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On the rank of the Doob graph and its complement

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Abstract

We compute the rank of the circulant Doob graph defined in Doob (2002). We also compute the rank and the determinant of its complement graph.

Key words: Doob graph; Complement graph, Circulant graph; Rank; Determinant.

Mathematical Subject Classifications: 05C50, 05C20, 15B36.

1. Introduction

Ranks and determinants of different types of circulant matrices are considered in Doob (2002), Garner (2004), and Sookyang *et al*. (2008). They can be used to determine the non-isomorphism between two graphs. In other words, if two adjacency matrices have different determinants or ranks, then the graphs are non-isomorphic. For more general results about the circulant matrices, refer to Wyn-jones (2013) and Gray (2006). Williams (2014) computed another algebraic invariant, namely the Smith normal form of various families of circulant graphs. The determinants and ranks of circulant graphs were used to check the results. In this paper, we compute the other algebraic invariants, specifically the rank and the determinant of the complement of the Doob graphs. This can be useful for verification of the Smith normal form of its complement.

2. Preliminaries

An $n \times n$ matrix A is said to be circulant if its first row is $(a_0, a_1, \dots, a_{n-1})$, and where row $i+1 (0 \le i \le n-2)$ is a cyclic shift of row i by one column. A circulant graph is a graph G whose vertices can be ordered so that the adjacency matrix $A = A(G)$ is a circulant matrix. We use $r(=\sum_{i=0}^{n-1} a_i)$ to denote the degree of this regular graph. From the theory of circulant matrices Davis (1979), we have:

Lemma 1. *Let* $p(x) = \sum_{i=0}^{n-1} a_i x^i$. *The eigenvalues of A are* $p(\varepsilon_k)$, *where* $\epsilon_k = e^{2i\pi k/n}$, $k = 0,1,\dots,n-1$.

The so-called Doob graphs $G(r,t)$ from Doob (2002) have $n = (r-1)t + 2$ vertices and are regular of degree r. They were constructed by taking a cycle of length n and joining each vertex with $r-2$ other vertices equally spaced around the circle. In other words, the vertices are $\{1,2,\dots,n\}$ with two vertices i and j adjacent if and only if $|i-j| \equiv 1 \bmod t$. Figure 1 shows a tangible example.

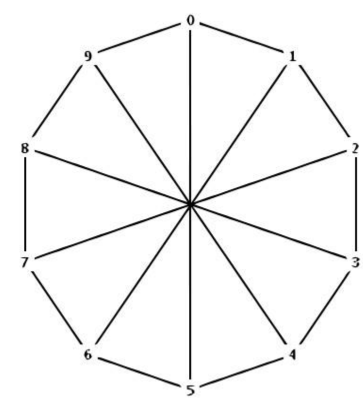

Fig. 1. Doob graph for $G(3,4)$

Let $\epsilon_k = e^{2i\pi k/n}$ and $\xi = e^{2i\pi t/n}$. The eigenvalues of $A(G(r,t)$ were computed in Doob (2002). They are given by the Theorem 1.

Theorem 1. *The eigenvalues of* $A(G(r,t))$ *are*

- *r, which is simple,*
- $\epsilon_k(\xi^{rk}-1)/(\xi^k-1)$ *for* $0 < k < n/2$,

each of which has multiplicity 2, and

- *for even n,* $\begin{cases} -1 \text{ if } t \text{ is even} \\ -1 \text{ if } t \text{ is odd} \end{cases}$

which are also simple.

3. The rank of the Doob graph

We now compute the rank of $G(r,t)$ by finding the nullity in its adjacency matrix. Before this, we need the following observation:

Let $\gcd(n,t)$ denote the greatest common divisor of n and t. Since $n=(r-1)t+2$, we have $\gcd(n,rt)=$

$$\gcd(rt,t-2)=\begin{cases}\gcd(r,t\text{-}2) & \text{if } t \text{ is odd}\\ \gcd(2r,t\text{-}2) & \text{if } t \text{ is even.}\end{cases}$$

Theorem 2. *The rank of the graph $G(r,t)$ (with $n=(r-1)t+2$ vertices) is given by*

$$\operatorname{rank} A(G(r,t))=\begin{cases} n+1\text{-}\gcd(r,t\text{-}2) & \text{if } t \text{ is odd}\\ n+2\text{-}\gcd(2r,t\text{-}2) & \text{if } t \text{ is even.}\end{cases}$$

Proof. The zero eigenvalues can appear only in the second part of Theorem 1. Therefore, $\xi^{rk}-1=0$ if and only if $rtk \equiv 0 \bmod n$. In order to solve this linear congruence, we let $d=\gcd(n,rt)$. If t is odd, since $d|0$ and $0<k<n/2$, we have exactly $(d-1)/2$ distinct solutions mod n. By the second part of Theorem 1, each eigenvalue has multiplicity 2. This implies that the number of zero eigenvalues is $d-1=\gcd(n,rt)-1=\gcd(r,t-2)-1$. Thus, we get the first part of the theorem. If t is even, since $d|0$ and $0<k<n/2$, we have exactly $(d-2)/2$ distinct solutions mod n. By the second part of Theorem 1, each eigenvalue has multiplicity 2, which implies that the number of zero eigenvalues is $d-2=\gcd(n,rt)-2=\gcd(2r,t-2)-2$. This completes the proof of the theorem.

Note that the rank formula generalizes the results found in Williams (2014) Corollary 3.8. Here, the Doob graphs are isomorphic to $F_{n,r,t}=\mathrm{circ}_n(\underbrace{v,\dots,v}_{r},\underbrace{0,\dots,0}_{n-tr})$, $(tr-(t-1)\le n, 1\le t)$ where $v=(1,\underbrace{0,\dots,0}_{t-1})$. Yet, in order to make sense $n-tr\ge 0$ and replacing $n=(r-1)t+2$ in the inequality, we get $(r-1)t+2-tr\ge 0$. This forces the values of t to be just 1 or 2. Therefore, Corollary 3.8 proposed in Williams (2014) is just valid for $t=1$ or $t=2$.

4. Complement of Doob graphs, ranks and determinants

In this section, the complement of $G(r,t)$ is denoted by $\overline{G(r,t)}$. We need the following lemma to compute the eigenvalues of the complement of the Doob graph, whose proof can be found in Theorem 2.1.2 of Cvetković *et al.* (2010).

Lemma 2. *If G is r-regular and $r=\lambda_1\ge\lambda_2\ge\dots\ge\lambda_n$ are the eigenvalues of G then the eigenvalues of $\bar{G}$ are $n-1-r$ and $\{-1-\lambda_i : 2\le i\le n\}$.*

Because of the lemma we can state the following.

Theorem 3. *The eigenvalues of $\overline{A(G(r,t))}$ are:*

- $n-r-1$, *which is simple,*
- $-1-\epsilon_k(\xi^{rk}-1)/(\xi^k-1)$ *for* $0<k<n/2$, *each of which has multiplicity* 2.
- *for even n,* $\begin{cases} r-1 & \text{if } t \text{ is even}\\ 0 & \text{if } t \text{ is odd}\end{cases}$ *which are also simple.*

We now compute the rank of the complement of the Doob graph by finding the nullity in its adjacency matrix.

Theorem 4. *The rank of the graph $\overline{G(r,t)}$ (with $n=(r-1)t+2$ vertices) is given by*

$$\operatorname{rank}\overline{A(G(r,t))}=\begin{cases} n+1-\gcd(t-1,n) & \text{if } t>1\\ 0 & \text{if } t=1.\end{cases}$$

Proof. In order to compute the rank of the complement of the Doob graph, we need to find the number of 0 eigenvalues. Therefore, we have to solve the following equation:

$$-1-\epsilon_k(\xi^{rk}-1)/(\xi^k-1)=0.$$

This expression simplifies to:

$$(\epsilon_k+1)(\xi^k-\epsilon_k)=0,$$

which can be written as:

$$(e^{\frac{2i\pi k}{n}}+1)(e^{\frac{2i\pi kt}{n}}-e^{\frac{2i\pi k}{n}})=0.$$

This means that either $e^{\frac{2i\pi k}{n}}+1=0$ or $e^{\frac{2i\pi kt}{n}}-e^{\frac{2i\pi k}{n}}=0$. Since $0<k<n$, the only solution for the first factor is when n is even and $k=n/2$.

For the second factor, we need to solve $e^{\frac{2i\pi kt}{n}}=e^{\frac{2i\pi k}{n}}$. This equation is equivalent to $kt\equiv k \bmod n$, which is equivalent to $(t-1)k\equiv 0 \bmod n$. Let $d=\gcd(t-1,n)$. Then since $d|0$, it has d distinct solutions mod n. But since $0<k<n$, we have $d-1$ distinct solutions mod n. This means that the number of zero eigenvalues is $d-1$. If $t=1$ then $n=r+1$, so $n-r-1=0$. Therefore, the number of all

zero eigenvalues is n, which completes the second part of the theorem.

We now compute the determinant by finding the product of the eigenvalues.

Theorem 5. *The determinant of the graph $\overline{G(r,t)}$ (with $n=(r-1)t+2$ vertices) is*

$$\overline{A(G(r,t))}=\begin{cases}(n-r-1) & \text{if } n \text{ and } t \text{ are even and } \gcd(t-1,n)=1\\ n-r-1 & \text{if } n \text{(hence } t\text{) are odd and } \gcd(t-1,n)=1\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Theorem 3, one of the eigenvalues is $n-r-1$. When n is even, we have another non-zero eigenvalue $r-1$ for $k=n/2$. By Theorem 4, if $\gcd(t-1,n)>1$ then we have at least one zero eigenvalue which forces the determinant to be zero. So let us consider the case $\gcd(t-1,n)=1$.

The remaining product of the eigenvalues is

$$\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\left(-1-\frac{\epsilon_k(\xi^{rk}-1)}{\xi^k-1}\right)$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{-\xi^k+1-\epsilon_k(\xi^{rk}-1)}{\xi^k-1}$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi tk}{n}}-e^{\frac{2i\pi k(1+tr)}{n}}+e^{\frac{2i\pi k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi tk}{n}}-e^{\frac{2i\pi k(n+t-1)}{n}}+e^{\frac{2i\pi k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi tk}{n}}-e^{\frac{2i\pi(t-1)k}{n}}+e^{\frac{2i\pi k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{(1-e^{\frac{2i\pi(t-1)k}{n}})(1+e^{\frac{2i\pi k}{n}})}{e^{\frac{2i\pi tk}{n}}-1}.$$

Note that $\gcd(n,t)=\gcd((r-1)t+2,t)=\gcd(t,2)$, which implies that if t is odd then $\gcd(n,t)=1$ and if t is even then $\gcd(n,t)=2$. Therefore if n and t are odd, we have $\gcd(n,t)=1$. If n and t are even, $\gcd(n,t)=2$.

Since $\gcd(t-1,n)=1$ we have $\{(t-1)k \bmod n \mid k=1,\dots,n\text{-}1\}=\{1,\dots,n\}$. So the above product becomes

$$\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{(1-e^{\frac{2i\pi k}{n}})(1+e^{\frac{2i\pi k}{n}})}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi 2k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}.$$

Now there are two cases to consider. The first one is when n and t are both even. The second one is when n and t are both odd.

Assume that n and t are even. Since n and t are even we have $\gcd(n,t)=2$. This implies $\{(tk \bmod n \mid k=1,\dots,n\text{-}1\}=\{0,2,4,\dots n-2\}=(2k \bmod n \mid k=1,\dots,n\text{-}1\}$. Note that $e^{\frac{2i\pi tk}{n}}-1\neq 0$ since $k\neq n/2$. Hence

$$\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi 2k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi 2k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

This proves the first part of the theorem.

Let n and t be odd. Then we have $\gcd(n,t)=1$ and $\gcd(n,2)=1$. This implies $\{tk \bmod n \mid k=1,\dots,n\text{-}1\}=\{1,\dots,n\text{-}1\}=\{2k \bmod n \mid k=1,\dots,n\text{-}1\}$. This implies

$$\prod_{\substack{0<k<n\\ k\neq\frac{n}{2}}}\frac{1-e^{\frac{2i\pi 2k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{0<k<n}\frac{1-e^{\frac{2i\pi 2k}{n}}}{e^{\frac{2i\pi tk}{n}}-1}$$

$$=\prod_{0<k<n}(-1)^{n-1}=1.$$

This proves the second part of the theorem.

References

Cvetković, D., Rowlinson, P. & Simić, S. (2010). Graph operations and modifications. In: An Introduction to the Theory of Graph Spectra, Pp. 25. London Mathematical Society Student Texts 75. Cambridge University Press, England.

Davis, P. J. (1979). Centralizers and circulants. In: Circulant Matrices, Pp. 208. American Mathematical Society Chelsea Publishing, Providence, RI.

Doob, M. (2002). Circulant graphs with det(−A(G))=−deg(G): Codeterminants with Kn. Linear Algebra and its Applications, **340**: 87–96.

Garner, C.R. (2004). Investigations into the ranks of regular graphs. PhD thesis, Rand Afrikaans University, Johannesburg, RSA.

Gray, R. M. (2006). Toeplitz and circulant matrices: A review. Foundations and Trends® in Communications and Information Theory, **2**(3): 155–239.

Sookyang, S., Arworn, S. & Wojtylak, P. (2008). Characterizations of non-singular cycles, path and trees. Thai Journal of Mathematics, **6**(2): 331–336.

Williams, G. (2014). Smith forms for adjacency matrices of circulant graphs. Linear Algebra and its Applications, **443**: 21–33.

Wyn-jones, A. (2013). Circulant matrices. In: Circulants, Pp. 12-21. Retrieved from CiteSeerX. http://www.circulants.org.

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الملخص

قمنا بحساب رتبة مخطط DOOB الذي هو عبارة عن نوع خاص من المخططات المتدفقة وكذلك قمنا بحساب رتبة ومحدد المخطط المتمم له.