Kuwait J. Sci. **45 (2)** pp 1-7, 2018

Approximation by rational functions in Morrey-Smirnov classes

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Abstract

In this article, we investigate the direct problem of approximation theory in Morrey-Smirnov classes of analytic functions, defined on a doubly-connected domain bounded by two sufficiently smooth curves.

Mathematical Subject Classifications (2010): 30E10, 41A20, 41A25, 46E30.

Keywords: Dini-smooth curve; modulus of smoothness; Morrey space; Morrey-Smirnov classes; rational approximation.

1. Introduction

The fundamental problem in approximation theory consists in finding for a complicated function f from a normed space, a simple function (polynomial or rational function) to approximate f. In 1885 Weierstrass proved the first theorem in approximation theory "Weierstrass Theorem" (Devore & Lorentz, 1993), stating that, each continuous real function defined on a closed and bounded interval can be approximated by polynomials. The approximation of complex functions by polynomials was studied by several mathematicians (Runge, 1885; Walsh, 1998; Keldysh, 1945; Lavrentiev, 1936). Finally, in 1951 Mergelyan completes their works by proving the following great theorem (Gaier, 1987).

Let K be a compact set in the complex plane with connected complement and let f be a function analytic in the interior of K and continuous on K. Then, f can be approximated on K by polynomials.

When K is a doubly connected domain, the approximation by polynomials is not possible and must be replaced with approximation by rational functions.

In the literature, there are many investigations relating to approximation problems in simply connected domains. For example, the problems of approximation theory for Smirnov classes, Smirnov-Orlicz classes and Morrey- Smirnov classes were studied (Kokilashvili, 1968; Israfilov, et al. 2005; Israfilov & Tozman, 2008).

The problems of approximation by trigonometric/algebraic polynomials in Orlicz spaces having non convex generating Young functions were obtained (Koç, 2016).

But the approximation problems in the doubly connected domains have not been investigated sufficiently.

In this work, rational approximation problem in Morrey-Smirnov classes of functions, defined on a doubly connected domain is investigated. Similar results were obtained in Jafarov (2015).

The concept of Morrey space, introduced by Morrey in 1938, has been studied intensively by various authors. Currently there are several investigations relating to the fundamental problems in this space. See for example, Israfilov & Tozman (2011), Bilalov & Quliyeva (2014) and Bilalov et al. (2016).

2. Preliminaries

Suppose that G is anarbitrary doubly connected domain in the complex plane , bounded by two rectifiable Jordan curves L_1 and L_2. Without loss of generality, we may assume that the closed curve is inside the closed curve L_1 and $0 \in \operatorname{int} L_2$.

Let $G_1^0 := \operatorname{int} L_1$, $G_1^\infty := \operatorname{ext} L_1$, $G_2^0 := \operatorname{int} L_2$, $G_2^\infty := \operatorname{ext} L_2$, $D := \left\{ w \in \pounds : |w| < 1 \right\}$, $D^- := \left\{ w \in \pounds : |w| > 1 \right\}$ and $\gamma_0 := \partial D := \left\{ w \in \pounds : |w| = 1 \right\}$.

We denote by $w = \phi(t)$ $\left(w = \phi_1(t)\right)$ the conformal mapping of $G_1^\infty \left(G_2^0\right)$ onto domain D^- normalized by the conditions

$$\phi(\infty) = \infty, \qquad \lim_{t \to \infty} \frac{\phi(t)}{t} > 0,$$
$$\phi_1(0) = \infty, \qquad \lim_{t \to 0} t\phi_1(t) > 0.$$

Moreover, by ψ and ψ_1 will denote the inverse mappings of ϕ and ϕ_1, respectively.

Let Γ be a rectifiable Jordan curve in the complex plane . By $L^p(\Gamma)$ $(1 \le p < \infty)$, we denote the set of all measurable complex valued functions f such that $|f|^p$ is Lebesgue integrable with respect to the arc length on Γ.

Throughout this paper, we assume that the letters $c_1, c_2, c_3, \ldots$ always remain to denote positive constants that may differ at each occurrence.

Definition 2.1. Le Γ be some rectifiable Jordan curve, $0 \le \alpha \le 1$ and $1 \le p < \infty$. We denote by $L^{p,\alpha}(\Gamma)$ the Morrey space, as the set of all locally integrable functions f, with a finite norm:

$$\|f\|_{L^{p,\alpha}(\Gamma)} := \left\{ \sup_B \frac{1}{|B\cap\Gamma|_\Gamma^{1-\alpha}} \int_{B\cap\Gamma} |f(t)|^p |dt| \right\}^{1/p} < \infty, \quad (1)$$

where B is an arbitrary disk centered on Γ and $|B\cap\Gamma|_\Gamma$ is the linear Lebesgue measure of the set $B\cap\Gamma$. We know that $L^{p,\alpha}(\Gamma)$ is a Banach space. If $\alpha=1$ then the class $L^{P,1}(\Gamma)$ coincides with the class $L^P(\Gamma)$, and for $\alpha=0$ the class $L^{P,0}(\Gamma)$ coincides with the class $L^\infty(\Gamma)$. Moreover, $L^{p,\alpha_1}(\Gamma)\subset L^{p,\alpha_2}(\Gamma)$ for $0\le\alpha_1\le\alpha_2$. Thus, $L^{p,\alpha}(\Gamma)\subset L^1(\Gamma)$, $\forall\alpha\in[0,1]$.

Definition 2.2. Let U be a finite simply connected domain with the rectifiable Jordan curve boundary Γ in the complex plane, and let Γ_r be the image of circle $\{w\in£ : |w|=r, 0<r<1\}$ under some conformal mapping of D onto U. By $E^1(U)$ we denote the class of analytic functions f in U, which satisfy

$$\int_{\Gamma_r} |f(t)||dt| < \infty$$

uniformly in .

It is known that every function of class $E^1(U)$ has nontangential boundary values almost everywhere on Γ and the boundary function belongs to $L^1(\Gamma)$ (Goluzin, 1969).

Definition 2.3. Let be a finite simply connected domain with the rectifiable Jordan curve boundary Γ in the complex plane , we define the Morrey Smirnov classes $E^{p,\alpha}(U)$, $0\le\alpha\le1$, and $1\le p<\infty$, of analytic functions in U as:

$$E^{p,\alpha}(U) := \left\{ f\in E^1(U) : f\in L^{p,\alpha}(\Gamma) \right\}.$$

Under the norm (1), $E^{p,\alpha}(U)$, $0\le\alpha\le1, 1\le p<\infty$, becomes a Banach space

$$\|f\|_{E^{p,\alpha}(U)} := \|f\|_{L^{p,\alpha}(\Gamma)}.$$

Definition 2.4. We define the $r\,th$ modulus of smoothness of a function $g\in L^{p,\alpha}(\gamma_0)$ for $r=1,2,3,...$ by the relation

$$\omega^r_{p,\alpha}(g,t) := \sup_{|h|\le t} \left\|\Delta^r_h(g,.)\right\|_{L^{p,\alpha}(\gamma_0)},$$

where

$$\Delta^r_h(g,.) = \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} g\left(.e^{ikh}\right).$$

Definition 2.5. Let Γ be a rectifiable Jordan curve in the complex plane . For a given $t\in\Gamma$ and $f\in L^1(\Gamma)$, the operator defined by

$$S_\Gamma(f)(t) := \lim_{\varepsilon\to0^+} \frac{1}{2\pi i} \int_{\Gamma\cap\{\xi:|\xi-t|>\varepsilon\}} \frac{f(\xi)}{\xi-t} d\xi$$

is called the Cauchy singular operator.

Definition 2.6. A smooth curve Γ is called Dini-smooth, if

$$\int_0^\delta \frac{\Omega(\sigma,s)}{s} ds < \delta, \quad \delta>0,$$

where $\sigma(s)$ is the angle, between the tangent line of Γ and the positive real axis expressed as a function of arc length s with the modulus of continuity $\Omega(\sigma,s)$, where

$$\Omega(\sigma,s) := \sup_{|s_1-s_2|\le s} |\sigma(s_1)-\sigma(s_2)|, \ s>0.$$

In Samko (2009); Kokilashvili & Meskhi (2008) it is proved that, if Γ is a Dini smooth curve, then the operator S_Γ is bounded in $L^{p,\alpha}(\Gamma)$ with $0<\alpha\le1$, $1<p<\infty$, i.e. there exists a positive constant c_1 such the following inequality holds for any $f\in L^{p,\alpha}(\Gamma)$

$$\|S_\Gamma(f)\|_{L^{p,\alpha}(\Gamma)} \le c_1 \|f\|_{L^{p,\alpha}(\Gamma)}. \quad (2)$$

To prove our main theorem, we need the following lemma. It can be found in (Goluzin, 1969).

Lemma 2.1. Let U be a simply connected domain in the complex plane , bounded by a rectifiable Jordan curve Γ, and let U^- be the exterior of Γ. Then for $f\in L^1(\Gamma)$, the functions f^+ and f^- defined by

$$f^+(t) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi-t} d\xi, \qquad t\in U,$$

$$f^-(t) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi-t} d\xi, \qquad t\in U^-$$

are analytic in U and U^-, respectively, $f^-(\infty)=0$, and satisfy the following Sokhotskiy – Plemel j formulas, i.e. on Γ.

$$f^+(t) = S_\Gamma f(t) + \frac{1}{2} f(t),$$

$$f^-(t) = S_\Gamma f(t) - \frac{1}{2} f(t),$$

$$f(t) = f^+(t) - f^-(t)$$

The level lines of the domains G_1^0 and G_2^0 are defined for $r, R>1$ by

$$C_r := \{t : |\phi(t)| = r\}, \quad C_R := \{t : |\phi_1(t)| = R\}.$$

The Faber polynomials $\Phi_k(t)$ of degree k are defined by the relation

$$\frac{\psi'(w)}{\psi(w)-t} = \sum_{k=0}^\infty \frac{\Phi_k(t)}{w^{k+1}}, \quad t\in G_1^0, w\in D^-,$$

and have the following integral representations (Markushevich, 1968):

If $t\in \operatorname{int} C_r$, then

$$\Phi_k(t) = \frac{1}{2\pi i}\int_{C_r} \frac{[\phi(\xi)]^k}{\xi-t} d\xi =$$

$$= \frac{1}{2\pi i}\int_{|w|=r} \frac{\psi'(w) w^k}{\psi(w)-t} dw. \quad (3)$$

If $t\in ext\, C_r$, then

$$\Phi_k(t)=\phi^k(t)+\frac{1}{2\pi i}\int_{C_r}\frac{[\phi(\xi)]^k}{\xi-t}d\xi. \tag{4}$$

Similarly, the Faber polynomials $F_k(1/t)$ of degree k with respect to $1/t$ are defined by the relation

$$\frac{\psi_1'(w)}{\psi_1(w)-t}=\sum_{k=0}^{\infty}\frac{F_k(1/t)}{w^{k+1}},\ \ t\in G_2^{\infty}, w\in D^-,$$

and satisfy the following relations:

If $t\in \operatorname{int} C_R$, then

$$F_k(1/t)=\phi_1^k(t)-\frac{1}{2\pi i}\int_{C_R}\frac{[\phi_1(\xi)]^k}{\xi-t}d\xi. \tag{5}$$

If $t\in ext\, C_R$, then

$$F_k(1/t)=-\frac{1}{2\pi i}\int_{C_R}\frac{[\phi_1(\xi)]^k}{\xi-t}d\xi$$
$$=-\frac{1}{2\pi i}\int_{|w|=R}\frac{\psi_1'(w)w^k}{\psi_1(w)-t}dw. \tag{6}$$

The basicity problems of Faber polynomials in Smirnov classes were studied in Bilalov & Najafov (2013).

If $f\in E^1(G)$, then $f(t)$ has the following formula (Suetin, 1998)

$$f(t)=\sum_{k=0}^{\infty}a_k\Phi_k(t)+\sum_{k=1}^{\infty}b_k\,F_k(1/t), \tag{7}$$

where

$$a_k=\frac{1}{2\pi i}\int_{|w|=r_1}\frac{f(\psi(w))}{w^{k+1}}dw,\ 1<r_1<r,\ k=0,1,2,...,$$

and

$$b_k=\frac{1}{2\pi i}\int_{|w|=R_1}\frac{f(\psi_1(w))}{w^{k+1}}dw,\ 1<R_1<R,\ k=1,2,...\ .$$

If G is an annulus domain, then the series (7) becomes the Laurent series for the function $f(t)$.

Taking the first terms of series (7) we obtain the rational function

$$R_n(f,t):=\sum_{k=0}^{n}a_k\Phi_k(t)+\sum_{k=1}^{n}b_kF_k(1/t). \tag{8}$$

For large values of n and if $f\in E^{p,\alpha}(G)$, we will prove that, such a rational function $R_n(f,t)$ can approximate the function $f(t)$ arbitrarily closely.

If L_1 and L_2 are Dini-smooth, then as per Warschawski (1932), it follows that

$$c_2<|\psi'(w)|<c_3,\ \ c_4<|\psi_1'(w)|<c_5, \tag{9}$$

where c_2, c_3, c_4 and c_5 are positive constants.

Let $L_i\,(i=1,2)$ be a Dini-smooth curve, $0<\alpha\le 1$, $1<p<\infty$, we define the following functions $f_0:=f\circ\psi$ for $f\in L^{p,\alpha}(L_1)$ and $f_1:=f\circ\psi_1$ for $f\in L^{p,\alpha}(L_2)$.

From Israfilov & Tozman (2008) it follows that $f_0\in L^{p,\alpha}(\gamma_0)$ and $f_1\in L^{p,\alpha}(\gamma_0)$. Further, we obtain $f_0^+, f_1^+\in E^{p,\alpha}(D)$ and $f_0^-, f_1^-\in E^{p,\alpha}(D^-)$ such that, $f_0^-(\infty)=0$, $f_1^-(\infty)=0$ and the following relations hold a.e. on γ_0

$$f_0(t)=f_0^+(t)-f_0^-(t), \tag{10}$$
$$f_1(t)=f_1^+(t)-f_1^-(t). \tag{11}$$

For $r=1,2,3,...$ let us consider the following quantities

$$\Omega^r_{\Gamma,p,\alpha}(f,\delta):=\omega^r_{p,\alpha}(f_0^+,\delta),\quad \delta>0,$$
$$\Omega^{*r}_{\Gamma,p,\alpha}(f,\delta):=\omega^r_{p,\alpha}(f_1^+,\delta),\quad \delta>0.$$

The following lemma was proved by Israfilov & Tozman (2008).

Lemma 2.2. Let $g\in E^{p,\alpha}(D)$ with $0<\alpha\le 1$ and $1<p<\infty$. If $\sum_{k=0}^{n}a_kw^k$ is the $n-$th partial sum of the Taylor series of g at the origin, then for any $r=1,2,...$ the following estimate

$$\left\|g(w)-\sum_{k=0}^{n}a_k(g)w^k\right\|_{L^{p,\alpha}(\gamma_0)}\le c_6\,\omega^r_{p,\alpha}\left(g,\frac{1}{n+1}\right)$$

holds, where c_6 is a positive constant.

3. Main result

In this section, we present the main result.

Theorem 3.1. Let G be a finite doubly- connected domain with the Dini- smooth boundary, $L=L_1\cup L_2^-$, $L^{p,\alpha}(L)$ be a Morrey space with $0<\alpha\le 1$ and $1<p<\infty$. If f is a function in $E^{p,\alpha}(G)$, then for every $n\in \bullet$ the estimate

$$\|f-R_n(f,.)\|_{E^{p,\alpha}(G)}\le c_7\left[\Omega^r_{L,p,\alpha}\left(f,\frac{1}{n+1}\right)+\Omega^{*r}_{L,p,\alpha}\left(f,\frac{1}{n+1}\right)\right]$$

holds, where c_7 is a positive constant and $R_n(f,t)$ is the rational function defined by (8).

Proof. Let $f\in E^{p,\alpha}(G)$, then $f_0, f_1\in L^{p,\alpha}(\gamma_0)$ and putting $\phi(\xi)$ and $\phi_1(\xi)$ in place of in (10) and (11) respectively, we obtain

$$f(\xi)=f_0^+(\phi(\xi))-f_0^-(\phi(\xi)),\quad \xi\in L_1, \tag{12}$$
$$f(\xi)=f_1^+(\phi_1(\xi))-f_1^-(\phi_1(\xi)),\ \ \xi\in L_2. \tag{13}$$

We suppose that $t\in ext\, L_1$, then using the relation (4), we have

$$\sum_{k=0}^{n}a_k\Phi_k(t)=\sum_{k=0}^{n}a_k\phi^k(t)+\frac{1}{2\pi i}\int_{L_1}\frac{\sum_{k=0}^{n}a_k\phi^k(\xi)}{\xi-t}d\xi,$$

and then by the relation (12)

$$\sum_{k=0}^{n} a_k \Phi_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t} d\xi + \frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)}{\xi - t} d\xi + \frac{1}{2\pi i}\int_{L_1} \frac{f_0^-(\phi(\xi))}{\xi - t} d\xi.$$

Since $f_0^-(\phi(\xi)) \in E^{p,\alpha}(G_1^\infty)$, we get

$$\frac{1}{2\pi i}\int_{L_1} \frac{f_0^-(\phi(\xi))}{\xi - t} d\xi = -f_0^-(\phi(t)).$$

So, we reach to the following relation:

$$\sum_{k=0}^{n} a_k \Phi_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t} d\xi + \frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)}{\xi - t} d\xi - f_0^-(\phi(t)). \tag{14}$$

Now for $t \in ext\, L_2$, and using the relations (6) and (13), we obtain

$$\sum_{k=1}^{n} b_k F_k(1/t) = -\frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t} d\xi = \frac{1}{2\pi i}\int_{L_2} \frac{f_1^+(\phi_1(\xi)) - \sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t} d\xi - \frac{1}{2\pi i}\int_{L_2} \frac{f(\xi)}{\xi - t} d\xi. \tag{15}$$

If $t \in ext\, L_1$, then we have

$$\frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)}{\xi - t} d\xi = \frac{1}{2\pi i}\int_{L_2} \frac{f(\xi)}{\xi - t} d\xi. \tag{16}$$

Since $ext\, L_1 \subset ext\, L_2$, the relations (14), (15) and (16) are valid for $t \in ext\, L_1$, and give

$$\sum_{k=0}^{n} a_k \Phi_k(t) + \sum_{k=1}^{n} b_k F_k(1/t) = = \sum_{k=0}^{n} a_k \phi^k(t) - f_0^-(\phi(t)) - \frac{1}{2\pi i}\int_{L_1} \frac{f_0^+(\phi(\xi)) - \sum_{k=0}^{n} a_k \phi^k(\xi)}{\xi - t} d\xi + \frac{1}{2\pi i}\int_{L_2} \frac{f_1^+(\phi_1(\xi)) - \sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t} d\xi.$$

Taking the limit as $t \to z \in L_1$ along non- tangential path outside L_1 for almost every $z \in L_1$, we get

$$f(z) - \left(\sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=1}^{n} b_k F_k(1/z)\right) = = f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k \phi^k(z) + \frac{1}{2}\left(f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k \phi^k(z)\right) + S_{L_1}\left(f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k \phi^k(z)\right) - \frac{1}{2\pi i}\int_{L_2} \frac{f_1^+(\phi_1(\xi)) - \sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t} d\xi. \tag{17}$$

By using (17), Minkowski's inequality and the relation (2), we have

$$\left\| f - R_n(f,.) \right\|_{L^{p,\alpha}(L_1)} \le c_8 \left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p,\alpha}(\gamma_0)} + c_9 \left\| f_1^+(w) - \sum_{k=0}^{n} b_k w^k \right\|_{L^{p,\alpha}(\gamma_0)}. \tag{18}$$

From the relation (18), and using lemma2.2, we get

$$\left\| f - R_n(f,.) \right\|_{L^{p,\alpha}(L_1)} \le c_{10} \left\{ \Omega^r_{L,p,\alpha}\left(f, \frac{1}{n+1}\right) + \Omega^{*r}_{\mathrm{L},p,\alpha}\left(f, \frac{1}{n+1}\right) \right\}. \tag{19}$$

For $t \in \operatorname{int} L_2$, by the relations (4) and (13), we get

$$\sum_{k=1}^{n} b_k F_k(1/t') = \sum_{k=1}^{n} b_k \phi_1^k(t') - \frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t'} d\xi = \sum_{k=1}^{n} b_k \phi_1^k(t') - \frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi) - f_1^+(\phi_1(\xi))}{\xi - t'} d\xi - \frac{1}{2\pi i}\int_{L_2} \frac{f(\xi)}{\xi - t'} d\xi - f_1^-(\phi_1(t')). \tag{20}$$

And for $t' \in \operatorname{int} L_1$, from (3) and (12), we have

$$\sum_{k=1}^{n} a_k \Phi_k(t') = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi)}{\xi - t'} d\xi = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t'} d\xi + \frac{1}{2\pi i}\int_{L} \frac{f(\xi)}{\xi - t'} d\xi. \tag{21}$$

Since $\operatorname{int} L_2 \subset \operatorname{int} L_1$, the relations (20) and (21) are valid for $t' \in \operatorname{int} L_2$, and give

$$\sum_{k=0}^{n} a_k \Phi_k(t') + \sum_{k=1}^{n} b_k F_k(1/t') =$$

$$= \frac{1}{2\pi i} \int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t'} d\xi$$

$$- \frac{1}{2\pi i} \int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi) - f_1^+(\phi_1(\xi))}{\xi - t'} d\xi$$

$$- f_1^-(\phi_1(t')) + \sum_{k=1}^{n} b_k \phi_1^k(t').$$

Taking the limit as $t' \to z \in L_2$ along non- tangential path inside L_2 for almost every $z \in L_2$, we get

$$f(z) - \left(\sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=1}^{n} b_k F_k(1/z) \right) =$$

$$= f_1^+(\phi_1(z)) - \frac{1}{2} \left(\sum_{k=1}^{n} b_k \phi_1^k(z) - f_1^+(\phi_1(z)) \right)$$

$$- S_{L_2} \left(\sum_{k=1}^{n} b_k \phi_1^k(z) - f_1^+(\phi_1(z)) \right)$$

$$- \frac{1}{2\pi i} \int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - z} d\xi. \tag{22}$$

By using (22), Minkowski's inequality and the relation (2), we obtain

$$\|f - R_n(f,.)\|_{L^{p,\alpha}(L_2)} \le c_{11} \left\| f_1^+(w) - \sum_{k=1}^{n} b_k w^k \right\|_{L^{p,\alpha}(\gamma_0)}$$

$$+ c_{12} \left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p,\alpha}(\gamma_0)}. \tag{23}$$

From the relation (23), and using lemma 2.2, we get

$$\|f - R_n(f,.)\|_{L^{p,\alpha}(L_2)} \le c_{13} \left\{ \Omega_{L,p,\alpha}^r \left(f, \frac{1}{n+1} \right) \right.$$

$$\left. + \Omega^{*r}_{L,p,\alpha} \left(f, \frac{1}{n+1} \right) \right\}. \tag{24}$$

Since $L = L_1 \cup L_2^-$, and $f \in E^{p,\alpha}(G)$, we get

$$\|f - R_n(f,.)\|_{L^{p,\alpha}(L)} \le \|f - R_n(f,.)\|_{L^{p,\alpha}(L_1)}$$

$$+ \|f - R_n(f,.)\|_{L^{p,\alpha}(L_2)}.$$

Then taking into account the relations (19) and (24), we reach

$$\|f - R_n(f,.)\|_{E^{p,\alpha}(G)} \le c_7 \left\{ \Omega_{L,p,\alpha}^r \left(f, \frac{1}{n+1} \right) \right.$$

$$\left. + \Omega^{*r}_{L,p,\alpha} \left(f, \frac{1}{n+1} \right) \right\}.$$

Thus, the theorem is proved.

4. Acknowledgments

This work was partially supported by Department of Mathematics in Tishreen University.

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Submitted: 17/06/2016
Revised : 25/05/2017
Accepted : 29/05/2017

التقريب بواسطة الدوال الكسرية في صفوف موري سميرنوف

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خلاصة

في هذه المقالة، بحثنا المسألة المباشرة في نظرية التقريب للدوال التحليلية من صفوف موري سميرنوف المعرفة على منطقة ثنائية الترابط محاطة بمنحنيين أملسين بما فيه الكفاية.