac{1}{2} a(\tilde{v}, \tilde{v}) - (\tilde{f}, \tilde{v})_{\mathcal{V}} + b(\tilde{v}, \mu)$$

Moreover \tilde{u} is the solution of the following elliptic variational inequality :

$$\tilde{u} \in \mathcal{V}, \quad a(\tilde{u}, \tilde{v} - \tilde{u}) \geq (\tilde{f}, \tilde{v} - \tilde{u})_{\mathcal{V}}, \quad \forall \tilde{v} \in \mathcal{V}.$$

It is also a solution of the minimization problem

$$\tilde{u} \in \mathcal{V}, \quad \mathcal{J}(\tilde{u}) = \min_{\tilde{v} \in \mathcal{V}} \mathcal{J}(\tilde{v}),$$

where

$$\mathcal{J}(\tilde{v}) = \frac{1}{2} a(\tilde{v}, \tilde{v}) - (\tilde{f}, \tilde{v})_{\mathcal{V}}.$$

MIXED FINITE ELEMENT APPROACH

In this section, we suppose for the sake of simplicity, that each subdomain Ω^ℓ , $\ell = 1, 2$ is a polygon and that Γ_3 is a straight line segment parallel to the x_1 -axis. The vertices of the contact region Γ_3 are $\{\mathbf{c}_1, \mathbf{c}_2\}$. We denote by \mathcal{T}_h^ℓ a triangulation of Ω^ℓ made of elements which are triangles (or quadrilateral) with a maximum size h_ℓ satisfying the usual admissibility assumption, i.e. the intersection of two different elements are either empty, a vertex, or a whole edge. Let $h = \max(h_1, h_2)$. In addition, \mathcal{T}_h^ℓ is assumed regular, hence there exists a constant c independent of the discretization parameter h_ℓ satisfying

$$\min_{K \in \mathcal{T}_h^\ell} \frac{\rho_K}{h_K} \geq c > 0, \quad \ell = 1, 2$$

where ρ_K is the diameter of the inscribed circle in K . We suppose that the end points \mathbf{c}_1 and \mathbf{c}_2 of the contact zone Γ_3 are common nodes of the triangulations \mathcal{T}_h^1 and \mathcal{T}_h^2 and that the one-dimensional traces of triangulations of \mathcal{T}_h^1 and \mathcal{T}_h^2 on Γ_3 are uniformly regular. The set of nodes on Γ_3 belonging to triangulation \mathcal{T}_h^ℓ is denoted Σ_h^ℓ and generally we have $\Sigma_h^1 \neq \Sigma_h^2$. Let $V_h(\Omega^\ell)$ and $W_h(\Omega^\ell)$ consist of continuous and piecewise affine functions on Ω^ℓ , respectively, that is,

$$\begin{aligned} V_h(\Omega^\ell) &= \left\{ \mathbf{v}_h^\ell \in [\mathcal{C}(\overline{\Omega^\ell})]^2; \mathbf{v}_{h|K}^\ell \in (\mathcal{P}_1(K))^2, \forall K \in \mathcal{T}_h^\ell, \mathbf{v}_{h|\Gamma_1}^\ell \equiv 0 \right\}, \\ W_h(\Omega^\ell) &= \left\{ \varphi_h^\ell \in \mathcal{C}(\overline{\Omega^\ell}); \varphi_{h|K}^\ell \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h^\ell, \varphi_{h|\Gamma_a}^\ell \equiv 0 \right\}, \end{aligned}$$

where $\mathcal{C}(\overline{\Omega^\ell})$ and $\mathcal{P}_1(K)$ denote the space of continuous functions on $\overline{\Omega^\ell}$ and the space of the polynomials with the global degree one on K , respectively. We define the spaces

$$\mathbf{V}_h = V_h(\Omega^1) \times V_h(\Omega^2), \quad W_h = W_h(\Omega^1) \times W_h(\Omega^2), \quad \mathcal{V}_h = \mathbf{V}_h \times W_h.$$

Next, we define the space X_h of continuous functions as follows:

$$X_h = \{ \psi_h \in \mathcal{C}(\overline{\Gamma_3}); \exists \mathbf{v}_h \in \mathbf{V}_h \text{ such that } \psi_h = [\mathbf{v}_{h\nu}] \text{ on } \Gamma_3 \},$$

piecewise polynomial of degree one on the mesh of Ω^ℓ on Γ_3 .

We introduce the projection operator π_h on X_h , defined for any function $\psi \in L^2(\Gamma_3)$ by:

$$\pi_h \psi \in X_h, \quad \int_{\Gamma_3} (\pi_h \psi - \psi) \mu_h d\Gamma_3 = 0 \quad \forall \mu_h \in X_h. \quad (25)$$

Let us now approximate the closed convex cone M by a subset of X_h . We introduce the set $X_h(\Gamma_3)$ where nonnegativity holds everywhere on Γ_3 :

$$X_h(\Gamma_3) = \{ \mu_h \in X_h; \mu_h \geq 0 \text{ on } \Gamma_3 \},$$

which corresponds to convex constraints of quadratic type. We then define another set denoted by $X_h(\Sigma_h^\ell)$ where nonnegativity holds every where on Σ_h^ℓ which leads to convex constraints of linear type:

$$X_h(\Sigma_h^\ell) = \{ \mu_h \in X_h; \mu_h(x) \geq 0 \quad \forall x \in \Sigma_h^\ell \}.$$

Next, we define the positive polar cones $\Lambda_h(\Gamma_3)$ and $\Lambda_h(\Sigma_h^\ell)$ of $X_h(\Gamma_3)$ and $X_h(\Sigma_h^\ell)$, respectively:

$$\Lambda_h(\Gamma_3) = \left\{ \boldsymbol{\mu}_h \in X_h; \int_{\Gamma_3} \boldsymbol{\mu}_h \psi_h \geq 0, \quad \forall \psi_h \in X_h(\Gamma_3) \right\},$$

$$\Lambda_h(\Sigma_h^\ell) = \left\{ \boldsymbol{\mu}_h \in X_h; \int_{\Gamma_3} \boldsymbol{\mu}_h \psi_h \geq 0, \quad \forall \psi_h \in X_h(\Sigma_h^\ell) \right\}.$$

From the inclusion $X_h(\Gamma_3) \subset X_h(\Sigma_h^\ell)$, by polarity it follows that $\Lambda_h(\Sigma_h^\ell) \subset \Lambda_h(\Gamma_3)$.

We then choose a discretized mixed formulation which uses either $\Lambda_h(\Gamma_3)$ or $\Lambda_h(\Sigma_h^\ell)$ as an approximation of Λ . The discretization of (23)-(24) is defined in a standard way:

Problem P_m^h

Find $\tilde{u}_h \in \mathcal{V}_h$ and $\lambda_h \in \Lambda_h$ such that

$$a(\tilde{u}_h, \tilde{v}_h) + b(\tilde{v}_h, \lambda_h) = (\tilde{f}, \tilde{v}_h)_{\mathcal{V}}, \quad \forall \tilde{v}_h \in \mathcal{V}_h, \quad (26)$$

$$b(\tilde{u}_h, \mu_h - \lambda_h) \leq 0, \quad \forall \mu_h \in \Lambda_h, \quad (27)$$

where $\Lambda_h = \Lambda_h(\Gamma_3)$ or $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$, $\ell = 1$ or 2 .

In order to prove the existence and the uniqueness of the saddle-point of (26)-(27), it is only necessary to verify that

$$\{\mu_h \in X_h; b(\tilde{v}_h, \mu_h) = 0, \quad \forall \tilde{v}_h \in \mathcal{V}_h\} = \{0\},$$

which is obvious. As a consequence, we obtain the following statement (Coorevits *et al.*, 2002):

Lemma 2 Let $\Lambda_h = \Lambda_h(\Gamma_3)$ or $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$, $\ell = 1$ or 2 . Then there exists a unique solution (\tilde{u}_h, λ_h) to Problem P_m^h , and it satisfies $(\tilde{u}_h, \lambda_h) \in \mathcal{V}_h \times \Lambda_h$.

We are now interested in obtaining a uniform inf-sup condition for $b(\cdot, \cdot)$ over $\mathcal{V}_h \times X_h$. The result is given in the following lemma. The proof of the lemma is the same as in the case of linear finite elements with continuous linear multipliers (Haslinger *et al.*, 1996) and consists in proving the stability of the projection operator

π_h^ℓ in the $H^{\frac{1}{2}}(\Gamma_3)$ -norm.

Lemma 3 Suppose that $\overline{\Gamma_1^\ell} \cap \overline{\Gamma_3} = \emptyset$ for $\ell = 1, 2$. Then the following inf-sup condition holds:

$$\inf_{\mu_h \in X_h} \sup_{\tilde{v}_h \in \mathcal{V}_h} \frac{b(\mu_h, \tilde{v}_h)}{\|\mu_h\|_H \cdot \| \tilde{v}_h \|_{H^{\frac{1}{2}}(\Gamma_3)}} \geq \beta > 0, \quad (28)$$

where β is independent of h .

Proof. Let $\psi \in L^2(\Gamma_3)$. Then we have

$$\|\pi_h \psi\|_{H^{\frac{1}{2}}(\Gamma_3)} \leq c \|\psi\|_{H^{\frac{1}{2}}(\Gamma_3)}, \quad \forall \psi \in H^{\frac{1}{2}}(\Gamma_3). \quad (29)$$

For any $\mu_h \in X_h$, there exists $\psi \in H^{\frac{1}{2}}(\Gamma_3)$ such that

$$\|\psi\|_{H^{\frac{1}{2}}(\Gamma_3)} = 1 \quad \text{and} \quad \int_{\Gamma_3} \mu_h \psi d\Gamma_3 = \|\mu_h\|_{H^{\frac{1}{2}}(\Gamma_3)}. \quad (30)$$

We then consider an extension operator R_h^ℓ from X_h into $V_h(\Omega^\ell)$ satisfying

$$R_h^\ell \psi_h|_{\Gamma_3} = \psi_h \text{ and } \left\| R_h^\ell \psi_h \right\|_{(H^1(\Omega^\ell))^2} \leq c_1 \left\| \psi_h \right\|_{H^{\frac{1}{2}}(\Gamma_3)}, \quad \forall \psi_h \in X_h. \quad (31)$$

Let $\tilde{\mathbf{v}}_h = ((v_h^1, v_h^2), 0) \in \mathcal{V}$ with $v_h^\ell = R_h^\ell(\pi_h \psi)$ and $v_h^{3-\ell} = 0$. According to (29) and (31), we have

$$\left\| \tilde{\mathbf{v}}_h \right\|_{\mathcal{V}} \leq c_2 \left\| R_h^\ell(\pi_h \psi) \right\|_{(H^1(\Omega^\ell))^2} \leq c^* \left\| \psi \right\|_{H^{\frac{1}{2}}(\Gamma_3)} = c^*. \quad (32)$$

Combining (25), (30) and (32), we obtain

$$\frac{1}{c^*} \left\| \mu_h \right\|_{H^{\frac{1}{2}}(\Gamma_3)} \leq \frac{\int_{\Gamma_3} \mu_h \psi d\Gamma_3}{\left\| \tilde{\mathbf{v}}_h \right\|_{\mathcal{V}}} = \frac{\int_{\Gamma_3} \mu_h \pi_h \psi d\Gamma_3}{\left\| \tilde{\mathbf{v}}_h \right\|_{\mathcal{V}}} \leq \sup_{\tilde{\mathbf{v}}_h \in \mathcal{V}_h} \frac{b(\mu_h, \tilde{\mathbf{v}}_h)}{\left\| \tilde{\mathbf{v}}_h \right\|_{\mathcal{V}}}.$$

Then the inf-sup condition (28) is proved.

Lemma 4 Let $\Lambda_h = \Lambda_h(\Gamma_3)$ or $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$ with $\ell = 1$ or 2 and let $(\tilde{\mathbf{u}}_h, \lambda_h) \in \mathcal{V}_h \times \Lambda_h$ be the solution of (26)-(27). Then $\tilde{\mathbf{u}}_h$ is also solution of the variational inequality:

$$\tilde{\mathbf{u}}_h \in K_h, \quad a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h - \tilde{\mathbf{u}}_h) \geq (\tilde{\mathbf{f}}, \tilde{\mathbf{v}}_h - \tilde{\mathbf{u}}_h)_{\mathcal{V}}, \quad \forall \tilde{\mathbf{v}}_h \in K_h, \quad (33)$$

where $K_h = K_h(\Gamma_3)$ if $\Lambda_h = \Lambda_h(\Gamma_3)$, $K_h = K_h(\Sigma_h^\ell)$ if $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$ with

$$K_h(\Gamma_3) = \{ \tilde{\mathbf{v}}_h = (\mathbf{v}_h, \varphi_h) \in \mathcal{V}_h; \pi_h[\mathbf{v}_{h\nu}] \leq 0 \text{ on } \Gamma_3 \}, \quad (34)$$

$$K_h(\Sigma_h^\ell) = \{ \tilde{\mathbf{v}}_h = (\mathbf{v}_h, \varphi_h) \in \mathcal{V}_h; (\pi_h[\mathbf{v}_{h\nu}]) (a) \leq 0, \quad \forall a \in \Sigma_h^\ell \}. \quad (35)$$

Proof. Let us first notice that $K_h(\Gamma_3)$ and $K_h(\Sigma_h^\ell)$ depend on ℓ which has been omitted to lighten the notations. Taking $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (27) leads to $b(\mathbf{u}_h, \lambda_h) = 0$ and to

$$b(\tilde{\mathbf{u}}_h, \mu_h) = \int_{\Gamma_3} \mu_h [\mathbf{u}_{h\nu}] d\Gamma_3 = \int_{\Gamma_3} \mu_h \pi_h [\mathbf{u}_{h\nu}] d\Gamma_3 \leq 0, \quad \forall \mu_h \in \Lambda_h.$$

The latter inequality implies by polarity that $-\pi_h[\mathbf{u}_{h\nu}] \in \Lambda_h^*$ (the notation Λ_h^* stands for the positive polar cone of Λ_h).

Then if $\Lambda_h = \Lambda_h(\Gamma_3)$ then $\Lambda_h^* = X_h(\Gamma_3)$ since $X_h(\Gamma_3)$ is a closed convex cone. Hence $-\pi_h[\mathbf{u}_{h\nu}] \in X_h(\Gamma_3)$ and $\tilde{\mathbf{u}}_h \in K_h(\Gamma_3)$. Consequently (26) and $b(\tilde{\mathbf{u}}_h, \lambda_h) = 0$ lead to

$$a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) = (\tilde{\mathbf{f}}, \tilde{\mathbf{u}}_h)_{\mathcal{V}} \quad (36)$$

and for any $\tilde{v}_h \in K_h(\Gamma_3)$, we get

$$a(\tilde{u}_h, \tilde{v}_h) - (\tilde{f}, \tilde{v}_h)_{\mathcal{V}} = - \int_{\Gamma_3} \lambda_h [\mathbf{v}_{h\nu}] d\Gamma_3 = - \int_{\Gamma_3} \lambda_h \pi_h [\mathbf{v}_{h\nu}] d\Gamma_3 \geq 0, \quad (37)$$

owing to $\lambda_h \in \Lambda_h(\Gamma_3)$,

Putting together (36) and (37) implies that \tilde{u}_h is a solution to the variational inequality (33) (with $K_h = K_h(\Gamma_3)$) which admits a unique solution according to Stampacchia's theorem.

If $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$ is treated similarly to the previous one.

ERROR ESTIMATES

Now we intend to analyze the convergence of both quadratic finite element approaches: discrete non-interpenetration condition of quadratic type (34) or of linear type (35). Before considering separately both methods, we begin with a common result.

Theorem 1 Set $\Lambda_h = \Lambda_h(\Gamma_3)$ and let $(\tilde{u}, \lambda) \in \mathcal{V} \times M$ be the solution of (23)-(24). Suppose that $\tilde{u} \in (H^{\eta+\frac{3}{2}}(\Omega^1))^3 \times (H^{\eta+\frac{3}{2}}(\Omega^2))^3$ with $0 < \eta < 1$. Let $(\tilde{u}_h, \lambda_h) \in \mathcal{V}_h \times \Lambda_h$ be the solution of (26)-(27). Then there exists a constant $C > 0$ independent of h and \tilde{u} such that

$$\|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq Ch^{\frac{1}{2}+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta}.$$

Proof. Let us denote by γ the ellipticity constant of $a(\cdot, \cdot)$ on \mathcal{V} . Let $\tilde{v}_h \in \mathcal{V}_h$, then by (23), (24), (26) and (27), it follows that

$$\begin{aligned} \gamma \|\tilde{u} - \tilde{u}_h\|^2 &\leq a(\tilde{u} - \tilde{u}_h, \tilde{u} - \tilde{v}_h) + a(\tilde{u}, \tilde{v}_h - \tilde{u}_h) - a(\tilde{u}_h, \tilde{v}_h - \tilde{u}_h) \\ &\leq a(\tilde{u} - \tilde{u}_h, \tilde{u} - \tilde{v}_h) - b(\tilde{v}_h - \tilde{u}_h, \lambda) + b(\tilde{v}_h - \tilde{u}_h, \lambda_h), \end{aligned}$$

hence, we obtain

$$\gamma \|\tilde{u} - \tilde{u}_h\|^2 \leq a(\tilde{u} - \tilde{u}_h, \tilde{u} - \tilde{v}_h) - b(\tilde{v}_h - \tilde{u}_h, \lambda - \lambda_h) - b(\tilde{u} - \tilde{u}_h, \lambda - \lambda_h).$$

The continuous and discrete complementary conditions imply

$$b(\tilde{u}, \lambda) = b(\tilde{u}_h, \lambda_h) = 0. \quad (38)$$

Hence

$$\gamma \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}}^2 \leq a(\tilde{u} - \tilde{u}_h, \tilde{u} - \tilde{v}_h) - b(\tilde{v}_h - \tilde{u}, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h) + b(\tilde{u}_h, \boldsymbol{\lambda}) + b(\tilde{u}, \boldsymbol{\lambda}_h).$$

Using the continuity of the bilinear form, we obtain

$$c_1 \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}}^2 \leq \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} \|\tilde{u} - \tilde{v}_h\|_{\mathcal{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \|\tilde{u} - \tilde{v}_h\| + b(\tilde{u}_h, \boldsymbol{\lambda}) + b(\tilde{u}, \boldsymbol{\lambda}_h). \quad (39)$$

Using (23), (26) and from $\mathcal{V}_h \subset \mathcal{V}$, we get

$$a(\tilde{u} - \tilde{u}_h, \tilde{v}_h) = b(\tilde{v}_h, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}).$$

Consequently, for any $\tilde{v}_h \in \mathcal{V}_h$ and any $\boldsymbol{\mu}_h \in X_h$

$$\begin{aligned} b(\tilde{v}_h, \boldsymbol{\lambda}_h - \boldsymbol{\mu}_h) &= a(\tilde{u} - \tilde{u}_h, \tilde{v}_h) + b(\tilde{v}_h, \boldsymbol{\lambda} - \boldsymbol{\mu}_h) \\ &\leq c_2 \left(\|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \right) \|\tilde{v}_h\|_{\mathcal{V}}. \end{aligned}$$

This estimate and the inf-sup condition (28) allow us to write

$$\|\boldsymbol{\lambda}_h - \boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq c_3 \left(\|\tilde{u} - \tilde{u}_h\| + \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \right).$$

By the triangular inequality we come to the conclusion that

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq c_4 \left(\|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + \inf_{\boldsymbol{\mu}_h \in X_h} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \right). \quad (40)$$

As a consequence (Ciarlet 1991; Crouzeix & Thome, 1987), we have

$$\inf_{\tilde{v}_h \in \mathcal{V}_h} \|\tilde{u} - \tilde{v}_h\| \leq Ch^{\frac{1}{2}+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta}, \quad (41)$$

and

$$\inf_{\boldsymbol{\mu}_h \in X_h} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq Ch^{\frac{1}{2}+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta}. \quad (42)$$

Since $[\mathbf{u}_\eta] \leq 0$ and $\lambda_h \in \Lambda_h$ with $\Lambda_h = \Lambda_h(\Gamma_3)$, we deduce

$$b(\tilde{u}, \boldsymbol{\lambda}_h) \leq 0. \quad (43)$$

Using (25), (27) and (38), we have

$$\int_{\Gamma_3} \mu_h \cdot \pi_h[u_{h\nu}] d\Gamma_3 = \int_{\Gamma_3} \mu_h [u_{h\nu}] d\Gamma_3 = b(\tilde{u}_h, \mu_h) = b(\tilde{u}_h, \mu_h - \boldsymbol{\lambda}_h) \leq 0.$$

It follows that

$$-\pi_h[\mathbf{u}_{h\nu}] \in \mathbf{\Lambda}_h^*, \quad (44)$$

where

$$\mathbf{\Lambda}_h^* = \left\{ \xi_h \in X_h; \int_{\Gamma_3} \xi_h \mu_h d\Gamma_3 \geq 0, \forall \mu_h \in \mathbf{\Lambda}_h \right\}.$$

From (44) and since $\lambda \geq 0$, we deduce

$$b(\tilde{u}_h, \boldsymbol{\lambda}) \leq \int_{\Gamma_3} \boldsymbol{\lambda} ([u_{h\nu}] - \pi_h[u_{h\nu}]) d\Gamma_3. \quad (45)$$

The projection operator π_h defined in (25) has the following approximation property: for any $0 \leq s \leq 3$ we have (Ciarlet, 1991)

$$h^{-\frac{1}{2}} \|\mu - \pi_h \mu\|_{H^{-\frac{1}{2}}(\Gamma_3)} + \|\mu - \pi_h \mu\|_{L^2(\Gamma_3)} \leq ch^s \|\mu\|_{H^s(\Gamma_3)}, \quad (46)$$

for all $\mu \in H^s(\Gamma_3)$. Let $\ell' = 3 - \ell$, using (25) and from $\mathbf{u}_{h\nu}^\ell|_{\Gamma_3} \in X_h$, we obtain

$$\int_{\Gamma_3} \boldsymbol{\lambda} (\mathbf{u}_{h\nu}^\ell - \pi_h \mathbf{u}_{h\nu}^\ell) d\Gamma_3 = \int_{\Gamma_3} (\pi_h \boldsymbol{\lambda}) (\mathbf{u}_{h\nu}^\ell - \pi_h \mathbf{u}_{h\nu}^\ell) d\Gamma_3 = 0, \quad (47)$$

where $\pi_h \mathbf{v}_{h\nu}^\ell = \pi_h[\mathbf{v}_{h\nu}]$ with $\mathbf{v}_h^{3-\ell} = 0$.

Now, by (45) and (47) it follows that

$$\begin{aligned} b(\tilde{u}_h, \boldsymbol{\lambda}) &\leq \int_{\Gamma_3} \boldsymbol{\lambda} (\mathbf{u}_{h\nu}^{\ell'} - \pi_h \mathbf{u}_{h\nu}^{\ell'}) d\Gamma_3 \\ &\leq \int_{\Gamma_3} (\boldsymbol{\lambda} - \pi_h \boldsymbol{\lambda}) (\mathbf{u}_{h\nu}^{\ell'} - \pi_h \mathbf{u}_{h\nu}^{\ell'}) d\Gamma_3 \\ &\leq \int_{\Gamma_3} (\boldsymbol{\lambda} - \pi_h \boldsymbol{\lambda}) ((\mathbf{u}_{h\nu}^{\ell'} - \mathbf{u}_\nu^{\ell'}) - \pi_h (\mathbf{u}_{h\nu}^{\ell'} - \mathbf{u}_\nu^{\ell'})) d\Gamma_3 \\ &\quad + \int_{\Gamma_3} (\boldsymbol{\lambda} - \pi_h \boldsymbol{\lambda}) (\mathbf{u}_\nu^{\ell'} - \pi_h \mathbf{u}_\nu^{\ell'}) d\Gamma_3 \\ &\leq \|\boldsymbol{\lambda} - \pi_h \boldsymbol{\lambda}\|_{L^2(\Gamma_3)} \cdot \|(\mathbf{u}_{h\nu}^{\ell'} - \mathbf{u}_\nu^{\ell'}) - \pi_h (\mathbf{u}_{h\nu}^{\ell'} - \mathbf{u}_\nu^{\ell'})\|_{L^2(\Gamma_3)} \\ &\quad + \|\boldsymbol{\lambda} - \pi_h \boldsymbol{\lambda}\|_{L^2(\Gamma_3)} \cdot \|\mathbf{u}_\nu^{\ell'} - \pi_h \mathbf{u}_\nu^{\ell'}\|_{L^2(\Gamma_3)}. \end{aligned}$$

Then, the approximation (46) and the trace theorem yield:

$$b(\tilde{u}_h, \boldsymbol{\lambda}) \leq ch^{\frac{1}{2}+\eta} \|\boldsymbol{\lambda}\|_{H^\eta(\Gamma_3)} \|\tilde{u}_h - \tilde{u}\|_{H^{\frac{1}{2}}(\Gamma_3)} + ch^{1+2\eta} \|\boldsymbol{\lambda}\|_{H^\eta(\Gamma_3)} \|\tilde{u}\|_{H^{1+\eta}(\Gamma_3)}. \quad (48)$$

Let us note that the trace theorem implies for any $s > 0$

$$\|\boldsymbol{\lambda}\|_{H^s(\Gamma_3)} \leq c \|\tilde{u}\|_{s+\frac{3}{2}} \quad (49)$$

Putting (48) and (49) and using the trace theorem gives

$$b(\tilde{u}_h, \boldsymbol{\lambda}) \leq ch^{\frac{1}{2}+\eta} \|\tilde{u} - \tilde{u}_h\| \|\tilde{u}\|_{\frac{3}{2}+\eta} + ch^{1+2\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta}^2. \quad (50)$$

The result is now a consequence of (39)-(43) and (50) with $ab \leq \beta a^2 + \frac{1}{4\beta} b^2$, $\beta > 0$.

Theorem 2 Set $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$, $\ell = 1$ or 2 and let $(\tilde{u}, \boldsymbol{\lambda}) \in \mathcal{V} \times M$ be the solution of (23)-(24). Suppose that $\tilde{u} \in (H^{\eta+\frac{3}{2}}(\Omega^1))^3 \times (H^{\eta+\frac{3}{2}}(\Omega^2))^3$ with $0 < \eta < 1$. Let $(\tilde{u}_h, \boldsymbol{\lambda}_h) \in \mathcal{V}_h \times \Lambda_h$ be the solution of (26)-(27). Then there exists a constant $C > 0$ independent of h and \tilde{u} such that

$$\|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq Ch^{\frac{1}{2}+\frac{\eta}{2}} \|\tilde{u}\|_{\frac{3}{2}+\eta}. \quad (51)$$

Proof. Using (39), (40) (41), (42) and estimate $ab \leq \beta a^2 + \frac{1}{4\beta} b^2$, $\beta > 0$ leads to the bound

$$\begin{aligned} \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} &\leq C \left\{ h^{\frac{1}{2}+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} + (\max(b(\tilde{u}, \boldsymbol{\lambda}_h), 0))^{\frac{1}{2}} \right. \\ &\quad \left. + (\max(b(\tilde{u}_h, \boldsymbol{\lambda}), 0))^{\frac{1}{2}} \right\}. \end{aligned} \quad (52)$$

The proof consists of estimating $b(\tilde{u}, \boldsymbol{\lambda}_h)$ and $b(\tilde{u}_h, \boldsymbol{\lambda})$.

Step 1. Estimation of $b(\tilde{u}, \boldsymbol{\lambda}_h)$.

Let us denote by i_h^ℓ the Lagrange interpolation operator of order one on the mesh of Ω^ℓ on Γ_3 .

So, there is a constant $c > 0$ satisfying for all $\boldsymbol{\mu} \in H^s(\Gamma_3)$ (Ciarlet, 1991)

$$\|\boldsymbol{\mu} - i_h^\ell \boldsymbol{\mu}\|_{L^2(\Gamma_3)} + h^{\frac{1}{2}} \|\boldsymbol{\mu} - i_h^\ell \boldsymbol{\mu}\|_{H^{\frac{1}{2}}(\Gamma_3)} \leq ch^s \|\boldsymbol{\mu}\|_{H^s(\Gamma_3)}, \quad (53)$$

for all $\boldsymbol{\mu} \in H^s(\Gamma_3)$. We write

$$b(\tilde{u}, \boldsymbol{\lambda}_h) = \int_{\Gamma_3} \boldsymbol{\lambda}_h[\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu] d\Gamma_3 + \int_{\Gamma_3} \boldsymbol{\lambda}_h i_h^\ell[\mathbf{u}_\nu] d\Gamma_3.$$

Obviously $i_h^\ell[\mathbf{u}_\nu] \leq 0$ on Γ_3 . From $\boldsymbol{\lambda}_h \in \Lambda_h(\Sigma_h^\ell)$, $i_h^\ell[\mathbf{u}_\nu] \in \Lambda_h(\Sigma_h^\ell)^*$ and (53), we deduce

$$\begin{aligned}
b(\tilde{u}, \boldsymbol{\lambda}_h) &\leq \int_{\Gamma_3} \boldsymbol{\lambda}_h ([\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu]) d\Gamma_3 \\
&\leq \int_{\Gamma_3} \boldsymbol{\lambda} ([\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu]) d\Gamma_3 + \\
&\quad \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \|[\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu]\|_{H^{\frac{1}{2}}(\Gamma_3)}.
\end{aligned}$$

Using again the approximation properties of i_h^ℓ implies

$$\begin{aligned}
b(\tilde{u}, \boldsymbol{\lambda}_h) &\leq \int_{\Gamma_3} \boldsymbol{\lambda} ([\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu]) d\Gamma_3 + \\
&ch^{\frac{1}{2}+\eta} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \|[\mathbf{u}_\nu]\|_{H^{1+\eta}(\Gamma_3)}. \tag{54}
\end{aligned}$$

The remaining integral term is estimated using (53):

$$\begin{aligned}
\int_{\Gamma_3} \boldsymbol{\lambda} ([\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu]) d\Gamma_3 &\leq \|\boldsymbol{\lambda}\|_{L^2(\Gamma_3)} \|[\mathbf{u}_\nu] - i_h^\ell[\mathbf{u}_\nu]\|_{L^2(\Gamma_3)} \\
&\leq ch^{1+\eta} \|[\mathbf{u}_\nu]\|_{H^{1+\eta}(\Gamma_3)} \|\boldsymbol{\lambda}\|_{L^2(\Gamma_3)}. \tag{55}
\end{aligned}$$

Putting (54) and (55) and using the trace theorem, we get

$$b(\tilde{u}, \boldsymbol{\lambda}_h) \leq C \|\tilde{u}\|_{\frac{3}{2}+\eta} \left(h^{\frac{1}{2}+\eta} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} + h^{1+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} \right). \tag{56}$$

Step 2. Estimation of $b(\tilde{u}_h, \boldsymbol{\lambda})$.

Let $X_h^\ell(\Gamma_3)$ be the space of the piecewise continuous functions on Γ_3 which are constant on the meshes of Ω^ℓ on Γ_3 . Define Π_h^ℓ as the projection operator for the $L^2(\Gamma_3)$ inner product on $X_h^\ell(\Gamma_3)$. Such an operator satisfies the following estimate for any $0 \leq r \leq 1$:

$$\|\mu - \Pi_h^\ell \mu\|_{L^2(\Gamma_3)} \leq ch^r \|\mu\|_{H^r(\Gamma_3)}, \quad \forall \mu \in H^r(\Gamma_3). \tag{57}$$

According to Lemma 4, we have $\pi_h^\ell[\mathbf{u}_{h\nu}](a) \leq 0$ for any $a \in \Sigma_h^\ell$. This implies that

$$\Pi_h^\ell (\pi_h^\ell[\mathbf{u}_{h\nu}]) \leq 0 \quad \text{on } \Gamma_3$$

As a consequence

$$\begin{aligned}
b(\tilde{u}_h, \boldsymbol{\lambda}) &= \int_{\Gamma_3} \boldsymbol{\lambda}[\mathbf{u}_{h\nu}] d\Gamma_3 \\
&= \int_{\Gamma_3} \boldsymbol{\lambda}([\mathbf{u}_{h\nu}] - \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3 + \int_{\Gamma_3} \boldsymbol{\lambda}(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3 \\
&\quad + \int_{\Gamma_3} \boldsymbol{\lambda} \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}] d\Gamma_3 \\
&\leq \int_{\Gamma_3} \boldsymbol{\lambda}([\mathbf{u}_{h\nu}] - \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3 + \int_{\Gamma_3} \boldsymbol{\lambda}(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3.
\end{aligned}$$

The term $\int_{\Gamma_3} \boldsymbol{\lambda}([\mathbf{u}_{h\nu}] - \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3$ has already been estimated in Theorem 1 and bounded in (50), hence we obtain

$$\begin{aligned}
b(\tilde{u}_h, \boldsymbol{\lambda}) &\leq ch^{\frac{1}{2}+\eta} \|\tilde{u} - \tilde{u}_h\| \|\tilde{u}\|_{\frac{3}{2}+\eta} + ch^{1+2\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta}^2 \\
&\quad + \int_{\Gamma_3} \boldsymbol{\lambda}(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3.
\end{aligned} \tag{58}$$

The remaining term is developed as follows:

$$\begin{aligned}
&\int_{\Gamma_3} \boldsymbol{\lambda}(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3 \\
&= \int_{\Gamma_3} \boldsymbol{\lambda}((\pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}]) - \Pi_h^\ell (\pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}])) d\Gamma_3 \\
&\quad + \int_{\Gamma_3} \boldsymbol{\lambda}([\mathbf{u}_{h\nu}] - \Pi_h^\ell [\mathbf{u}_{h\nu}]) d\Gamma_3 \\
&= \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ((\pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}]) - \Pi_h^\ell (\pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}])) d\Gamma_3 \\
&\quad + \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_{h\nu}] - \Pi_h^\ell [\mathbf{u}_{h\nu}]) d\Gamma_3.
\end{aligned}$$

Next, we apply Cauchy-Schwartz inequality to deduce

$$\begin{aligned}
&\int_{\Gamma_3} \boldsymbol{\lambda}(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}]) d\Gamma_3 \\
&\leq \|\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}\|_{L^2(\Gamma_3)} \left\| (\pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}]) - \Pi_h^\ell (\pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}]) \right\|_{L^2(\Gamma_3)} \\
&\quad + \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) (([\mathbf{u}_{h\nu}] - [\mathbf{u}_\nu]) - \Pi_h^\ell ([\mathbf{u}_{h\nu}] - [\mathbf{u}_\nu])) d\Gamma_3 \\
&\quad + \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) d\Gamma_3.
\end{aligned}$$

Therefore by (57), we get

$$\begin{aligned} \int_{\Gamma_3} \boldsymbol{\lambda} \left(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}] \right) d\Gamma_3 &\leq \\ c \left\| \boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda} \right\|_{L^2(\Gamma_3)} \left\| \pi_h[\mathbf{u}_{h\nu}] - [\mathbf{u}_{h\nu}] \right\|_{L^2(\Gamma_3)} & \\ + \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) \left(([\mathbf{u}_{h\nu}] - [\mathbf{u}_\nu]) - \Pi_h^\ell ([\mathbf{u}_{h\nu}] - [\mathbf{u}_\nu]) \right) d\Gamma_3 & \\ + \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) d\Gamma_3. & \end{aligned}$$

Now, we use (46), (49), (57) and the trace theorem yield:

$$\begin{aligned} \int_{\Gamma_3} \boldsymbol{\lambda} \left(\pi_h[\mathbf{u}_{h\nu}] - \Pi_h^\ell \pi_h[\mathbf{u}_{h\nu}] \right) d\Gamma_3 &\leq Ch^\eta \|\tilde{u}\|_{\frac{3}{2}+\eta} \left(\|\pi_h[\mathbf{u}_\nu] - [\mathbf{u}_\nu]\|_{L^2(\Gamma_3)} \right. \\ &\quad \left. + \|\pi_h([\mathbf{u}_{h\nu}] - [\mathbf{u}_\nu]) - ([\mathbf{u}_{h\nu}] - [\mathbf{u}_\nu])\|_{L^2(\Gamma_3)} \right) \\ + Ch^{\frac{1}{2}+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} &+ \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) d\Gamma_3 \\ &\leq C \|\tilde{u}\|_{\frac{3}{2}+\eta} \left(h^{\frac{1}{2}+\eta} \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + h^{1+2\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} \right) \\ &\quad + \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) d\Gamma_3. \end{aligned} \quad (59)$$

Using again the approximation properties of Π_h^ℓ gives

$$\begin{aligned} &\int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) d\Gamma_3 \\ &\leq \left\| \boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda} \right\|_{L^2(\Gamma_3)} \left\| [\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu] \right\|_{L^2(\Gamma_3)} \\ &\leq Ch^{1+\eta} \|\boldsymbol{\lambda}\|_{H^\eta(\Gamma_3)} \|\mathbf{u}_\nu\|_{H^1(\Gamma_3)}. \end{aligned}$$

Here, we observe a loss of optimality when approximating the function $[\mathbf{u}_\nu] \in H^{1+\eta}(\Gamma_3)$ with $\Pi_h^\ell [\mathbf{u}_\nu]$. As a consequence

$$b(\tilde{u}_h, \boldsymbol{\lambda}) \leq C \|\tilde{u}\|_{\frac{3}{2}+\eta} \left(h^{\frac{1}{2}+\eta} \|\tilde{u} - \tilde{u}_h\|_{\mathcal{V}} + h^{1+\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} \right). \quad (60)$$

Step3. End of the proof.

The estimate (51) of the theorem is proved by combining (52), (56) and (60) with

$$ab \leq \beta a^2 + \frac{1}{4\beta} b^2, \beta > 0.$$

Theorem 3 Set $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$, $\ell = 1$ or 2 and let $(\tilde{\mathbf{u}}, \lambda) \in \mathcal{V} \times M$ be the solution of (23)-(24). Suppose that $\tilde{\mathbf{u}} \in (H^{\eta+\frac{3}{2}}(\Omega^1))^3 \times (H^{\eta+\frac{3}{2}}(\Omega^2))^3$ with $\frac{1}{2} < \eta < 1$. Assume that the set of points of Γ_3 where the change from $[\mathbf{u}_\nu] < 0$ to $[\mathbf{u}_\nu] = 0$ occurs is finite. Let $(\tilde{\mathbf{u}}_h, \lambda_h) \in \mathcal{V}_h \times \Lambda_h$ be the solution of (26)-(27). Then there exists a constant $C > 0$ independent of h and $\tilde{\mathbf{u}}$ such that

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathcal{V}} + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq Ch^{\frac{1}{2}+\eta} \|\tilde{\mathbf{u}}\|_{\frac{3}{2}+\eta}. \quad (61)$$

Proof. Consider again estimate (59) and suppose now that $\frac{1}{2} < \eta < 1$. Let $N(h)$ represent as in the previous theorem the number of $(\mathbf{l} D)$ -segments denoted T_i ($1 \leq i \leq N(h)$), of the triangulation of Ω^ℓ on Γ_3 where the change from $[\mathbf{u}_\nu] < 0$ to $[\mathbf{u}_\nu] = 0$ occurs.

The integral term in (59) is now estimated as follows:

$$\begin{aligned} \int_{\Gamma_3} (\lambda - \Pi_h^\ell \lambda) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) d\Gamma_3 &= - \int_{\Gamma_3} \lambda \Pi_h^\ell [\mathbf{u}_\nu] d\Gamma_3 \\ &\leq \sum_{i=1}^{N(h)} \int_{T_i} |\lambda| |\Pi_h^\ell [\mathbf{u}_\nu]| d\Gamma_3 \\ &\leq h \sum_{i=1}^{N(h)} \|\lambda\|_{L^\infty(T_i)} \|\Pi_h^\ell [\mathbf{u}_\nu]\|_{L^\infty(T_i)} \\ &\leq h \sum_{i=1}^{N(h)} \|\lambda\|_{L^\infty(T_i)} \|\mathbf{u}_\nu\|_{L^\infty(T_i)}. \end{aligned} \quad (62)$$

From the definition of the segment T_i , we deduce that

$$\|\lambda\|_{L^\infty(T_i)} \leq h^{\eta-\frac{1}{2}} \|\lambda\|_{C^{0,\eta-\frac{1}{2}}(T_i)}$$

and

$$\|D^1[\mathbf{u}_\nu]\|_{L^\infty(T_i)} \leq h^{\eta-\frac{1}{2}} \|D^1[\mathbf{u}_\nu]\|_{C^{0,\eta-\frac{1}{2}}(T_i)}.$$

So

$$\begin{aligned}
& \int_{\Gamma_3} (\boldsymbol{\lambda} - \Pi_h^\ell \boldsymbol{\lambda}) ([\mathbf{u}_\nu] - \Pi_h^\ell [\mathbf{u}_\nu]) \, d\Gamma_3 \\
& \leq h \sum_{i=1}^{N(h)} h^{\eta-\frac{1}{2}} \|\boldsymbol{\lambda}\|_{C^{0,\eta-\frac{1}{2}}(T_i)} h \|D^1[\mathbf{u}_\nu]\|_{L^\infty(T_i)} \\
& \leq h^{1+2\eta} \sum_{i=1}^{N(h)} \|\boldsymbol{\lambda}\|_{C^{0,\eta-\frac{1}{2}}(T_i)} \|[\mathbf{u}_\nu]\|_{C^{1,\eta-\frac{1}{2}}(T_i)} \\
& \leq N(h) h^{1+2\eta} \|\boldsymbol{\lambda}\|_{C^{0,\eta-\frac{1}{2}}(\Gamma_3)} \|[\mathbf{u}_\nu]\|_{C^{1,\eta-\frac{1}{2}}(\Gamma_3)} \\
& \leq N(h) h^{1+2\eta} \|\boldsymbol{\lambda}\|_{H^\eta(\Gamma_3)} \|[\mathbf{u}_\nu]\|_{H^{1+\eta}(\Gamma_3)}, \tag{63}
\end{aligned}$$

where the embedding properties of Sobolev and Hölder spaces (Zhong, 1993) have been used. If $N(h)$ is uniformly bounded in h , we obtain thanks to the trace theorem, (58), (59) and (63):

$$b(\tilde{u}_h, \boldsymbol{\lambda}) \leq c \|\tilde{u}\|_{\frac{3}{2}+\eta} \left(h^{\frac{1}{2}+\eta} \|\tilde{u} - \tilde{u}_h\| + h^{1+2\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} \right). \tag{64}$$

The remaining integral term in (54) is estimated as

$$\begin{aligned}
\int_{\Gamma_3} \boldsymbol{\lambda} ([\mathbf{u}_\nu] - i_h^\ell [\mathbf{u}_\nu]) \, d\Gamma_3 &= - \int_{\Gamma_3} \boldsymbol{\lambda} i_h^\ell [\mathbf{u}_\nu] \, d\Gamma_3 \\
&\leq \sum_{i=1}^{N(h)} \int_{\Gamma_3} |\boldsymbol{\lambda}| |i_h^\ell [\mathbf{u}_\nu]| \, d\Gamma_3 \\
&\leq h \sum_{i=1}^{N(h)} \|\boldsymbol{\lambda}\|_{L^\infty(T_i)} \|i_h^\ell [\mathbf{u}_\nu]\|_{L^\infty(T_i)} \\
&\leq h \sum_{i=1}^{N(h)} \|\boldsymbol{\lambda}\|_{L^\infty(T_i)} \|[\mathbf{u}_\nu]\|_{L^\infty(T_i)}.
\end{aligned}$$

The latter term has already been estimated in (62). Hence, from (54), we deduce

$$b(\tilde{u}, \boldsymbol{\lambda}_h) \leq c \|\tilde{u}\|_{\frac{3}{2}+\eta} \left(h^{\frac{1}{2}+\eta} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_3)} + h^{1+2\eta} \|\tilde{u}\|_{\frac{3}{2}+\eta} \right). \tag{65}$$

Finally, the estimate (61) of the theorem is proved by combining (52), (64), (65) and by using $ab \leq \beta a^2 + \frac{1}{4\beta} b^2$, $\beta > 0$.

CONCLUSION

In this paper we present a mixed variational formulation for frictionless contact problems between two electro-elastic bodies, in which the unknowns are the displacement field, the electric potential field and the contact pressure. We have proposed and studied two mixed finite element methods, in which the discrete non-interpenetration conditions are either an exact non-interpenetration condition “ $\Lambda_h = \Lambda_h(\Gamma_3)$ ” or only a nodal condition “ $\Lambda_h = \Lambda_h(\Sigma_h^\ell)$ ” and proved that they can lead to optimal convergence rates under reasonable hypotheses.

REFERENCES

- Batra, R. C. & Yang, J. S. 1995.** Saint-Venant’s principle in linear piezoelectricity. *Journal of Elasticity* 38: 209-218.
- Bisenga, P., Lebon, F. & Maceri, F. 2002.** The unilateral frictional contact of a piezoelectric body with a rigid support, in *Contact Mechanics*, J.A.C. Martins and Manuel D.P. Monteiro Marques (Eds.). Kluwer, Dordrecht: 347-354.
- Ciarlet, P. G. 1991.** The finite element method for elliptic problems, In *Handbook of Numerical Analysis, Volume II, Part I*, Eds. P.G. Ciarlet and J.L. Lions. North Holland Publishing, Amsterdam.
- Coorevits, P. Hild, P. Lahalouani, K. & Sassi, T. 2002.** Mixed finite element methods for unilateral contact problems: convergence analysis and numerical studies. *Mathematics of Computation* 71(237): 1–25.
- Crouzeix, M. & Thome, V. 1987.** The stability in L^p and $W^{1,p}$ of the L^2 -projection on finite element function spaces. *Mathematics of computation* 48: 521-532.
- Haslinger, J., Hlaváček, I. & Nečas, J. 1996.** Numerical methods for unilateral problems in solid mechanics. in *Handbook of Numerical Analysis, Volume IV, Part 2*, Eds. Ciarlet, P. G. and Lions, J. L., North Holland Amsterdam: 313-485.
- Nečas, J. & Hlaváček, I. 1981.** *Mathematical Theory of Elastic and Elastico-Plastic Bodies*. Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York.
- Mindlin, R. D. 1968.** Polarisation gradient in elastic dielectrics. *International Journal of Solids and Structures* 4: 637-663.
- Mindlin, R. D. 1972.** Elasticity, piezoelectricity and crystal lattice dynamics. *Journal of Elasticity*, 4: 217-280.
- Toupin, R. A. 1963.** A dynamical theory of elastic dielectrics. *International Journal of Engineering Science*, 1: 101-126.
- Voigt, W. 1910.** *Lehrbuch der Kristall-Physik*. Teubner, Leipzig.
- Zhong, Z. H. 1993.** *Finite element procedures for contact-impact problems*. Oxford University Press.

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تقريب عنصر منته مختلط لمسألة تلامس في المرونة كهربية

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خلاصة

يهتم هذا البحث بمسألة تلامس عديم الاحتكاك بين جسمين مرنين كهريياً في مجال ثنائي البعدية. نحصل هنا على صياغة مختلطة بحيث تكون المجاهيل هي حقل الازاحة، حقل الكمون الكهربي وضغط التلامس. نقوم باستخدام طريقة العنصر المنته المختلط للوصول إلى حلول تقريبية. ثم نحسب تقديرات الخطأ لهذه الحلول التقريبية و نقوم من خلال ذلك بالحصول على تقارب الخوارزمية تحت تأثير شروط ملائمة لإنظام الحل الدقيق.