Kuwait J. Sci. **45 (3)** pp 6-13, 2018

Some notes on the space of *p*-summable sequences

Sukran Konca

Dept. of Mathematics, Faculty of Science and Arts, Bitlis Eren University, Bitlis, Turkey.

Corresponding author: skonca@beu.edu.tr

Abstract

Recently, Konca *et al*. (2015a) have revisited the space ℓ^p of p-summable sequences of real numbers and have shown that this space is actually contained in a weighted inner product space called ℓ_v^2. In another paper, Konca *et al*. (2015b) have investigated the space ℓ^p to show that it is also contained in a weighted 2-inner product space. In this work, we show in the details that ℓ^p is dense in ℓ_v^2 as a normed space, and as a 2-normed space. Further, we prove that ℓ_v^2 is separable and conclude that it is isometric to the completion of ℓ^p.

Mathematical Subject Classifications (2010): 46C50, 46B20, 46C05, 46C15, 46B99.

Keywords: Completion; dense; isometric; separable; the space of p-summable sequences.

1. Introduction

Let N, R and Q be the sets of all natural, real numbers, respectively and we denote the space of all p-summable sequences of real numbers by $\ell^p = \ell^p(R)$. We know that for $1 \le p < \infty$, $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$ is the usual norm on ℓ^p. As an infinite dimensional normed space, ℓ^p can be equipped with another norm $\|x\|_{2,v} := \left[\sum_{k=1}^{\infty} v_k^{p-2} |x_k|^2\right]^{1/2}$, which is not equivalent to the usual norm, where $v = (v_k) \in \ell^p$ with $v_k > 0$, $k \in N$ and $2 < p < \infty$ (Konca *et al*. 2015a).

For $1 \le p < 2$ we know that $\ell^p \subset \ell^2$ and ℓ^p is dense in $\left(\ell^2, \|.\|_2\right)$, where $\|.\|_2$ is the usual norm on it. As seen in the third section in (Konca *et al*. 2015a), for every sequence $x \in \ell^p$, we have $\|x\|_{2,v} < \infty$. This suggests that ℓ^p is situated inside a larger space, consisting of all sequences x with $\|x\|_{2,v} < \infty$. Konca *et al*. (2015a) have denoted by ℓ_v^2 the space given below:

$$\ell_v^2 := \left\{ x = (x_k) : \sum_{k=1}^{\infty} v_k^{p-2} |x_k|^2 < \infty, v = (v_k) \in \ell^p, v_k > 0,\ k \in N. \right\}$$

The concept of 2-normed spaces was initially developed by Gähler (1964), while that of *n*-normed spaces was introduced by Misiak (1989). Raj and Sharma (2013) and Raj and Jamwal (2015) are some recent papers on *n*-normed space.

The function $\|.,.\|$ which satisfies the following four properties

(1) $\|x,z\| \geq 0$, for $x,z \in X$, $\|x,z\| = 0$ if and only if x and z are linearly dependent,

(2) $\|x,z\| = \|z,x\|$, for $x,z \in X$,

(3) $\|\alpha x,z\| = |\alpha| \|x,z\|$, for $x,z \in X$ and $\alpha \in R$,

(4) $\|x+y,z\| \leq \|x,z\| + \|y,z\|$, for $x,y,z \in X$.

is called a 2-norm, and the pair $(X, \|.,.\|)$ is called a 2-normed space (Gähler, 1964).

We know that, as a 2-normed space, the space ℓ^p, which is equipped with the usual 2-norm $\|.,.\|_p$ (Gunawan , 2001) can be equipped with another 2-norm $\|.,.\|_{2,v}$ given below (Konca *et al*. 2015b).

$$\|x,z\|_{2,v} := \left[\frac{1}{2} \sum_{k_1} \sum_{k_2} \left| v_{k_1} v_{k_2} \right|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^2 \right]^{\frac{1}{2}}.$$

For a deeper understanding of these concepts, we also recommend the papers related to the space ℓ^p given in Konca *et al*. (2014 and 2016).

Recall that a topological space is called separable if it contains a countable, dense subset. In other words, there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence. If we refer to a countable set, we mean that the set is in one-to-one corresponding with the set of natural numbers. For further research on this, see Roy (1970) and Bella and Constantini (2015).

Let (X,d) and $(\tilde{X},\tilde{d})$ be metric spaces. Then a mapping T of X into $\tilde{X}$ is said to be an isometry if T preserves distances. This is if for all $x,y \in X$, $\tilde{d}(Tx,Ty) = d(x,y)$, where Tx and Ty are the images of x and y, respectively. The space X is said to be isometric with the space $\tilde{X}$ if there exists a bijective isometry of X onto $\tilde{X}$. The spaces X and $\tilde{X}$ are then called isometric spaces. For a metric space (X,d) there exists a complete metric space $(\hat{X},\hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$. This space $\hat{X}$ is unique up to isometry, that is, if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric. The space $\hat{X}$ is called the completion of the given space X

Let $\left(X,\|.\|\right)$ be a normed space, then we can extend the norm to $\hat{X}$ by setting $\|\hat{x}\|:=\hat{d}\left(\hat{0},\hat{x}\right)$ for every $\hat{x}\in\hat{X}$ (Kreyszig, 1989, pp. 41-45, 59).

Now, we need the following known lemma to prove our main results.

Lemma 1.1 (Kreyszig, 1989, pp. 21-23) Let $\left(X,d\right)$ be a metric space and $Y\subset X$. $\overline{Y}=X$ if and only if for every $x\in X$ and for every $\varepsilon>0$ there exists $y\in Y$ such that $d\left(x,y\right)<\varepsilon$.

In this paper, we show that ℓ^p is dense in ℓ_v^2, which is defined as a normed space with respect to the new norm $\|.\|_{2,v}$ (Konca *et al.* 2015a) and as a 2-normed space with respect to the new 2-norm $\|.,.\|_{2,v}$ defined on it (Konca *et al.* 2015b). Further, we show that ℓ_v^2 is separable. Thus, we can conclude that ℓ_v^2 is isometric to the completion of ℓ^p

Throughout the paper, by $\overline{A}$ and $\left\|\begin{matrix}x_1 & y_1\\ x_2 & y_2\end{matrix}\right\|:=\left(abs\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\right)$ we mean the closure of the set A and the absolute value of the determinant $\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}$, respectively.

2. Main results

As we have seen in Konca *et al.* (2015a), $\ell^p\subset\ell_v^2$ and $\left(\ell_v^2,\|.\|_{2,v}\right)$ is complete. We also know that as a set in $\left(\ell_v^2,\|.\|_{2,v}\right)$, ℓ^p is not closed.

Theorem 2.1. $\left(\ell_v^2,\|.\|_{2,v}\right)$ is separable and ℓ_p is dense in $\left(\ell_v^2,\|.\|_{2,v}\right)$.

Proof. Let $\tilde{c}_{00}$ be the space of all sequences such that $(y_1,y_2,...,y_n,0,...)$, where $y_1,y_2,...,y_n$ are any rational numbers and $n\in N$. Then $\tilde{c}_{00}$ is countable. To show that $\tilde{c}_{00}$ is dense in $\left(\ell_v^2,\|.\|_{2,v}\right)$, let $(x_1,x_2,...,x_n,...)$ be an arbitrary sequence in ℓ_v^2. Since $\sum_{k=1}^{\infty}v_k^{p-2}\left|x_k\right|^2$ is convergent series, then for every $\varepsilon>0$ there exists $n_\varepsilon\in N$ such that for every $n_0\geq n_\varepsilon$ we have $\sum_{k=n_0+1}^{\infty}v_k^{p-2}\left|x_k\right|^2<\frac{\varepsilon^2}{2}$. Since Q is dense in R, for every $x_k\in R$ $\left(k=1,2,...,n_0\right)$ there exists $y_k\in Q$ such that

$$\left|x_k-y_k\right|<\frac{\varepsilon}{2^{\frac{1}{2}}n_0^{\frac{1}{2}}v_k^{\frac{p-2}{2}}}\Rightarrow v_k^{p-2}\left|x_k-y_k\right|^2<\frac{\varepsilon^2}{2n_0}.$$

Then we have $\sum_{k=1}^{n_0}v_k^{p-2}\left|x_k-y_k\right|^2<\frac{\varepsilon^2}{2}$. Let $(y_1,y_2,...,y_{n_0},0,...)\in\tilde{c}_{00}$. Hence,

$$\|y-x\|_{2,v}^2=\sum_{k=1}^{\infty}v_k^{p-2}|y_k-x_k|^2$$
$$=\sum_{k=1}^{n_0}v_k^{p-2}|y_k-x_k|^2$$
$$+\sum_{k=n_0+1}^{\infty}v_k^{p-2}|x_k|^2$$
$$<\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}=\varepsilon^2.$$

This shows us that $\tilde{c}_{00}$ is dense in ℓ_v^2, that is, $\overline{\tilde{c}}_{00}=\ell_v^2$. Since $\tilde{c}_{00}\subset\ell^p\subset\ell_v^2$, we obtain $\overline{\tilde{c}}_{00}=\ell_v^2\subset\overline{\ell^p}\subset\overline{\ell_v^2}=\ell_v^2$ and then $\overline{\ell^p}=\ell_v^2$. Thus ℓ^p is dense in ℓ_v^2. This completes the proof.

Note that by $d_{2,v}(y,x)=\|y-x\|_{2,v}$, a metric can be defined on ℓ_v^2.

Corollary 2.2. Let $\tilde{X}$ be the completion of ℓ^p. Then the spaces $\tilde{X}$ and $\left(\ell_v^2,\|.\|_{2,v}\right)$ are isometric.

Now, we obtain the following result for ℓ_p as a 2-normed space when $2<p<\infty$.

Theorem 2.3. $\tilde{c}_{00}$ is dense in $\left(\ell_v^2,\|.,.\|_{2,v}\right)$ and ℓ_p is dense in $\left(\ell_v^2,\|.,.\|_{2,v}\right)$.

Proof. To show that $\tilde{c}_{00}$ is dense in $\left(\ell_v^2,\|.,.\|_{2,v}\right)$, let $(x_1,x_2,...,x_n,...)$ be an arbitrary sequence in ℓ_v^2 and $\{z_1,z_2\}$ be a linearly independent set in ℓ_v^2 such that $z_1=(1,0,...)$ and $z_2=(0,1,0,...)$. Since $\sum_{k=1}^{\infty}v_k^{p-2}|x_k|^2$ is a convergent series, then for every $\varepsilon>0$ there exists $n_\varepsilon\in N$ such that for every $n_0\geq n_\varepsilon$ we have $\sum_{k=n_0+1}^{\infty}v_k^{p-2}|x_k|^2<\frac{\varepsilon^2}{2|v_i|^{p-2}}$. Since Q is dense in R, for every $x_k\in R$ $(k=1,2,...,n_0)$ there exists $y_k\in Q$ such that $|x_k-y_k|<\frac{\varepsilon}{2^{\frac{1}{2}}n_0^{\frac{1}{2}}v_k^{\frac{p-2}{2}}|v_i|^{\frac{p-2}{2}}}$ for each $i=1,2$. Hence, we obtain $v_k^{p-2}|x_k-y_k|^2<\frac{\varepsilon^2}{2n_0|v_i|^{p-2}}$. Then we have
$$\sum_{k=1}^{n_0}v_k^{p-2}|x_k-y_k|^2<\frac{\varepsilon^2}{2|v_i|^{p-2}}$$ for each $i=1,2$.

Let $(y_1,y_2,...,y_{n_0},0,...)\in\tilde{c}_{00}$. Thus, for each $i=1,2$ we have

$$\|y-x,z_i\|_{2,v}^2=\frac{1}{2}\sum_{k_1}\sum_{k_2}|v_{k_1}v_{k_2}|^{p-2}\left\|\begin{matrix}y_{k_1}-x_{k_1} & z_{ik_1}\\ y_{k_2}-x_{k_2} & z_{ik_2}\end{matrix}\right\|^2$$
$$=\frac{1}{2}\sum_{k\neq i}|v_iv_k|^{p-2}\left\|\begin{matrix}y_i-x_i & z_{ii}\\ y_k-x_k & z_{ik}\end{matrix}\right\|^2$$
$$+\frac{1}{2}\sum_{k\neq i}|v_kv_i|^{p-2}\left\|\begin{matrix}y_k-x_k & z_{ik}\\ y_i-x_i & z_{ii}\end{matrix}\right\|^2$$
$$\leq\sum_{k=1}^{\infty}|v_iv_k|^{p-2}|y_k-x_k|^2=|v_i|^{p-2}\sum_{k=1}^{\infty}v_k^{p-2}|y_k-x_k|^2$$
$$=|v_i|^{p-2}\left[\sum_{k=1}^{n_0}v_k^{p-2}|y_k-x_k|^2+\sum_{k=n_0+1}^{\infty}v_k^{p-2}|x_k|^2\right]$$
$$<|v_i|^{p-2}\left(\frac{\varepsilon^2}{2|v_i|^{p-2}}+\frac{\varepsilon^2}{2|v_i|^{p-2}}\right)=\varepsilon^2.$$

Hence, $\|y-x,z_i\|_{2,v}<\varepsilon$. This shows that $\tilde{c}_{00}$ is dense in ℓ_v^2. That is, $\overline{\tilde{c}}_{00}=\ell_v^2$. Since $\tilde{c}_{00}\subset\ell^p\subset\ell_v^2$, then $\overline{\tilde{c}}_{00}=\ell_v^2\subset\overline{\ell^p}\subset\overline{\ell_v^2}=\ell_v^2$ and hence $\overline{\ell^p}=\ell_v^2$. Thus ℓ^p is dense in ℓ_v^2 and this completes the proof.

Corollary 2.4. $\left(\ell_v^2, \|.,.\|_{2,v}\right)$ is separable.

Proof. Let $\tilde{c}_{00}$ be the space of all sequences such that $(y_1, y_2, ..., y_n, 0, ...)$, where $y_1, y_2, ..., y_n$ are any rational numbers and $n \in N$. Then $\tilde{c}_{00}$ is countable. From Theorem 2.3, we know that $\tilde{c}_{00}$ is dense in $\left(\ell_v^2, \|.,.\|_{2,v}\right)$. Thus $\left(\ell_v^2, \|.,.\|_{2,v}\right)$ is separable.

Corollary 2.5. Let $\tilde{X}$ be the completion of ℓ^p. Then the spaces $\tilde{X}$ and $\left(\ell_v^2, \|.,.\|_{2,v}\right)$ are isometric.

We have the following result for $1 \le p < 2$.

Theorem 2.6. $\tilde{c}_{00}$ is dense in $\left(\ell^2, \|.,.\|_2\right)$.

Proof. To show that $\tilde{c}_{00}$ is dense in $\left(\ell^2, \|.,.\|_2\right)$, where $\|.,.\|_2$ is the usual 2-norm on ℓ^2 such that

$$\|x,z\|_2 := \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|\right]^{\frac{1}{2}},$$

let $(x_1, x_2, ..., x_n, ...)$ be an arbitrary sequence in ℓ^2 and $\{z_1, z_2\}$ be a linearly independent set in ℓ^2 such that $z_1 = (1,0,0,...)$ and $z_2 = (0,1,0,...)$. Since $\sum_{k=1}^{\infty}|x_k|^2$ is a convergent series, then for every $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that for every $n_0 \ge n_\varepsilon$ we have $\sum_{k=n_0+1}^{\infty}|x_k|^2 < \frac{\varepsilon^2}{2}$. Since Q is dense in R, for every $x_k \in R$ $(k = 1, 2, ..., n_0)$ there exists $y_k \in Q$ such that $|x_k - y_k| < \frac{\varepsilon}{2^{\frac{1}{2}} n_0^{\frac{1}{2}}} \Rightarrow |x_k - y_k|^2 < \frac{\varepsilon^2}{2n_0}$.

Then we have $\sum_{k=1}^{n_0}|x_k - y_k|^2 < \frac{\varepsilon^2}{2}$. Let $(y_1, y_2, ..., y_{n_0}, 0, ...) \in \tilde{c}_{00}$. Thus, for each $i = 1, 2$ we have

$$\|y - x, z_i\|_2^2 = \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} y_{k_1} - x_{k_1} & z_{ik_1} \\ y_{k_2} - x_{k_2} & z_{ik_2}\end{matrix}\right\|^2$$

$$= \frac{1}{2}\sum_{k \ne i}\left\|\begin{matrix} y_i - x_i & z_{ii} \\ y_k - x_k & z_{ik}\end{matrix}\right\|^2$$

$$+ \frac{1}{2}\sum_{k \ne i}\left\|\begin{matrix} y_k - x_k & z_{ik} \\ y_i - x_i & z_{ii}\end{matrix}\right\|^2$$

$$\le \sum_{k=1}^{\infty}|y_k - x_k|^2 = \sum_{k=1}^{n_0}|y_k - x_k|^2 + \sum_{k=n_0+1}^{\infty} x_k^{\ 2}$$

$$< \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.$$

Hence, $\|y - x, z_i\|_2 < \varepsilon$. This shows that $\tilde{c}_{00}$ is dense in ℓ^2. That is $\overline{\tilde{c}}_{00} = \ell^2$. Since, $\tilde{c}_{00} \subset \ell^p \subset \ell^2 \Rightarrow \overline{\tilde{c}}_{00} = \ell^2 \subset \overline{\ell^p} \subset \overline{\ell^2} = \ell^2$ then $\overline{\ell^p} = \ell^2$. Therefore, ℓ^p is dense in ℓ^2.

Corollary 2.7. $\left(\ell^2, \|.,.\|_2\right)$ is separable.

Proof. Let $\tilde{c}_{00}$ be the space of all sequences such that $(y_1, y_2, ..., y_n, 0, ...)$, where $y_1, y_2, ..., y_n$ are any rational numbers and $n \in N$. Then $\tilde{c}_{00}$ is countable. From Theorem 2.9, we know that $\tilde{c}_{00}$ is dense in $\left(\ell^2, \|.,.\|_2\right)$, i.e. $\left(\ell^2, \|.,.\|_2\right)$ is separable.

Corollary 2.8. Let $\tilde{X}$ be the completion of ℓ^p. Then the spaces $\tilde{X}$ and $\left(\ell^2, \|.,.\|_2\right)$ are isometric.

3. Conclusions

It has been shown that the space ℓ^p can be equipped with a weighted inner product and its induced norm (Konca *et al*. 2015b). Using the inner product, one may define orthogonality on ℓ^p, carry out the Gram-Schmidt process to get an orthogonal set, then define the volume of an n-dimensional parallelepiped on ℓ^p (Gunawan *et al*. 2005), and so on. In this work, we show with the details that ℓ^p is dense in ℓ_v^2 as a normed space and as a 2-normed space for $2 < p < \infty$. In addition, we explain that ℓ^p is dense in ℓ^2 for $1 \le p \le 2$. Therefore ℓ^2 and ℓ_v^2 are separable. We conclude that ℓ^2 for $1 \le p \le 2$ and ℓ_v^2 for $2 < p < \infty$ are isometric to the completion of ℓ^p.

Separability is especially important in numerical analysis and constructive mathematics, since many theorems that can be proved for nonseparable spaces have constructive proofs only for separable spaces. Such constructive proofs can be turned into algorithms for use in numerical analysis. Furthermore, they are the only sorts of proofs acceptable in constructive analysis. A famous example of a theorem of this sort is the Hahn-Banach Theorem.

4. Acknowledgements

The authors would like to thank two anonymous referees for many helpful comments and suggestions that directly lead to the improvement of the original manuscript.

5. References

Bella, A. & Costantini, C. (2015). Sequential separability vs selective sequential separability. Filomat, **29**(1): 121-124.

Gähler, S. (1964). Lineare 2-normierte räume. Mathematische Nachrichten, **28**(1-2): 1-43.

Gunawan, H. (2001). The space of p-summable sequences and its natural n-norm. Bulletin of the Australian Mathematical Society, **64**(1): 137-147.

Gunawan, H., Setya-Budhi, W. & Mashadi, S. Gemawati. (2005). On volumes of *n*-dimensional parallelepipeds on ℓ^p spaces. Univerzitet u Beogradu Publikacije Elektrotehnickog Fakulteta. Serija Matematika, **16**:48-54.

Konca, S., Idris, M. & Gunawan, H. (2015a). *p*-Summable sequence spaces with inner products. Bitlis Eren University Journal of Science and Technology, **5**(1): 37-41.

Konca, S., Idris, M., Gunawan, H. & Basarir, M. (2015b). *p*-Summable sequence spaces with 2-inner products. Contemporary Analysis and Applied Mathematics, 3(2): 213-226.

Konca, S., Idris, M. & Gunawan, H. (2016). A new 2-inner product on the space of p-summable sequences. Journal of Egyptian Mathematical Society, **24**: 244-249.

Konca, S., Gunawan, H. & Basarir, M. (2014). Some remarks on ℓ^p as an n-normed space. Mathematical Sciences Applications E-Notes, **2**(2):45-50. Retrieved online from http://dergipark.ulakbim.gov.tr/mathenot/article/view/5000105312.

Kreyszig, E. (1989). Introductory functional analysis with applications. Wiley Classics Library. Wiley: New York. pp. 21-23, 41-45, 59.

Misiak, A. (1989). n-inner product spaces. Mathematische Nachrichten, **140:** 299-319.

Raj, K., & Sharma, S.K. (2013). Some new sequence spaces. Applications and Applied Mathematics, **8**(2): 596-613.

Raj, K. & Jamwal, S. (2015). On some generalized statistically convergent sequence spaces. Kuwait Journal of Science, **42**(3): 86-104.

Roy, P. (1970). Separability of metric spaces. Transaction of the American Mathematical Society, **149**(1):19-43.

Submitted *: 13/04/2016*
Revised *: 09/06/2016*
Accepted : *03/07/2016*

بعض الملاحظات على فضاء متواليات p-summable

زوكران كونكا
قسم الرياضيات، جامعة بيتليس إرين، 13000، بيتليس، تركيا
skonca@beu.edu.tr

الملخص

قام كونكا وآخرون[1] مؤخراً بإعادة النظر في فضاء المتواليات اللامتناهية p-summable للأرقام الحقيقية ℓ^p وبينوا أن هذا الفضاء موجود فعلياً في فضاء ضرب داخلي يسمى ℓ_v^2. وفي بحث آخر لهم[2] قاموا بالبحث في الفضاء ℓ^p لإثبات وجوده كذلك في فضاء ضرب داخلي2-. وفي هذا البحث، نوضح أن ℓ^p كثيفة في ℓ_v^2 كفضاء معاير وكفضاء معاير2-، وبالتالي ℓ_v^2 قابل للانفصال. نستنتج من ذلك أن ℓ_v^2 يتناظر مكمله ℓ^p.

1) مجلة بي إي يو "BEU" للعلوم والتكنولوجيا، 2015

2) مجلة التحليل المعاصر والرياضيات التطبيقية، 2015