*Kuwait J. Sci.* **42 (3)** pp. 86-104, 2015

# On some generalized statistically convergent sequence spaces

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## ABSTRACT

In this paper we construct some generalized new difference statistically convergent sequence spaces defined by a Musielak-Orlicz function over $n-$normed spaces. We also study several properties relevant to topological structures and inclusion relations between these spaces.

**Keywords:** Generalized difference sequence space; Musielak-Orlicz function; $n-$normed space; paranorm; statistical convergence.

## INTRODUCTION AND PRELIMINARIES

The notion of difference sequence spaces was introduced by Kızmaz(1981) who studied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et & Çolak(1995) in which they introduced the spaces $l_\infty(\Delta^v), c(\Delta^v)$ and $c_0(\Delta^v)$. Later the concept has been studied by Bektaş *et al.* (2004); Et & Esi (2000). Another type of generalization of the difference sequence spaces is due to Tripathy & Esi (2006) who studied the spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$. Recently, Esi *et al.* (2007); Tripathy *et al.* (2005) have introduced a new type of generalized difference operators and unified those as follows.

Let $m, v$ be non-negative integers, then for $Z$ a given sequence space, we have

$$Z(\Delta_m^v) = \{x = (x_k) \in w : (\Delta_m^v x_k) \in Z\}$$

For $Z = c, c_0$ and $\ell_\infty$ where $\Delta_m^v x = (\Delta_m^v x_k) = (\Delta_m^{v-1} x_k - \Delta_m^{v-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^v x_k = \sum_{\nu=0}^{v} (-1)^\nu \binom{v}{\nu} x_{k+m\nu.}$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^v), c(\Delta^v)$ and $c_0(\Delta^v)$ studied by Et & Çolak (1995). Taking $m = v = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz (1981). For more details about sequence spaces see (Mursaleen, 1996; Raj *et al.,* 2011; Raj & Sharma, 2013a; Tripathy *et al.,* 2008; Tripathy *et al*. 2012b) and references therein.

Başar & Altay (2003) introduced the generalized difference matrix $B(r,s) = (b_{vk}(r,s))$ which is a generalization of $\Delta^1_{(1)}$ $-$difference operator as follows:

$$b_{vk}(r,s) = \begin{cases} r & (k = v), \\ s & (k = v-1), \\ 0 & (0 \leq k < v-1) \text{ or } (k > v) \end{cases}$$

for all $k, v \in \mathbb{N}$, $r, s \in \mathbb{R} \setminus \{0\}$. Recently, Başarir & Kayıkçı (2009) have defined the generalized difference matrix $B^v$ of order $v$ which reduced the difference operator $\Delta^1_{(1)}$ in case $r = 1, s = -1$ the binomial representation of this operator is

$$B^v x_k = \sum_{\nu=0}^{v} \binom{v}{\nu} r^{v-\nu} s^{\nu} x_{k-\nu},$$

where $r, s \in \mathbb{R} \setminus \{0\}$ and $v \in \mathbb{N}$. Thus for any sequence space $Z$, the space $Z(B^v)$ is more general and more comprehensive than the corresponding space $Z(\Delta^v_{(1)})$. For details one may refer to (Başarir & Nuray, 1991; Kayıkçı & Başarir, 2010) and references therein.

The idea of statistical convergence was given by Zygmund (2011). The concept was further studied by Fast (1951) ; Schoenberg (1959) independently for the real sequences. Later on, it was further investigated from sequence point of view and linked with the summability theory by Fridy (1985). The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows. The natural density of a subset $E$ of $\mathbb{N}$ is denoted by

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \left| \{ k \in E : k \leq n \} \right|,$$

where the vertical bar denotes the cardinality of the enclosed set.

An Orlicz function $\mathcal{M}$ is a function, which is continuous, non-decreasing and convex $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$ for $x > 0$ and $\mathcal{M}(x) \to \infty$ as $x \to \infty$.

Lindenstrauss & Tzafriri (1971) used the idea of Orlicz function to define the following sequence space:

$$\ell_{\mathcal{M}} = \left\{ x \in w : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called as an Orlicz sequence space. The space $\ell_{\mathcal{M}}$ is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown by Lindenstrauss & Tzafriri (1971) that every Orlicz sequence space $\ell_{\mathcal{M}}$ contains a subspace isomorphic to $\ell_p(p \geq 1)$. In the later stage different Orlicz sequence spaces were introduced and studied see (Parashar & Choudhary, 1994; Esi & Et, 2000; Tripathy & Mahanta 2003; Mursaleen, 1996; Sen & Roy, 2013; Esi, 2013) and many others. The $\Delta_2$ –condition is equivalent to $\mathcal{M}(Lx) \leq kL\mathcal{M}(x)$ for all values of $x \geq 0$ and for $L > 1$.

A sequence $\mathsf{M} = (\mathcal{M}_k)$ of Orlicz functions is called a Musielak-Orlicz function (Maligranda, 1989; Musielak, 1983). A sequence $\mathsf{N} = (N_k)$ defined by

$$N_k(v) = \sup\left\{|v|u - \mathcal{M}_k(u) : u \geq 0\right\}, k = 1,2,....$$

is called the complementary function of a Musielak-Orlicz function $\mathsf{M}$. For a given Musielak-Orlicz function $\mathsf{M}$, the Musielak-Orlicz sequence space $t_{\mathsf{M}}$ and its subspace $h_{\mathsf{M}}$ are defined as follows:

$$t_{\mathsf{M}} = \left\{x \in w : I_{\mathsf{M}}(cx) < \infty \text{ for some } c > 0\right\},$$

$$h_{\mathsf{M}} = \left\{x \in w : I_{\mathsf{M}}(cx) < \infty \text{ for all } c > 0\right\},$$

where $I_{\mathsf{M}}$ is a convex modular defined by

$$I_{\mathsf{M}}(x) = \sum_{k=1}^{\infty} \mathcal{M}_k(x_k), \quad x = (x_k) \in t_{\mathsf{M}}.$$

We consider $t_{\mathsf{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_{\mathsf{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\Big(1 + I_{\mathsf{M}}(kx)\Big) : k > 0\right\}.$$

For more details about Orlicz functions see (Et *et al.*, 2006; Tripathy & Dutta, 2012)

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,

(2) $p(-x) = p(x)$ for all $x \in X$,

(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,

(4) if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair

$(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm see (Wilansky, 1984) Theorem 10.4.2, pp. 183.

By $w$ we denote the set of all real or complex sequences $x = (x_k)$. Let $c, c_0, \bar{c}, \bar{c}_0, l_\infty, m$ and $m_0$ denote the sets of all convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null sequences, respectively.

A sequence space $E$ is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k, x_k) \in E$; for all sequences of scalars $(\alpha_k)$ with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

The concept of 2-normed spaces was initially developed by Gähler (1965) in the mid of 1960's, while that of $n-$normed spaces was introduced by Misiak (1989). Since then, many others have studied these concepts and obtained various results, see (Gunawan, 2001a; Gunawan, 2001b; Gunawan & Mashadi, 2001). Let $n \in \mathbb{N}$ and $X$ be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\|$ on $X_n$ satisfying the following four conditions:

(1) $\|(x_1, x_2, \ldots, x_n)\| = 0$ f and only if $(x_1, x_2, \ldots, x_n)$ are linearly dependent in $X$,

(2) $\|(x_1, x_2, \ldots, x_n)\|$ is invariant under permutation,

(3) $\|(\alpha x_1, x_2, \ldots, x_n)\| = |\alpha| \, \|(x_1, x_2, \ldots, x_n)\|$ for any $\alpha \in \mathbb{K}$, and

(4) $\|(x + x', x_2, \ldots, x_n)\| \leq \|(x, x_2, \ldots, x_n)\| + \|(x', x_2, \ldots, x_n)\|$

is called an $n-$norm on $X$, and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an $n-$normed space over the field $\mathbb{K}$.

**Example 1.1.** We may take $X = \mathbb{R}^n$ equipped with the $n-$norm $\|(x_1, x_2, \ldots, x_n)\|_E =$ the volume of the $n-$dimensional parallelopiped spanned by the vectors $(x_1, x_2, \ldots, x_n)$ which may be given explicitly by the formula

$$\|(x_1, x_2, \ldots, x_n)\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i - 1, 2, \ldots, n$. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an $n-$normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in $X$. Then the following function $\|\cdot, \ldots, \cdot\|_\infty$ on $X^{n-1}$ defined by

$$\|(x_1, x_2, \ldots, x_{n-1})\|_\infty = \max\{\|(x_1, x_2, \ldots, x_{n-1}), a_i\| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)-$ norm on $X$ with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence $(x_k)$ in an $n-$normed space $(X,\|\cdot,\dots,\cdot\|)$ is said to converge to some $L\in X$ if

$$\lim_{k\to\infty}\|x_k - L, z_1,\dots, z_{n-1}\| = 0 \text{ for every } z_1,\dots,z_{n-1}\in X.$$

A sequence $(x_k)$ in an $n-$normed space $(X,\|\cdot,\dots,\cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty}\|x_k - x_p, z_1,\dots, z_{n-1}\| = 0 \text{ for every } z_1,\dots,z_{n-1}\in X.$$

If every Cauchy sequence in $X$ converges to some $L\in X$, then $X$ is said to be complete with respect to the $n-$norm. Any complete $n-$normed space is said to be $n-$Banach space. For more details about $n-$normed space see ( Altin *et al.,* 2004; Raj & Sharma, 2012; Raj *et al.,* 2010; Tripathy *et al.,* 2012a; Tripathy & Borgogain, 2013) and references therein.

A sequence $(x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$, the set $\{k\in \mathbb{N}: \|x_k - L, z_1,\dots, z_{n-1}\|\geq \varepsilon\}$ has natural density zero for each nonzero $z_1,\dots, z_{n-1}\in X$, in other words $(x_k)$ statistically converges to $L$ in $n-$normed space $(X,\|\cdot,\dots,\cdot\|)$ if

$$\lim_{k\to\infty}\frac{1}{k}\left|\{k\in \mathbb{N}: \|x_k - L, z_1,\dots, z_{n-1}\|\geq \varepsilon\}\right| = 0,$$

for each nonzero $z_1,\dots, z_{n-1}\in X$. For $L>0$ we say this is statistically null.

First we give the following lemma, which we need to establish our main results.

**Lemma 1.2.** Every closed linear subspace $F$ of an arbitrary linear normed space $E$ (different from $E$ is a nowhere dense set in $E$ (Tripathy & Dutta, 2010).

Throughout the paper $w(X), c(x), c_0(X), \bar{c}(X), \bar{c}_0(X), \ell_\infty(X), m(X)$ and $m_0(X)$ denote the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null $X$ valued sequence spaces respectively, where $(X,\|\cdot,\dots,\cdot\|)$ is an n-normed space. By $\theta = (0,0,\dots)$ we mean the zero element of $X$. By $x_k \xrightarrow{stat} 0$ we mean that $x_k$ is statistically convergent to zero.

Let $m, v$ be non-negative integers, $\mathcal{M} = (\mathcal{M}_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Let $(X,\|\cdot,\dots,\cdot\|)$ be an $n-$normed space. In this paper we define the following sequence spaces:

$$\bar{c}\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) =$$

$$\left\{ x = (x_k) \in w(X) : \sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \xrightarrow{stat} 0 \right.$$

$$\left. \text{for every non zero } z_1, \ldots, z_{n-1} \in X \text{ and some } L \in X,\ \rho > 0 \right\}$$

$$\bar{c}_0\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) =$$

$$\left\{ x = (x_k) \in w(X) : \sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k)}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \xrightarrow{stat} 0 \right.$$

$$\left. \text{for every non zero } z_1, \ldots, z_{n-1} \in X \text{ and some } \rho > 0 \right\}$$

$$\ell_\infty\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) =$$

$$\left\{ x = (x_k) \in w(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k)}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} < \infty \right.$$

$$\left. \text{for every non zero } z_1, \ldots, z_{n-1} \in X \text{ and some } \rho > 0 \right\}$$

$$c\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) =$$

$$\left\{ x = (x_k) \in w(X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0 \right.$$

$$\left. \text{for every non zero } z_1, \ldots, z_{n-1} \in X \text{ and some } \rho > 0 \right\}$$

$$c_0\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) =$$

$$\left\{ x = (x_k) \in w(X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k)}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0 \right.$$

$$\left. \text{for every non zero } z_1, \ldots, z_{n-1} \in X \text{ and some } \rho > 0 \right\}$$

$$W\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) =$$

$$\left\{ x = (x_k) \in w(X) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0 \right.$$

$$\left. \text{for every non zero } z_1, \ldots, z_{n-1} \in X \text{ and some } \rho > 0 \right\}$$

$$m\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) = \bar{c}\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) \cap \ell_\infty\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$$

and

$$m_0\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) = \bar{c}_0\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) \cap \ell_\infty\left(\mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right),$$

where $B^{v}_{(m)}x = B^{v}_{(m)}x_k = rB^{v-1}_{(m)}x_k + sB^{v-1}_{(m)}x_{k-m}$ and $B^{0}_{(m)}x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the binomial representation as follows:

$$B^{v}_{(m)}x_k = \sum_{\nu=0}^{v} \binom{v}{\nu} r^{v-\nu} s^{\nu} x_{k-m\nu}.$$

In this representation, we obtain the matrix $B^{v}_{(1)}$ defined in Başarir & Kayıkçı (2009) for $v > 1$ and in Başar & Altay (2003) for $v = 1$.

1) If we take $v = 0$ then the above sequence spaces are reduced to $\bar{c}(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, $\bar{c}_0(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, $\ell_\infty(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, $c(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, $c_0(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, $W(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, $m(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$ and $m_0(\mathcal{M}, u, p, \|\cdot,\ldots,\cdot\|)$, respectively.

2) If we take $r = 1$ and $s = -1$ then the above sequence spaces are reduced to $\bar{c}(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $\bar{c}_0(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $\ell_\infty(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $c(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $c_0(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $W(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $m(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$ and $m_0(\mathcal{M}, \Delta^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, respectively.

3) By taking $p = (p_k) = 1$ and $u = (u_k) = 1$ for all $k$ then these sequence spaces are reduces to $\bar{c}(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, $\bar{c}_0(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, $\ell_\infty(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, $c(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, $c_0(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, $W(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, $m(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$ and $m_0(\mathcal{M}, B^{v}_{(m)}, \|\cdot,\ldots,\cdot\|)$, respectively.

4) If we take $\mathcal{M} = I$ where $I$ is the Identity map, then we get the sequence spaces $\bar{c}(B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $\bar{c}_0(B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $\ell_\infty(B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$, $c(B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|)$,

$c_0\left(B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$, $W\left(B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$, $m\left(B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$ and $m_0\left(B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$, respectively.

5) If we replace the base space $X$ which is a linear $n-$normed space by $\mathbb{C}$, complete normed linear space and take $m=1$ and take $r=1$, $s=-1$ then the above spaces reduces to $\bar{c}\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, $\bar{c}_0\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, $\ell_\infty\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, $c\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, $c_0\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, $W\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, $m\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$ and $m_0\left(\mathcal{M}, \Delta_{(1)}^v, u, p, \|.,\ldots,.\|\right)$, respectively.
6) If we replace the base space $X$ which is an $n-$normed space by 2-normed space, $\mathcal{M}=I, u=(u_k)=1$ for all $k$ and $\rho=1$ we obtained the sequence spaces introduced by (Başarir *et al.,* 2013).

7) Moreover if we take $X=\mathbb{C}$, $v=0$, $\mathcal{M}=I$, $p=(p_k)=1$ and $u=(u_k)=1$ for all $k$ we get the spaces $\bar{c}, \bar{c}_0, \ell_\infty, c, c_0, W, m$ and $m_0$, respectively.

The following inequality will be used throughout the paper.

Let $p=(p_k)$ be a sequence of positive real numbers with $0<p_k\le\sup_k p_k=H$ and let $D=\max\left\{1,2^{H-1}\right\}$. Then, for the sequences $(a_k)$ and $(b_k)$ in the complex plane, we have

$$|a_k+b_k|^{p_k}\le D\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{1}$$

and $|\lambda|^{p_k}\le\max\left\{|\lambda|^h,|\lambda|^H\right\}$ for $\lambda\in\mathbb{C}$, where $h=\inf_k p_k$.

The main purpose of this paper is to introduce and study the above defined sequence spaces. We make an effort to study some topological and algebraic properties of these spaces. Further we show that the sequence spaces $m\left(\mathcal{M}, B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$ and $m_0\left(\mathcal{M}, B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$ are complete paranormed spaces, when the base space is $n-$Banach space. We have also studied some inclusion relations between these spaces.

## MAIN RESULTS

**Theorem 2.1.** Let be a $\mathcal{M}=(\mathcal{M}_k)$ Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of positive real numbers. Then the sequence spaces $Z\left(\mathcal{M}, B_{(m)}^v, u, p, \|.,\ldots,.\|\right)$ are linear spaces over the field of complex number $\mathbb{C}$, where $Z=\bar{c}, \bar{c}_0, l_\infty, W, m, m_0$.

**Proof.** We prove the theorem only for the space $\bar{c}(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$ and for the other spaces it will follow on applying similar argument.

Let $x=(x_k), y=(y_k)\in \bar{c}(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists $L, J \in X$ and positive real numbers $\rho_1, \rho_2$ such that for every $z_1,\dots,z_{n-1}\in X$

$$\sum_{k=1}^{\infty}\mathcal{M}_k\left(\left\|\frac{u_kB^v_{(m)}(x_k)-L}{\rho_1}, z_1,\dots,z_{n-1}\right\|\right)^{p_k} \xrightarrow{stat} 0$$

and

$$\sum_{k=1}^{\infty}\mathcal{M}_k\left(\left\|\frac{u_kB^v_{(m)}(y_k)-L}{\rho_2}, z_1,\dots,z_{n-1}\right\|\right)^{p_k} \xrightarrow{stat} 0.$$

Let $\rho_3=\max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M}_k's$ are non-decreasing convex so by using the inequality (1) we have

$$\sum_{k=1}^{\infty}\mathcal{M}_k\left(\left\|\frac{u_kB^v_{(m)}(\alpha x_k+\beta y_k)-(\alpha L+\beta J)}{\rho_3}, z_1,\dots,z_{n-1}\right\|\right)^{p_k}$$

$$=\sum_{k=1}^{\infty}\mathcal{M}_k\left(\left\|\frac{u_kB^v_{(m)}\alpha(x_k-L)}{\rho_3}, z_1,\dots,z_{n-1}+\frac{u_kB^v_{(m)}\beta(y_k-J)}{\rho_3}, z_1,\dots,z_{n-1}\right\|\right)^{p_k}$$

$$\leq D\sum_{k=1}^{\infty}\mathcal{M}_k\left(\left\|\frac{u_kB^v_{(m)}(x_k)-L}{\rho_1}, z_1,\dots,z_{n-1}\right\|\right)^{p_k}$$

$$+D\sum_{k=1}^{\infty}\mathcal{M}_k\left(\left\|\frac{u_kB^v_{(m)}(y_k)-J}{\rho_2}, z_1,\dots,z_{n-1}\right\|\right)^{p_k}$$

$$\xrightarrow{stat} 0 \text{ as } k\to\infty.$$

Thus, $\alpha x+\beta y\in \bar{c}(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$. Hence $\bar{c}(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$ is a linear space over the field of complex number $\mathbb{C}$.

**Theorem 2.2.** Let $\mathcal{M}=(\mathcal{M}_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Let $(X,\|.,\dots,.\|)$ be a $n-$Banach space. Then the spaces $m(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$ and $m_0(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$ are complete paranormed sequence spaces, paranormed by

$$g(x) = \sup_{\substack{k \in \mathbb{N} \\ z_1,\ldots,z_{n-1} \in X}} \sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k)}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{\frac{p_k}{M}}, \quad \text{for some } \rho > 0$$

where $M = \max(1, H)$, $H = \sup_k p_k$ and $h = \inf_k p_k$.

**Proof.** We shall prove the theorem for the space $m_0\left( \mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$.

Clearly, $g(\theta) = 0$, where $\theta = (0,0,\ldots,0)$ is the zero sequence and $g(-x) = g(x)$. Let $x = (x_k)$, $y = (y_k) \in m_0\left( \mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$. Then there exists positive real numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k)}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \xrightarrow{stat} 0, \text{ for some } \rho_1 > 0$$

and

$$\sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(y_k)}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \xrightarrow{stat} 0, \text{ for some } \rho_2 > 0.$$

Let $\rho = \rho_1 + \rho_2$, thus

$$\begin{aligned} g(x+y) &= \sup_{\substack{k \in \mathbb{N} \\ z_1,\ldots,z_{n-1} \in X}} \left[ \sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k + y_k)}{\rho_1 + \rho_2}, z_1, \ldots, z_{n-1} \right\| \right)^{\frac{p_k}{\mathcal{M}}} \right] \\ &\le \sup_{\substack{k \in \mathbb{N} \\ z_1,\ldots,z_{n-1} \in X}} \left[ \sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(x_k)}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)^{\frac{p_k}{\mathcal{M}}} \right] \\ &+ \sup_{\substack{k \in \mathbb{N} \\ z_1,\ldots,z_{n-1} \in X}} \left[ \sum_{k=1}^{\infty} \mathcal{M}_k \left( \left\| \frac{u_k B^{v}_{(m)}(y_k)}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right)^{\frac{p_k}{\mathcal{M}}} \right]. \end{aligned}$$

This implies that $g(x+y) = g(x) + g(y)$. To prove the continuity of scalar multiplication, assume that $(x^{v})$ be any sequence of points in $m_0\left( \mathcal{M}, B^{v}_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$ such that $g(x^{v} - x) \to 0$ as $v \to \infty$ and $(\lambda_v)$ be a sequence of scalars such that $\lambda_v \to \lambda$. Since the inequality $g(x^{v}) \le g(x) + g(x^{v} - x)$ holds by subadditivity of $g$, $(g(x^{v}))$ is bounded. Thus, $g(\lambda_v x^{v} - \lambda x)$

$$= \sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(\lambda_v x^v_k - \lambda x_k)}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}}$$

$$\leq (\max\{|\lambda_n - \lambda|^h, |\lambda_n - \lambda|^H\})^{\frac{1}{\mathcal{M}}} \sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x^v_k)}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}}$$

$$+ (\max\{|\lambda|^h, |\lambda|^H\})^{\frac{1}{\mathcal{M}}} \sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \left(\left\|\frac{u_k B^v_{(m)}(x^v_k - x)}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}}$$

$$= (\max\{|\lambda_v - \lambda|^h, |\lambda_v - \lambda|^H\})^{\frac{1}{\mathcal{M}}} g(x^v) + (\max\{|\lambda|^h, |\lambda|^H\})^{\frac{1}{\mathcal{M}}} g(x^v - x)$$

$$\to 0 \ as \ v \to \infty .$$

Thus, $g$ is a paranorm. To prove the completeness of space $m_0(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$ assume that $(x^i)$ be a Cauchy sequence in $m_0(\mathcal{M}, B^v_{(m)}, u, p, \|.,\dots,.\|)$. Then for a given $\varepsilon > 0$ there exists a positive integer $N_0$ such that $g(x^i - x^j) < \varepsilon$ for all $i, j \geq N_0$. This implies that

$$\sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \sum_{k=1}^{\infty} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x^i{}_k) - u_k B^v_{(m)}(x^j_{(k)})}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}} < \varepsilon, \text{ for all } i, j \geq N_0.$$

It follows that for every nonzero $z_1,\dots,z_{n-1} \in X$.

$$\left\|\frac{u_k B^v_{(m)}(x^i_k) - u_k B^v_{(m)}(x^j_k)}{\rho}\right\| < \varepsilon, \text{ for each } k \geq 1 \text{ and for all } i, j \geq N_0.$$

Therefore, $(u_k B^v_{(m)}(x^i_k))$ is a Cauchy sequence in $X$ for all $k \in \mathbb{N}$. Since $X$ is $n-$ Banach space, $(u_k B^v_{(m)}(x^i_k))$ is convergent in $X$ for all $k \in \mathbb{N}$. We write $u_k B^v_{(m)}(x^i_k) \to u_k B^v_{(m)}(x_k)$ as $i \to \infty$. Now we have for all $i, j \geq N_0$.

$$\sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \sum_{k=1}^{\infty} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x^i_k - x^j_k)}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}} < \varepsilon$$

$$\Rightarrow \lim_{j\to\infty}\left\{\sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \sum_{k=1}^{\infty} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x^i_k - x^j_k)}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}}\right\} < \varepsilon$$

$$\Rightarrow \sup_{\substack{k\in\mathbb{N}\\ z_1,\dots,z_{n-1}\in X}} \sum_{k=1}^{\infty} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x^i_k - x_k)}{\rho}, z_1,\dots,z_{n-1}\right\|\right)^{\frac{p_k}{\mathcal{M}}} < \varepsilon, \quad \text{for all } i \geq N_0.$$

It follows that $\left(x^i - x\right) \in m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ and $m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ is a linear space, so we have $x = x^i - \left(x^i - x\right) \in m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$. Similarly we can prove for the space $m\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$. This completes the proof.

**Theorem 2.3.** The space $Z\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ is not solid in general, where $Z = \bar{c}, \bar{c}_0, m, m_0$.

**Proof**. To show the space is not solid in general, consider the following examples.

**Example 2.4.** Let $m = 3, v = 1, r = 1, s = -1$ and consider the $n-$normed space as defined in example (1.1). Let $p_k = 5$ , $u_k = 1$ and $\mathcal{M}_k = I$, (the identity map); for all $k \in \mathbb{N}$. Consider the sequence $(x_k)$, where $x_k = \left(x^i_k\right)$ is defined by $\left(x^i_k\right) = (k, k, k, \ldots\ldots.)$ for each fixed $k \in \mathbb{N}$. Then $x_k \in Z\left(\mathcal{M}, B^1_{(3)}, u, p, \|.,\ldots,.\|\right)$ for $Z = \bar{c}, m$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z\left(\mathcal{M}, B^1_{(3)}, u, p, \|.,\ldots,.\|\right)$ .Thus $Z\left(\mathcal{M}, B^1_{(3)}, u, p, \|.,\ldots,.\|\right)$ for $Z = \bar{c}, m$ is not solid in general.

**Example 2.5.** Let $m = 3, v = 1, r = 1, s = -1$ and consider the $n-$normed space as defined in example (1.1). Let $p_k = 1$, for all odd $k$ and $p_k = 2$ for all even $k$ , $u_k = 1$ and $\mathcal{M}_k = I$, the identity map, for all $k \in \mathbb{N}$. Consider the sequence $(x_k)$, where $x_k = \left(x^i_k\right)$ is defined by $\left(x^i_k\right) = (3,3,3,\ldots\ldots.)$ for each fixed $k \in \mathbb{N}$. Then $x_k \in Z\left(\mathcal{M}, B^1_{(3)}, u, p, \|.,\ldots,.\|\right)$ for $Z = \bar{c}_0, m_0$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z\left(\mathcal{M}, B^1_{(3)}, u, p, \|.,\ldots,.\|\right)$. Thus $Z\left(\mathcal{M}, B^1_{(3)}, u, p, \|.,\ldots,.\|\right)$ for $Z = \bar{c}_0, m_0$ is not solid in general.

**Theorem 2.6.** The spaces $m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ and $m\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ are nowhere dense subsets of $\ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$.

**Proof.** From Theorem 2.2, it follows that $m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ and $m\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ are closed subspace of $\ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$. Since the inclusion relations

$$m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right) \subset \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right),$$

and

$$m\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right) \subset \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right),$$

are strict, the spaces $m_0\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ and $m\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ are nowhere dense subsets of $\ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$ by lemma 1.2.

**Theorem 2.7.** Let $p = (p_k)$ be a non-negative bounded sequence of positive real numbers such that $\inf_k p_k > 0$. Then

$$W\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) \cap \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) \subset m\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right).$$

**Proof.** Let $(x_k) \in W\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) \cap \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$. Then for a given $\varepsilon > 0$, we have

$$\frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{p_k}$$

$$\geq \frac{1}{n} \sum_{\substack{k=1 \\ \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{p_k} \geq \varepsilon}}^{n} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{p_k}$$

$$\geq \varepsilon\frac{1}{n}\left|\left\{k \leq n : \sum_{k=1}^{n} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{p_k} \geq \varepsilon\right\}\right|.$$

If we take the limit for $n \to \infty$. It follows that $(x_k) \in W\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$ from the above inequality. Since $(x_k) \in W\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$. we have the result.

**Theorem 2.8.** If $0 < p_k < q_k < \infty$, for each $k$. Then

$$\ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right) \subseteq \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, q, \|\cdot,\ldots,\cdot\|\right).$$

**Proof.** Let $(x_k) \in \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|\cdot,\ldots,\cdot\|\right)$ then their exists some $\rho > 0$ such that

$$\sup_n \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{p_k} < \infty.$$

This implies that

$$\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho} z_1, \ldots, z_{n-1}\right\|\right)^{p_k} \leq 1,$$

for sufficiently large value of $k$. Since $\mathcal{M}_k' s$ are non-decreasing, we get

$$\sup_n \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{q_k}$$

$$\leq \sup_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k)-L}{\rho}, z_1,\ldots,z_{n-1}\right\|\right)^{p_k}$$

$$< \infty.$$

Thus, $(x_k)\in \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, q, \|.,\ldots,.\|\right)$. This completes the proof.

**Theorem 2.9.** (i) If $0<\inf p_k \leq p_k < 1$, then

$$\ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right) \subseteq \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, \|.,\ldots,.\|\right).$$

(ii) If $1\leq p_k \leq \sup p_k < \infty$, then

$$\ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, \|.,\ldots,.\|\right) \subseteq \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right).$$

**Proof.** (i) Let $(x_k)\in \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$. Since $0<\inf p_k \leq 1$, we have

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k)-L}{\rho}, z_1,\ldots,z_{n-1}\right\|\right)$$

$$\leq \sup_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k)-L}{\rho}, z_1,\ldots,z_{n-1}\right\|\right)^{p_k}.$$

Hence $(x_k)\in \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, \|.,\ldots,.\|\right)$.

(ii) Let $p_k \geq 1$ for each $k\in \mathbb{N}$ and $\sup p_k < \infty$. Let $(x_k)\in \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, \|.,\ldots,.\|\right)$. Then for each $0<\varepsilon<1$, there exists a positive integer $\mathbb{N}$ such that

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k)-L}{\rho}, z_1,\ldots,z_{n-1}\right\|\right) \leq \varepsilon < 1$$

for all $k\in \mathbb{N}$. This implies that

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k)-L}{\rho}, z_1,\ldots,z_{n-1}\right\|\right)^{p_k}$$

$$\leq \sup_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_k\left(\left\|\frac{u_k B^v_{(m)}(x_k)-L}{\rho}, z_1,\ldots,z_{n-1}\right\|\right).$$

Hence $(x_k)\in \ell_\infty\left(\mathcal{M}, B^v_{(m)}, u, p, \|.,\ldots,.\|\right)$.

**Theorem 2.10.** Let $\mathbb{M}' = (\mathcal{M}'_k)$ and $\mathbb{M}'' = (\mathcal{M}''_k)$ be two Musielak-Orlicz functions. Then we have

$$\ell_\infty(\mathbb{M}', B^v_{(m)}, u, p, \|.,\ldots,.\|) \cap \ell_\infty(\mathbb{M}'', B^v_{(m)}, u, p, \|.,\ldots,.\|)$$
$$\subseteq \ell_\infty(\mathbb{M}' + \mathbb{M}'', B^v_{(m)}, u, p, \|.,\ldots,.\|).$$

**Proof.** Let $(x_k) \in \ell_\infty(\mathbb{M}', B^v_{(m)}, u, p, \|.,\ldots,.\|) \cap \ell_\infty(\mathbb{M}'', B^v_{(m)}, u, p, \|.,\ldots,.\|)$. Then

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}'_k \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} < \infty$$

and

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}''_k \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} < \infty .$$

Adding the above inequalities from $k = 1$ to $\infty$, we have

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} (\mathcal{M}'_k + \mathcal{M}''_k) \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k}$$
$$\leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[ \mathcal{M}'_k \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right.$$
$$\left. + \sup_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}''_k \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right]$$
$$\leq D \sup_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}'_k \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k}$$
$$+ D \sup_n \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}''_k \left( \left\| \frac{u_k B^v_{(m)}(x_k) - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k}$$
$$\leq \infty.$$

Thus, we get $(x_k) \in \ell_\infty(\mathbb{M}' + \mathbb{M}'', B^v_{(m)}, u, p, \|.,\ldots,.\|)$. This completes the proof.

## CONCLUSION

In this paper we have introduced some new statistically convergent sequence spaces defined by a Musielak-Orlicz function over $n-$normed spaces. We have studied some topological properties and interesting inclusion relations between these sequence spaces. There are many applications of sequence spaces in Science and Engineering see (Raj & Sharma, 2013b). The solutions obtained here are potentially significant and important for the explanation of some practical physical problems.

## REFERENCES

**Altin, Y., Et, M. & Tripathy, B. C. 2004.** The sequence space $|\overline{N}_p|(\mathcal{M}, r, q, s)$ on seminormed spaces, Applied Mathematics and Computation, **154:** 423-430.

**Bektaş, Ç. A., Et, M. & Çolak, R. 2004.** Generalized difference sequence spaces and their dual spaces, Journal of Mathematical Analysis and Applications, **292:** 423-432.

**Başarir, M., Konca, S. & Kara, E. E. 2013.** Some generalized difference statistically convergent sequence spaces in 2-normed space, Journal of Inequalities and Applications, **2013:** 177.

**Başarir, M. & Nuray, F. 1991.** Paranormed difference sequence spaces generated by infinite matrices, Pure & Applied Mathematika Sciences, **34:** 87-90.

**Başarir, M. & Kayıkçı, M. 2009.** On the generalized $B^m-$Riesz difference sequence space and $\beta-$property, Journal of Inequalities and Applications, Article ID 385029.

**Başar, F. & Altay, B. 2003.** On the space of sequences of $p-$bounded variation and related matrix mappings, Ukrainian Mathematical Journal, **55:** 136-147.

**Esi, A. 2013.** Lacunary strong $A-$convergent sequence spaces defined by sequences of modulus, Kuwait Journal of Science, **40:** 57-65.

**Esi, A., Tripathy, B. C. & Sarma, B. 2007.** On some new type generalized difference sequence spaces, Mathematica Slovaca, **57:** 475-482.

**Esi, A. & Et, M. 2000.** Some new sequence spaces defined by a sequence of Orlicz functions, Indian Journal of Pure and Applied Mathematics, **31:** 967-972.

**Et, M. & Çolak, R. 1995**. On generalized difference sequence spaces, Soochow Journal of Mathematics, **21:** 377-386.

**Et, M. & Esi, A. 2000.** On Köthe - Toeplitz duals of generalized difference sequence spaces, Bulletin of the Malaysian Mathematical Sciences Society, **23:** 25-32.

**Et, M., Altin, Y., Choudhary, B. & Tripathy, B. C. 2006.** On some classes of sequences defined by sequences Orlicz functions, Mathematical Inequalities and Applications, **9:** 335-342.

**Fast, H. 1951.** Sur la convergence statistique, Colloquium Mathematicum, **2:** 241-244.

**Fridy, J. A. 1985.** On statistical convergence, Analysis, **5:** 301-313.

**Gähler, S. 1965.** Linear 2-normietre Rume, Mathematische Nachrichten, **28:** 1-43.

**Gunawan, H. 2001a.** On $n-$inner product, $n-$norms, and the Cauchy-Schwartz inequality, Scientiae Mathematicae Japonicae, **5:** 47-54.

**Gunawan, H. 2001b.** The space of $p-$summable sequence and its natural $n-$norm, Bulletin of the Australian Mathematical Society, **64:** 137-147.

**Gunawan, H. & Mashadi, M. 2001.** On $n-$normed spaces, International Journal of Mathematics and Mathematical Sciences, **27:** 631-639.

**Kizmaz, H. 1981.** On certain sequence spaces, Canadian Mathematical Bulletin, **24:** 169-176.

**Kirişçi, M. & Başar, F. 2010.** Some new sequence spaces derived by the domain of generalized diffrence matrix, Computers & Mathematics with Applications, **60:** 1299-1309.

**Lindenstrauss, J. & Tzafriri, L. 1971.** On Orlicz sequence spaces, Israel Journal of Mathematics, **10:** 379-390.

**Misiak, A. 1989.** $n-$Inner product spaces, Mathematische Nachrichten, **140:** 299-319.

**Mursaleen, M. 1996.** Generalized spaces of difference sequences, Journal of Mathematical Analysis and Applications, **203:** 738-745.

**Maligranda, L. 1989.** Orlicz spaces and interpolation, Seminars in Mathematics 5, Polish Academy of Science.

**Musielak, J. 1983.** Orlicz spaces and modular spaces, Lecture Notes in Mathematics, 1034.

**Parasher, S. D. & Choudhary, B. 1994.** Sequence spaces defined by Orlicz function, Indian Journal of Pure and Applied Mathematics, **25:** 419-428.

**Raj, K., Sharma, A. K. & Sharma, S. K. 2011.** A sequence space defined by Musielak-Orlicz functions, International Journal of Pure and Applied Mathematics, **67:** 475-484.

**Raj, K. & Sharma, S. K. 2012.** A new sequence space defined by a sequence of Orlicz functions over $n-$normed spaces, Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium, Mathematica, **51:** 89-100.

**Raj, K., Sharma, S. K. & Sharma, A. K. 2010**. Some difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function, Armenian journal of Mathematics, **3:** 127-141.

**Raj, K. & Sharma, S. K. 2013a.** Some difference sequence spaces defined by Musielak-Orlicz functions, Mathematica Pannonica, **24:** 33-43.

**Raj, K. & Sharma, S. K. 2013b.** Some new sequence spaces, Applications and Applied Mathematics, **8:** 596-613.

**Schoenberg, I. J. 1959.** The integrability of certain functions and related summability methods, American Mathematical Monthly, **66:** 361-375.

**Sen, M. & Roy, S. 2013.** On paranormed type fuzzy I-convergent double multiplier sequences, Kuwait Journal of Science, **40:** 1-12.

**Tripathy, B. C. & Dutta, H. 2010.** On some new paranormed difference sequence spaces defined by Orlicz functions, Kyungpook Mathematical Journal, 50: 59-69.

**Tripathy, B. C., Esi, A. & Tripathy, B. 2005.** On a new type of generalized difference Ces`aro sequence spaces, Soochow Journal of Mathematics, **31:** 333-340.

**Tripathy, B. C. & Esi, A. 2006.** A new type of difference sequence spaces, International Journal of Science and Technology, **1:** 11-14.

**Tripathy, B. C. & Mahanta, S. 2003**. On a class of sequences related to the lp space defined by Orlicz functions, Soochow Journal of Mathematics, **29:** 379-391.

**Tripathy, B. C., Altin, Y. & Et, M. 2008.** Generalized difference sequence spaces on seminormed spaces defined by Orlicz functions, Mathematica Slovaca, **58:** 315-324.

**Tripathy, B. C. & Dutta, H. 2012.** On some lacunary difference sequence spaces defined by sequence of Orlicz functions and $q-$lacunary $\Delta_m^n-$Statistical convergence, Analele Stiintifice ale Universitatii Ovidius, Seria Matematica, **20:** 417-430.

**Tripathy, B. C., Sen, M. & Nath, S. 2012a.** $I-$convergence in probabilistic $n-$normed space, Soft Computing, **16:** 1021-1027, DOI 10.1007/s00500-011-0799-8.

**Tripathy, B. C., Baruah, A., Et, M. & Gungor, M. 2012b.** On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers, Iranian Journal of Science and Technology Transaction A: Science, **36:** 147-155.

**Tripathy, B. C. & Borgogain, S. 2013.** On a class of $n-$normed sequences related to the $l_p-$ space, Boletim da Sociedade Paranaense de Matemàtica, **31:** 167-173.

**Wilansky, A. 1984.** Summability through functional analysis, North-Holland Mathematics Studies

**Zygmund, A. 2011.** Trigonometric Series, Cambridge University Press, Cambridge **51:** 233-239.

***Submitted*** **:** 06/02/2014
***Revised*** **:** 02/04/2014
***Accepted*** **:** 03/04/2014

# حول بعض فضاءات متتاليات متقاربة إحصائياً

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## خلاصة

نقوم في هذا البحث بإنشاء بعض فضاءات جديدة معممة لفضاءات متتاليات متقاربة إحصائياً. ويقوم بناءنا على استخدام دالة موزيلاك – اورتلتز على فضاءات معيرة بعدتيها m . كما ندرس خصائص عدة متعلقة بالبنى الطوبولوجية وعلاقات الاحتواء بين هذه الفضاءات.