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# Some convergence and data dependence results for various fixed point iterative methods

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## Abstract

We have compared rate of convergence among various iterative methods. Also, we have established an equivalency result between convergence of two recently introduced iterative methods and we have proven a data dependence result for one of them.

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**Keywords:** Convergence; data dependence of fixed points; equivalence of convergence; iterative methods; rate of convergence.

## 1. Introduction

Throughout this paper $\mathbb{N}$ denotes the set of all nonnegative integers. Let $B$ be a Banach space, $S$ be a non empty closed convex subset of $B$ and $T$ be a self map of $S$. Let $\{\alpha_n^i\}_{n=0}^{\infty}$, $i \in \{1,2,3\}$ be real sequences in $[0,1]$ satisfying certain control condition(s).

The following iteration methods are referred to as $CR$ (Chugh *et al.*, 2012), and $S^*$ (Karahan & Ozdemir, 2013) iteration methods, respectively:

$$\begin{cases} u_0 \in S, \\ u_{n+1} = (1-\alpha_n^1)v_n + \alpha_n^1 T v_n, \\ v_n = (1-\alpha_n^2)Tu_n + \alpha_n^2 T y_n, \\ y_n = (1-\alpha_n^3)u_n + \alpha_n^3 T u_n, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

$$\begin{cases} p_0 \in S, \\ p_{n+1} = (1-\alpha_n^1)Tp_n + \alpha_n^1 T q_n, \\ q_n = (1-\alpha_n^2)Tp_n + \alpha_n^2 T r_n, \\ r_n = (1-\alpha_n^3)p_n + \alpha_n^3 T p_n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

Convergence analysis of iterative methods has an important role in the study of iterative approximation of fixed point theory. Fixed point iteration methods may exhibit radically different behaviors for various classes of mappings. While a particular fixed point iteration method is convergent for an appropriate class of mappings, it may not be convergent for the others. Due to various reasons, it is important to determine whether an iteration method converges to fixed point of a mapping. In many cases, there can be two or more than two iteration procedures approximating to the same fixed point of a mapping, for example, (Karakaya *et al.*, 2013; Rhoades & Şoltuz, 2004; Xue, 2007). In such cases, the critical and important point is to compare rate of convergence of these iterations to find out which ones converge faster to the same fixed point of a mapping, e.g., (Berinde, 2004; Hussain *et al.*, 2011; Sahu, 2011; Xue, 2008).

Recently, several authors introduced several types of iteration methods and they have proven that their iterations converge faster than Picard (1890), Mann (1953), and Ishikawa (1974) iteration methods, e.g., (Chugh *et al.*, 2012; Karahan & Ozdemir, 2013; Karakaya *et al.*, 2013; Sahu, 2011).

In this paper, we are concerned with two recent iteration methods defined by (1.1) and (1.2). We show that iteration method (1.2) converges to fixed point of a contraction mapping satisfying

$$\|Tx - Ty\| \leq \delta\|Tx - Ty\|, \delta \in (0,1), \text{for all } x, y \in B. \tag{1.3}$$

Also, we prove that CR (1.1) and $S^*$ (1.2) iteration methods are equivalent when converging to the fixed point of a contraction mapping. In addition, we show that CR iteration method (1.1) is faster than $S^*$ iteration (1.2) in the sense of converging to the fixed point of a contraction mapping. Finally, we give a data dependence result for the fixed point of contraction mappings using iteration method (1.2).

In order to obtain our main results, we need following lemmas and definitions.

Definition 1.1 (Berinde, 2007) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits $a$ and $b$, respectively. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l \tag{1.4}$$

(i) If $l = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to $a$ than $\{b_n\}_{n=0}^{\infty}$ to $b$,

(ii) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 1.2 (Berinde, 2007) Let $T, \tilde{T}: B \to B$ be two operators. We say that $\tilde{T}$ is an approximate operator of $T$ if for any fixed $\varepsilon > 0$ and for $x \in B$ we have

$$\left\|Tx - \tilde{T}x\right\| \leq \varepsilon. \tag{1.5}$$

Lemma 1.3 (Weng, 1991) Let $\{a_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:

$$a_{n+1} \leq (1-\eta_n)a_n + \rho_n, \tag{1.6}$$

where $\eta_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=0}^{\infty} \eta_n = \infty$, and $\frac{\rho_n}{\eta_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4 (Şoltuz & Grosan, 2008) Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ the following inequality holds

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\eta_n, \tag{1.7}$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$, and $\eta_n \geq 0$, $\forall n \in \mathbb{N}$. Then the following inequality holds

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \eta_n. \tag{1.8}$$

## 2. Main results

Theorem 2.1 Let $S$ be a nonempty closed convex subset of a Banach space $B$ and $T: S \to S$ be a contraction map satisfying condition (1.3). Let $\{p_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1.2) with real sequences $\{\alpha_n^i\}_{n=0}^{\infty}$, $i \in \{1,2,3\}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n^1 = \infty$. Then $\{p_n\}_{n=0}^{\infty}$ converges to the unique fixed point of $T$, say $x_*$.

Proof. Picard-Banach theorem guarantees the existence and uniqueness of $x_*$. We will show that $p_n \to x_*$ as $n \to \infty$. From Equations (1.2) and (1.3) we have

$$\begin{aligned}\|r_n - x_*\| &= \|(1-\alpha_n^3)p_n + \alpha_n^3 Tp_n - (1-\alpha_n^3+\alpha_n^3)x_*\| \\ &\leq (1-\alpha_n^3)\|p_n - x_*\| + \alpha_n^3\|Tp_n - x_*\| \\ &\leq (1-\alpha_n^3)\|p_n - x_*\| + \alpha_n^3\delta\|p_n - x_*\| \\ &= [1-\alpha_n^3(1-\delta)]\|p_n - x_*\|, \end{aligned} \tag{2.1}$$

$$\begin{aligned}\|q_n - x_*\| &\leq (1-\alpha_n^2)\|Tp_n - x_*\| + \alpha_n^2\|Tr_n - Tx_*\| \\ &\leq (1-\alpha_n^2)\delta\|p_n - x_*\| + \alpha_n^2\delta\|r_n - x_*\| \\ &\leq \{(1-\alpha_n^2)\delta + \alpha_n^2\delta[1-\alpha_n^3(1-\delta)]\}\|p_n - x_*\|, \end{aligned} \tag{2.2}$$

and

$$\|p_{n+1}-x_*\| \leq (1-\alpha_n^1)\|Tp_n - Tx_*\| + \alpha_n^1\|Tq_n - Tx_*\|$$

$$\leq (1-\alpha_n^1)\delta\|p_n - x_*\| + \alpha_n^1\delta\|q_n - x_*\|$$

$$\leq \{(1-\alpha_n^1)\delta + \alpha_n^1\delta\{(1-\alpha_n^2)\delta + \alpha_n^2\delta[1-\alpha_n^3(1-\delta)]\}\}\|p_n - x_*\|. \quad (2.3)$$

Since $\delta \in (0,1)$ and $\alpha_n^i \in [0,1]$, for all $n \in \mathbb{N}$ and for each $i \in \{1,2,3\}$

$$1-\alpha_n^2(1-\delta) < 1 \text{ and } 1-\alpha_n^3(1-\delta) < 1. \quad (2.4)$$

By using $\delta \in (0,1)$ and Equation (2.4) in (2.3), we obtain

$$\begin{aligned}\|p_{n+1}-x_*\| &\leq [(1-\alpha_n^1)\delta\{(1-\alpha_n^2)\delta + \alpha_n^2\delta\} + \alpha_n^1\delta]\|p_n - x_*\| \\ &\leq [(1-\alpha_n^1)\delta\{1-\alpha_n^2(1-\delta)\} + \alpha_n^1\delta]\|p_n - x_*\| \\ &\leq [1-\alpha_n^1(1-\delta)]\|p_n - x_*\| \\ &\leq \cdots \leq \textstyle\prod_{n=0}^k[1-\alpha_n^1(1-\delta)]\|p_0 - x_*\|. \quad (2.5)\end{aligned}$$

It is well-known from the classical analysis that $1-x \leq e^{-x}$ for all $x \in [0,1]$. By considering this fact together with Equation (2.5), we obtain

$$\|p_{n+1}-x_*\| \leq \prod_{n=0}^{k}[1-\alpha_n^1(1-\delta)]\|p_0 - x_*\|$$

$$\leq \frac{\prod_{n=0}^k[1-\alpha_n^1(1-\delta)]\|p_0-x_*\|}{e^{(1-\delta)\sum_{n=0}^k \alpha_n^1}}. \quad (2.6)$$

Taking the limit of both sides of inequality (2.6) yields $p_n \to x_*$ as $k \to \infty$.

Theorem 2.2 Let $S$, $B$ and $T$ with a fixed point $x_*$ be as in Theorem 2.1. Let $\{u_n\}_{n=0}^\infty$, $\{p_n\}_{n=0}^\infty$ be two iterative sequences defined by (1.1) for $u_0 \in S$ and (1.2) for $p_0 \in S$ with the same real sequences $\{\alpha_n^i\}_{n=0}^\infty$, $i \in \{1,2,3\}$ in [0,1] satisfying $\sum_{n=0}^\infty \alpha_n^1 = \infty$. Then the following are equivalent:

(i) the $S^*$ iteration method (1.2) converges to the fixed point $x_*$ of $T$;

(ii) the $CR$ iteration method (1.1) converges to the fixed point $x_*$ of $T$.

Proof. We will prove (i) ⇒ (ii), that is, if $S^*$ iteration method (1.2) converges to $x_*$, then CR iteration method (1.1) does too. Now by using Equations (1.1), (1.2), and (1.3), we have

$$\|p_{n+1}-u_{n+1}\| = \|(1-\alpha_n^1)Tp_n + \alpha_n^1Tq_n - (1-\alpha_n^1)v_n - \alpha_n^1Tv_n\|$$

$$\leq (1-\alpha_n^1)\|Tp_n - v_n\| + a_n^1\|Tq_n - Tv_n\|$$

$$\leq (1-\alpha_n^1)\|p_n - v_n\| + a_n^1\delta\|q_n - v_n\| + (1-\alpha_n^1)\|p_n - Tp_n\|$$

$$= (1-\alpha_n^1)\|(1-\alpha_n^2+\alpha_n^2)p_n-(1-\alpha_n^2)Tu_n-\alpha_n^2Ty_n\|$$

$$+a_n^1\delta\|q_n-v_n\|+(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\|p_n-Tu_n\|+(1-\alpha_n^1)\alpha_n^2\|p_n-Ty_n\|$$

$$+a_n^1\delta\|q_n-v_n\|+(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\{\|p_n-Tp_n\|+\|Tp_n-Tu_n\|\}$$

$$+(1-\alpha_n^1)\alpha_n^2\{\|p_n-Tp_n\|+\|Tp_n-Ty_n\|\}$$

$$+a_n^1\delta\|q_n-v_n\|+(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\delta\|p_n-u_n\|+(1-\alpha_n^1)(1-\alpha_n^2)\|p_n-Tp_n\|$$

$$+(1-\alpha_n^1)a_n^2\|p_n-Tp_n\|+(1-\alpha_n^1)a_n^2\delta\|p_n-y_n\|$$

$$+\alpha_n^1\delta\|q_n-v_n\|+(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$= (1-\alpha_n^1)(1-\alpha_n^2)\delta\|p_n-u_n\|$$

$$+(1-\alpha_n^1)a_n^2\delta\|(1-a_n^3+a_n^3)p_n-(1-a_n^3)u_n-a_n^3Tu_n\|$$

$$+a_n^1\delta\|q_n-v_n\|+2(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\delta\|p_n-u_n\|$$

$$+(1-\alpha_n^1)a_n^2\delta(1-\alpha_n^3)\|p_n-u_n\|+(1-\alpha_n^1)a_n^2\delta\alpha_n^3\|p_n-Tu_n\|$$

$$+a_n^1\delta\|q_n-v_n\|+2(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\delta\|p_n-u_n\|+(1-\alpha_n^1)a_n^2\delta(1-\alpha_n^3)\|p_n-u_n\|$$

$$+(1-\alpha_n^1)a_n^2\delta\alpha_n^3\|p_n-Tp_n\|+(1-\alpha_n^1)a_n^2\delta\alpha_n^3\delta\|p_n-u_n\|$$

$$+a_n^1\delta\|q_n-v_n\|+2(1-\alpha_n^1)\|p_n-Tp_n\|$$

$$= \{(1-\alpha_n^1)(1-\alpha_n^2)\delta+(1-\alpha_n^1)a_n^2\delta[1-\alpha_n^3(1-\delta)]\}\|p_n-u_n\|$$

$$+a_n^1\delta\|q_n-v_n\|+(1-\alpha_n^1)(2+\alpha_n^2\delta\alpha_n^3)\|p_n-Tp_n\|, \quad (2.7)$$

$$\|q_n-v_n\| = \|(1-\alpha_n^2)Tp_n+\alpha_n^2Tr_n-(1-\alpha_n^2)Tu_n-\alpha_n^2Ty_n\|$$

$$\leq (1-\alpha_n^2)\|p_n - u_n\| + \alpha_n^2\delta\|r_n - y_n\|$$

$$= (1-\alpha_n^2)\|p_n - u_n\|$$

$$+\alpha_n^2\delta\|(1-a_n^3)p_n + a_n^3 Tp_n - (1-a_n^3)u_n - a_n^3 Tu_n\|$$

$$\leq (1-\alpha_n^2)\|p_n - u_n\| + \alpha_n^2\delta[1-a_n^3(1-\delta)]\|p_n - u_n\|$$

$$\leq (1-\alpha_n^2)\|p_n - u_n\| + \alpha_n^2\delta\|p_n - u_n\|$$

$$= [1-a_n^2(1-\delta)]\|p_n - u_n\|. \quad (2.8)$$

Substituting Equation (2.8) in (2.7)

$$\|p_{n+1} - u_{n+1}\| \leq \{(1-\alpha_n^1)(1-\alpha_n^2)\delta + (1-\alpha_n^1)a_n^2\delta[1-\alpha_n^3(1-\delta)]$$

$$+\alpha_n^1\,\delta[1-\alpha_n^2(1-\delta)]\}\|p_n - u_n\|$$

$$+(1-\alpha_n^1)(2+\alpha_n^2\delta\alpha_n^3)\|p_n - Tp_n\|. \quad (2.9)$$

Since $\delta \in [0,1)$, $\alpha_n^i \in [0,1]$ for all $n \in \mathbb{N}$ and for each $i \in \{1,2,3\}$,

$$1-\alpha_n^2(1-\delta) < 1 \text{ and } 1-\alpha_n^3(1-\delta) < 1. \quad (2.10)$$

By applying inequality (2.10) to (2.9), we obtain

$$\|p_{n+1} - u_{n+1}\| \leq [1-\alpha_n^1(1-\delta)]\|p_n - u_n\|$$

$$+(1-\alpha_n^1)(2+\alpha_n^2\delta\alpha_n^3)\|p_n - Tp_n\|. \quad (2.11)$$

Using the fact $x_* = Tx_*$ and triangle inequality for norms, we derive

$$\|p_n - Tp_n\| = \|p_n - x_* + Tx_* - Tp_n\|$$

$$\leq \|p_n - x_*\| + \|Tx_* - Tp_n\|$$

$$\leq (1+\delta)\|p_n - x_*\|. \quad (2.12)$$

Substituting inequality (2.12) in (2.11)

$$\|p_{n+1} - u_{n+1}\| \leq [1-\alpha_n^1(1-\delta)]\|p_n - u_n\|$$

$$+(1-\alpha_n^1)(2+\alpha_n^2\delta\alpha_n^3)(1+\delta)\|p_n - x_*\|. \quad (2.13)$$

Denote that

$$a_n = \|p_n - u_n\|,$$

$$\eta_n = \alpha_n^1(1-\delta) \in (0,1),$$

$$\rho_n = (1-\alpha_n^1)(2+\alpha_n^2\delta\alpha_n^3)(1+\delta)\|p_n - x_*\|. \tag{2.14}$$

Thus, an application of Lemma 1.3 to inequality (2.13) yields $a_n = \|p_n - u_n\| \to 0$ as $n \to \infty$. Also, since $\|u_n - x_*\| \leq \|p_n - u_n\| + \|p_n - x_*\|$, we have $\|u_n - x_*\| \to 0$ as $n \to \infty$.

Next, we will prove (ii) ⇒ (i). Assume that $\|u_n - x_*\| \to 0$ as $n \to \infty$. It follows from Equations (1.1), (1.2), and (1.3) that

$$\|p_{n+1} - u_{n+1}\| = \|(1-\alpha_n^1)v_n + \alpha_n^1 Tv_n - (1-\alpha_n^1)Tp_n - \alpha_n^1 Tq_n\|$$

$$\leq (1-\alpha_n^1)\|v_n - Tp_n\| + a_n^1\|Tv_n - Tq_n\|$$

$$\leq (1-\alpha_n^1)\|(1-\alpha_n^2)Tu_n + \alpha_n^2 Ty_n - Tp_n\| + a_n^1\delta\|q_n - v_n\|$$

$$= (1-\alpha_n^1)\|(1-\alpha_n^2)Tu_n + \alpha_n^2 Ty_n - (1-\alpha_n^2+\alpha_n^2)Tp_n\|$$

$$+a_n^1\delta\|(1-\alpha_n^2)Tu_n + \alpha_n^2 Ty_n - (1-\alpha_n^2)Tp_n - \alpha_n^2 Tr_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\|Tu_n - Tp_n\| + (1-\alpha_n^1)\alpha_n^2\|Ty_n - Tp_n\|$$

$$+a_n^1\delta(1-\alpha_n^2)\|Tu_n - Tp_n\| + \alpha_n^1\delta\alpha_n^2\|Ty_n - Tr_n\|$$

$$\leq (1-\alpha_n^1)(1-\alpha_n^2)\delta\|u_n - p_n\| + (1-\alpha_n^1)\alpha_n^2\delta\|y_n - p_n\|$$

$$+a_n^1\delta(1-\alpha_n^2)\delta\|u_n - p_n\| + a_n^1\delta\alpha_n^2\delta\|y_n - r_n\|$$

$$= \{(1-\alpha_n^1)(1-\alpha_n^2)\delta + a_n^1\delta(1-\alpha_n^2)\delta\}\|u_n - p_n\|$$

$$+(1-\alpha_n^1)\alpha_n^2\delta\|(1-a_n^3)u_n + a_n^3 Tu_n - (1-a_n^3+a_n^3)p_n\|$$

$$+a_n^1\delta\alpha_n^2\delta\|(1-a_n^3)u_n + a_n^3 Tu_n - (1-a_n^3)p_n - a_n^3 Tp_n\|$$

$$\leq \{(1-\alpha_n^1)(1-\alpha_n^2)\delta + a_n^1\delta(1-\alpha_n^2)\delta\}\|u_n - p_n\|$$

$$+(1-\alpha_n^1)\delta\alpha_n^2\delta(1-a_n^3)\|u_n - p_n\| + (1-\alpha_n^1)\alpha_n^2\delta a_n^3\|Tu_n - p_n\|$$

$$+a_n^1\delta\alpha_n^2\delta(1-a_n^3)\|u_n - p_n\| + a_n^1\delta\alpha_n^2\delta a_n^3\|Tu_n - Tp_n\|$$

$$\leq \{(1-\alpha_n^1)(1-\alpha_n^2)\delta + a_n^1\delta(1-\alpha_n^2)\delta$$

$$+(1-\alpha_n^1)\delta\alpha_n^2\delta(1-a_n^3) + (1-\alpha_n^1)\alpha_n^2\delta a_n^3$$

$$+a_n^1\delta\alpha_n^2\delta(1-a_n^3) + a_n^1\delta\alpha_n^2\delta a_n^3\}\|u_n - p_n\|$$

$$+(1-\alpha_n^1)\alpha_n^2\delta a_n^3\|u_n - Tu_n\|$$

$$= \{(1-\alpha_n^1)(1-\alpha_n^2)\delta + a_n^1\delta(1-\alpha_n^2)\delta$$

$$+(1-\alpha_n^1)\alpha_n^2\delta(1-a_n^3) + (1-\alpha_n^1)\alpha_n^2\delta a_n^3$$

$$+a_n^1\delta\alpha_n^2\delta[1-a_n^3(1-\delta)]\}\|u_n - p_n\|$$

$$+(1-\alpha_n^1)\alpha_n^2\delta a_n^3\|u_n - Tu_n\|, \quad (2.15)$$

or,

$$\|u_{n+1} - p_{n+1}\| \leq \{[1-a_n^1(1-\delta)](1-\alpha_n^2)\delta + (1-\alpha_n^1)\alpha_n^2\delta$$

$$+a_n^1\delta\alpha_n^2\delta[1-a_n^1(1-\delta)]\}\|u_n - p_n\|$$

$$+(1-\alpha_n^1)\alpha_n^2\delta a_n^3\|u_n - Tu_n\|. \quad (2.16)$$

Since $\delta \in [0,1)$, $a_n^i \in [0,1]$ for all $n \in \mathbb{N}$ and for each $i \in \{1,2,3\}$,

$$1-\alpha_n^2(1-\delta) < 1 \text{ and } 1-\alpha_n^3(1-\delta) < 1. \quad (2.17)$$

By use of inequality (2.17) in (2.16), we get

$$\|u_{n+1} - p_{n+1}\| \leq [1-\alpha_n^1(1-\delta)]\|u_n - p_n\| + (1-\alpha_n^1)\alpha_n^2\delta\alpha_n^3\|u_n - Tu_n\|. \quad (2.18)$$

Using the fact $x_* = Tx_*$ and triangle inequality for norms, we derive

$$\|u_n - Tu_n\| \leq (1+\delta)\|u_n - x_*\|. \quad (2.19)$$

Hence, inequality (2.18) becomes

$$\|u_{n+1} - p_{n+1}\| \leq [1-\alpha_n^1(1-\delta)]\|u_n - p_n\| + (1-\alpha_n^1)\alpha_n^2\delta\alpha_n^3(1+\delta)\|u_n - x_*\|. \quad (2.20)$$

Define

$$a_n = \|u_n - p_n\|,$$

$$\eta_n = \alpha_n^1(1-\delta) \in (0,1),$$

$$\rho_n = (1-\alpha_n^1)\alpha_n^2\delta\alpha_n^3(1+\delta)\|u_n - x_*\|. \quad (2.21)$$

Thus, an application of Lemma 1.3 to inequality (2.20) yields $a_n = \|u_n - p_n\| \to 0$ as $n \to \infty$.

Also, since $\|p_n - x_*\| \leq \|p_n - u_n\| + \|u_n - x_*\|$, we have $\|p_n - x_*\| \to 0$ as $n \to \infty$.

Theorem 2.2 Let $S$, $B$ and $T$ with a fixed point $x_*$ be as in Theorem 2.1. Let $\{\alpha_n^i\}_{n=0}^{\infty}$, $i \in \{1,2,3\}$ in $[0,1]$ satisfying (i) $0 < a_i = inf_{n\in\mathbb{N}}\alpha_n^i$. For given $u_0 = p_0 \in S$, consider iterative sequences $\{u_n\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ defined by (1.1) and (1.2), respectively. Then $\{u_n\}_{n=0}^{\infty}$ converges to $x_*$ faster than $\{p_n\}_{n=0}^{\infty}$ does.

Proof. The following equality was obtained in (Theorem 1 of Karahan & Ozdemir (2013))

$$b_n = \|p_0 - x_*\|\delta^{n+1}\left[1 - a_1\left(1 - \delta\left(1 - a_2 a_3(1-\delta)\right)\right)\right]^{n+1}. \qquad (2.22)$$

Using now iteration method (1.2) and condition (1.3) we have

$$\begin{aligned}
\|u_{n+1} - x_*\| &= \|(1-\alpha_n^1)v_n + \alpha_n^1 T v_n - x_*\| \\
&\leq (1-\alpha_n^1)\|v_n - x_*\| + a_n^1\|Tv_n - x_*\| \\
&\leq [(1-\alpha_n^1) + a_n^1]\|v_n - x_*\| \\
&\leq [(1-\alpha_n^1) + a_n^1]\|(1-\alpha_n^2)Tu_n + \alpha_n^2 T y_n - x_*\| \\
&\leq [(1-\alpha_n^1) + a_n^1](1-\alpha_n^2)\|Tu_n - x_*\| \\
&\quad +[(1-\alpha_n^1) + a_n^1]\alpha_n^2\|Ty_n - x_*\| \\
&\leq [(1-\alpha_n^1) + a_n^1](1-\alpha_n^2)\delta\|u_n - x_*\| \\
&\quad +[(1-\alpha_n^1) + a_n^1]\alpha_n^2\delta\|y_n - x_*\| \\
&\leq [(1-\alpha_n^1) + a_n^1](1-\alpha_n^2)\delta\|u_n - x_*\| \\
&\quad +[(1-\alpha_n^1) + a_n^1]\alpha_n^2\delta(1-a_n^3)\|u_n - x_*\| \\
&\quad +[(1-\alpha_n^1) + a_n^1]\alpha_n^2\delta a_n^3\delta\|u_n - x_*\| \\
&= [(1-\alpha_n^1) + a_n^1]\{(1-\alpha_n^2)\delta \\
&\quad + \alpha_n^2\delta(1-a_n^3)\alpha_n^2\delta a_n^3\delta\}\|u_n - x_*\|
\end{aligned}$$

$$= [1 - a_n^1(1-\delta)][1 - \alpha_n^2 a_n^3(1-\delta)]\delta\|u_n - x_*\|$$

$$\leq \cdots$$

$$\leq \prod_{k=0}^{n}[1 - a_k^1(1-\delta)]\,[1 - \alpha_k^2 a_k^3(1-\delta)]\delta\|u_0 - x_*\|. \qquad (2.23)$$

From assumption (i), we obtain

$$\|u_{n+1} - x_*\| \leq \|u_0 - x_*\|\delta^{n+1}[1 - a_1(1-\delta)]^{n+1}[1 - a_2 a_3(1-\delta)]^{n+1}. \quad (2.24)$$

Let

$$a_n = \|u_0 - x_*\|\delta^{n+1}[1 - a_1(1-\delta)]^{n+1}[1 - a_2 a_3(1-\delta)]^{n+1}. \quad (2.25)$$

Define

$$\theta_n = \frac{a_n}{b_n} = \frac{\|u_0 - x_*\|\delta^{n+1}[1-a_1(1-\delta)]^{n+1}[1-a_2a_3(1-\delta)]^{n+1}}{\|p_0 - x_*\|\delta^{n+1}\left[1-a_1\left(1-\delta\left(1-a_2a_3(1-\delta)\right)\right)\right]^{n+1}}$$

$$= \frac{[1-a_1(1-\delta)]^{n+1}[1-a_2a_3(1-\delta)]^{n+1}}{\left[1-a_1\left(1-\delta\left(1-a_2a_3(1-\delta)\right)\right)\right]^{n+1}}. \qquad (2.26)$$

Since $\delta \in (0,1)$ and $a_i \in (0,1)$ for each $i \in \{1,2,3\}$

$$a_1 < 1$$

$$\Rightarrow a_1a_2a_3(1-\delta) < a_2a_3(1-\delta)$$

$$\Rightarrow a_2a_3(-1+\delta) + a_1a_2a_3(1-\delta) < 0$$

$$\Rightarrow -a_2a_3 + a_2a_3\delta + a_1a_2a_3 - a_1a_2a_3\delta < 0$$

$$\Rightarrow \begin{cases} 1 - a_1 + a_1\delta - a_2a_3 + a_2a_3\delta + a_1a_2a_3 - 2a_1a_2a_3\delta + a_1a_2a_3\delta^2 \\ \qquad < 1 - a_1 + a_1\delta - a_1a_2a_3\delta + a_1a_2a_3\delta^2 \end{cases}$$

$$\Rightarrow \begin{cases} 1 - a_1 + a_1\delta - a_2a_3(1-\delta) + a_1a_2a_3(1-\delta)^2 \\ \qquad < 1 - a_1 + a_1\delta - a_1a_2a_3\delta(1-\delta) \end{cases}$$

$$\Rightarrow \begin{cases} 1 - a_1(1-\delta) - [1 - a_1(1-\delta)]a_2a_3(1-\delta) \\ \qquad < 1 - a_1 + a_1\delta(1 - a_2a_3(1-\delta)) \end{cases}$$

$$\Rightarrow [1 - a_1(1-\delta)][1 - a_2a_3(1-\delta)] < 1 - a_1(1 - \delta(1 - a_2a_3(1-\delta)))$$

$$\Rightarrow \frac{[1-a_1(1-\delta)][1-a_2a_3(1-\delta)]}{1-a_1(1-\delta(1-a_2a_3(1-\delta)))} < 1. \tag{2.27}$$

Thus $\lim_{n\to\infty} \theta_n = 0$ which implies that $\{u_n\}_{n=0}^{\infty}$ is faster than $\{p_n\}_{n=0}^{\infty}$.

Example 2.4 Let $B = \mathbb{R}$ and $S = [0,\infty)$. Let $T: S \to S$ be the map defined by $Tx = \frac{1}{2} + \frac{1}{3}\arctan x$. It is clear that $T$ satisfies condition (1.3) with $\delta = \frac{1}{3}$ and $x_* = 0.7046028071450$. Take $\alpha_n^1 = \alpha_n^2 = \alpha_n^3 = \frac{n+1}{n+2}$, $\forall n \in \mathbb{N}$ with initial value $x_0 = 30$. The following figure and tables show that *CR* iterative scheme (1.1) converge to $x_*$ faster than all Picard (Picard, 1890), Mann (Mann, 1953), Ishikawa (Ishikawa, 1974), Noor (Noor, 2000), S (Agarwal *et al.*, 2007), SP (Phuengrattana & Suantai, 2011), Normal-S (Sahu, 2011), $S^*$ (1.2) iterative schemes up to the accuracy of thirteen decimal places.

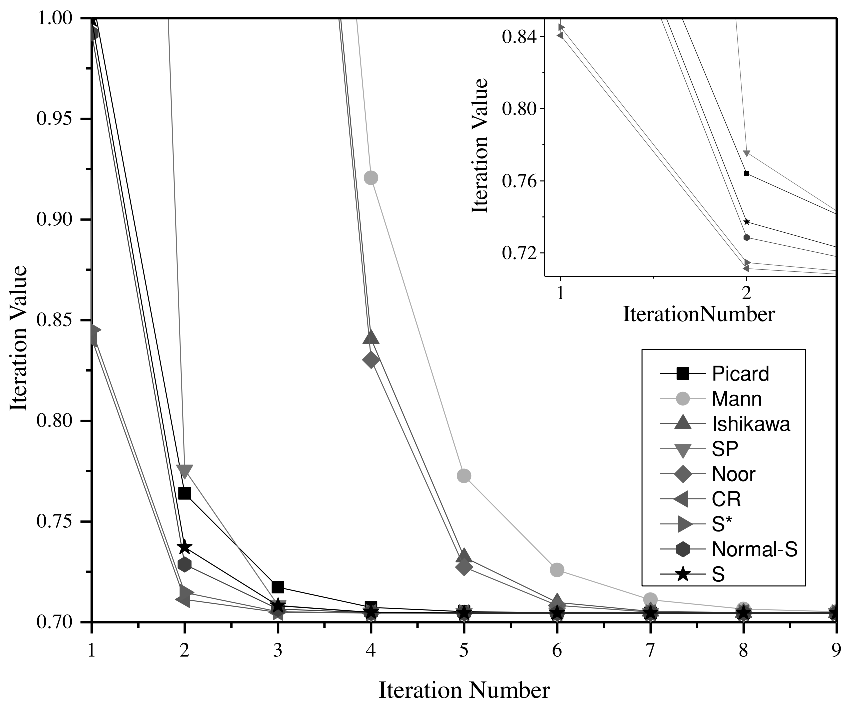


**Fig. 1.** Comparison of the rate of convergence among various iterations

**Table 1.** Comparison rate of convergence among various iterations

| Iter. No. | ***CR*** | ***S**** | ***S*** |
|---|---|---|---|
| 1 | 0.8851626784486 | 0.8877526754688 | 1.007311782932 |
| 2 | 0.7163190923162 | 0.7206984735219 | 0.7435069717741 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 9 | 0.7046028071457 | 0.7046028071614 | 0.7046028105023 |
| 10 | 0.7046028071450 | 0.7046028071456 | 0.7046028074125 |
| 11 | ⋮ | 0.7046028071450 | 0.7046028071658 |
| ⋮ | | ⋮ | ⋮ |
| 14 | | | 0.7046028071451 |
| 15 | | | 0.7046028071450 |
| ⋮ | | | ⋮ |

**Table 2.** Comparison rate of convergence among various iterations

| Iter. No. | ***SP*** | **Normal-*S*** | **Noor** |
|---|---|---|---|
| 1 | 4.618799966137 | 1.002131788892 | 15.50106231755 |
| 2 | 0.9747965600970 | 0.7333057242463 | 5.811553726253 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | 0.7046028071451 | 0.7046028071468 | 0.7046033352781 |
| 12 | 0.7046028071450 | 0.7046028071451 | 0.7046028624322 |
| 13 | ⋮ | 0.7046028071450 | 0.7046028125691 |
| ⋮ | | ⋮ | ⋮ |
| 17 | | | 0.7046028071452 |
| 18 | | | 0.7046028071450 |
| ⋮ | | | ⋮ |

**Table 3.** Comparison rate of convergence among various iterations

| Iter. No. | Picard | Mann | Ishikawa |
|---|---|---|---|
| 1 | 1.012491776972 | 15.50624588849 | 15.50106589445 |
| 2 | 0.7638684010641 | 5.836836488758 | 5.811703880012 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | 0.7046028843365 | 0.7047125607878 | 0.7046045362514 |
| 12 | 0.7046028243392 | 0.7046338153151 | 0.7046030406020 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 19 | 0.7046028071455 | 0.7046028103103 | 0.7046028071451 |
| 20 | 0.7046028071451 | 0.7046028079673 | 0.7046028071450 |
| 21 | 0.7046028071450 | 0.7046028073572 | ⋮ |
| ⋮ | ⋮ | ⋮ | |
| 27 | | 0.7046028071451 | |
| 28 | | 0.7046028071450 | |
| ⋮ | | ⋮ | |

We are now able to establish the following data dependence result.

Theorem 2.5 Let $\tilde{T}$ be an approximate operator of $T$ satisfying condition (1.3). Let $\{p_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1.2) for $T$ and define an iterative sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} \tilde{p}_0 \in S, \\ \tilde{p}_{n+1} = (1-\alpha_n^1)\tilde{T}\tilde{p}_n + \alpha_n^1\tilde{T}\tilde{q}_n, \\ \tilde{q}_n = (1-\alpha_n^2)\tilde{T}\tilde{p}_n + \alpha_n^2\tilde{T}\tilde{r}_n, \\ \tilde{r}_n = (1-\alpha_n^3)\tilde{p}_n + \alpha_n^3\tilde{T}\tilde{p}_n, n \in \mathbb{N}, \end{cases} \tag{2.28}$$

where $\{\alpha_n^i\}_{n=0}^{\infty}$, $i \in \{1,2,3\}$ are real sequences in [0,1] satisfying (i) $\frac{1}{2} \leq \alpha_n^1$ for all $n \in \mathbb{N}$. If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ such that $\tilde{p}_n \to \tilde{p}$ as $n \to \infty$, then we have

$$\|p-\tilde{p}\| \leq \frac{5\varepsilon}{1-\delta}, \tag{2.29}$$

where $\varepsilon > 0$ is a fixed number.

Proof. It follows from Equations (1.2), (1.3), and (2.28) that

$$\|r_n - \tilde{r}_n\| = \left\|(1-\alpha_n^3)p_n + \alpha_n^3 Tp_n - (1-\alpha_n^3)\tilde{p}_n - \alpha_n^3\tilde{T}\tilde{p}_n\right\|$$

$$\leq (1-\alpha_n^3)\|p_n - \tilde{p}_n\| + \alpha_n^3\left\|Tp_n - \tilde{T}\tilde{p}_n\right\|$$

$$\leq (1-\alpha_n^3)\|p_n - \tilde{p}_n\| + \alpha_n^3\left\{\|Tp_n - T\tilde{p}_n\| + \left\|T\tilde{p}_n - \tilde{T}\tilde{p}_n\right\|\right\}$$

$$\leq [1-\alpha_n^3(1-\delta)]\|p_n - \tilde{p}_n\| + \alpha_n^3\varepsilon, \tag{2.30}$$

$$\|q_n - \tilde{q}_n\| = \left\|(1-\alpha_n^2)Tp_n + \alpha_n^2 Tr_n - (1-\alpha_n^2)\tilde{T}\tilde{p}_n - \alpha_n^2\tilde{T}\tilde{r}_n\right\|$$

$$\leq (1-\alpha_n^2)\left\|Tp_n - \tilde{T}\tilde{p}_n\right\| + \alpha_n^2\left\|Tr_n - \tilde{T}\tilde{r}_n\right\|$$

$$\leq (1-\alpha_n^2)\left\{\|Tp_n - T\tilde{p}_n\| + \left\|T\tilde{p}_n - \tilde{T}\tilde{p}_n\right\|\right\}$$

$$+\alpha_n^2\left\{\|Tr_n - T\tilde{r}_n\| + \left\|T\tilde{r}_n - \tilde{T}\tilde{r}_n\right\|\right\}$$

$$\leq (1-\alpha_n^2)\delta\|p_n - \tilde{p}_n\| + \alpha_n^2\delta\|r_n - \tilde{r}_n\|$$

$$+ (1-\alpha_n^2)\varepsilon + \alpha_n^2\varepsilon, \tag{2.31}$$

$$\|p_{n+1} - \tilde{p}_{n+1}\| = \left\|(1-\alpha_n^1)Tp_n + \alpha_n^1 Tq_n - (1-\alpha_n^1)\tilde{T}\tilde{p}_n - \alpha_n^1\tilde{T}\tilde{q}_n\right\|$$

$$\leq (1-\alpha_n^1)\left\|Tp_n - \tilde{T}\tilde{p}_n\right\| + \alpha_n^1\left\|Tq_n - \tilde{T}\tilde{q}_n\right\|$$

$$\leq (1-\alpha_n^1)\left\{\|Tp_n - T\tilde{p}_n\| + \left\|T\tilde{p}_n - \tilde{T}\tilde{p}_n\right\|\right\}$$

$$+\alpha_n^1\left\{\|Tq_n - T\tilde{q}_n\| + \left\|T\tilde{q}_n - \tilde{T}\tilde{q}_n\right\|\right\}$$

$$\leq (1-\alpha_n^1)\{\delta\|p_n - \tilde{p}_n\| + \varepsilon\} + \alpha_n^1\{\delta\|q_n - \tilde{q}_n\| + \varepsilon\}$$

$$= (1-\alpha_n^1)\delta\|p_n - \tilde{p}_n\| + \alpha_n^1\delta\|q_n - \tilde{q}_n\|$$

$$+ (1-\alpha_n^1)\varepsilon + \alpha_n^1\varepsilon. \tag{2.32}$$

Combining inequalities (2.30), (2.31), and (2.32)

$$\|p_{n+1} - \tilde{p}_{n+1}\| \leq \left\{(1-\alpha_n^1)\delta + \alpha_n^1\delta\{(1-\alpha_n^2)\delta + \alpha_n^2\delta[1-\alpha_n^3(1-\delta)]\}\right\}\|p_n - \tilde{p}_n\|$$

$$+\alpha_n^1\delta\alpha_n^2\delta\alpha_n^3\varepsilon + \alpha_n^1\delta(1-\alpha_n^2)\varepsilon + \alpha_n^1\delta\alpha_n^2\varepsilon + (1-\alpha_n^1)\varepsilon + \alpha_n^1\varepsilon. \tag{2.33}$$

Since $\delta \in (0,1)$ and $\alpha_n^i \in [0,1]$ for each $i \in \{1,2,3\}$ and for all $n \in \mathbb{N}$,

$$1-\alpha_n^3(1-\delta)<1, 1-\alpha_n^2(1-\delta)<1, \alpha_n^2\alpha_n^3\delta^2<1, (1-\alpha_n^2)\delta<1, \text{ and } \alpha_n^2\delta<1, \quad (2.34)$$

and by assumption (i) we have

$$1-\alpha_n^1 \leq \alpha_n^1. \quad (2.35)$$

Thus using inequalities (2.34) and (2.35) in (2.33) yields

$$\|p_{n+1}-\tilde{p}_{n+1}\| \leq [1-\alpha_n^1(1-\delta)]\|p_n-\tilde{p}_n\| + \alpha_n^1(1-\delta)\frac{5\varepsilon}{1-\delta}. \quad (2.36)$$

Let us denote

$$a_n = \|p_n-\tilde{p}_n\|,\ \mu_n = \alpha_n^1(1-\delta) \in (0,1), \eta_n = \frac{5\varepsilon}{1-\delta} \geq 0. \quad (2.37)$$

It follows from Lemma 1.4 that

$$0 \leq \limsup_{n\to\infty}\|p_n-\tilde{p}_n\| \leq \limsup_{n\to\infty}\frac{5\varepsilon}{1-\delta}. \quad (2.38)$$

From Theorem 2.1 we know that $\lim_{n\to\infty} p_n = p$. Thus, using this fact together with the assumption $\lim_{n\to\infty}\tilde{p}_n = \tilde{p}$ we obtain

$$\|p_n-\tilde{p}_n\| \leq \frac{5\varepsilon}{1-\delta}. \quad (2.39)$$

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# بعض النتائج حول التقارب و تبعية المعطيات لطريقة النقطة الصامدة التكريرية المتنوعة

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## خلاصة

نقوم في هذا البحث بمقارنة الطرق التكريرية المتنوعة. كما نقوم بإيجاد تكافؤ بين طريقتي تكرير حديثتين، و نثبت إحدى النتائج حول تبعية المعطيات لواحدة منهما.