Some notes on the space of \( p \)-summable sequences

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Abstract

Recently, Konca et al. (2015a) have revisited the space \( \ell^p \) of \( p \)-summable sequences of real numbers and have shown that this space is actually contained in a weighted inner product space called \( \ell^2_v \). In another paper, Konca et al. (2015b) have investigated the space \( \ell^p \) to show that it is also contained in a weighted 2-inner product space. In this work, we show in the details that \( \ell^p \) is dense in \( \ell^2_v \) as a normed space, and as a 2-normed space. Further, we prove that \( \ell^2_v \) is separable and conclude that it is isometric to the completion of \( \ell^p \). Mathematical Subject Classifications (2010): 46C50, 46B20, 46C05, 46C15, 46B99.

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1. Introduction

Let \( N, R \) and \( Q \) be the sets of all natural, real numbers, respectively and we denote the space of all \( p \)-summable sequences of real numbers by \( \ell^p = \ell^p(R) \). We know that for \( 1 \leq p < \infty \), \( \|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \) is the usual norm on \( \ell^p \). As an infinite dimensional normed space, \( \ell^p \) can be equipped with another norm \( \|x\|_{2,v} := \left[ \sum_{k=1}^{\infty} v_k^{p-2} |x_k|^2 \right]^{1/2} \), which is not equivalent to the usual norm, where \( v = (v_k) \in \ell^p \) with \( v_k > 0, k \in N \) and \( 2 < p < \infty \) (Konca et al. 2015a).

For \( 1 \leq p < 2 \) we know that \( \ell^p \subset \ell^2 \) and \( \ell^p \) is dense in \( \left( \ell^2, \|\cdot\|_2 \right) \), where \( \|\cdot\|_2 \) is the usual norm on it. As seen in the third section in (Konca et al. 2015a), for every sequence \( x \in \ell^p \), we have \( \|x\|_{2,v} < \infty \). This suggests that \( \ell^p \) is situated inside a larger space, consisting of all sequences \( x \) with \( \|x\|_{2,v} < \infty \). Konca et al. (2015a) have denoted by \( \ell^2_v \) the space given below:
\[ \ell^2 := \left\{ x = (x_k) \mid x = \sum_{k \in \mathbb{N}} v_k x_k < \infty, v_k = (v_k) \in \ell^\infty, v_k > 0, k \in \mathbb{N} \right\} \]

The concept of 2-normed spaces was initially developed by Gähler (1964), while that of \( n \)-normed spaces was introduced by Misiak (1989). Raj and Sharma (2013) and Raj and Jamwal (2015) are some recent papers on \( n \)-normed space.

The function \( \| \cdot \| \) which satisfies the following four properties:

1. \( \| x, z \| \geq 0 \), for \( x, z \in X \), \( \| x, z \| = 0 \) if and only if \( x \) and \( z \) are linearly dependent,
2. \( \| x, z \| = \| z, x \| \), for \( x, z \in X \),
3. \( \| \alpha x, z \| = \| x, z \| \), for \( x, z \in X \) and \( \alpha \in \mathbb{R} \),
4. \( \| x + y, z \| \leq \| x, z \| + \| y, z \| \), for \( x, y, z \in X \).

is called a 2-norm, and the pair \( (X, \| \cdot, \cdot \|) \) is called a 2-normed space (Gähler, 1964).

We know that, as a 2-normed space, the space \( \ell^p \), which is equipped with the usual 2-norm \( \| \cdot \|_p \) (Gunawan, 2001) can be equipped with another 2-norm \( \| \cdot, \cdot \|_{2,v} \) given below (Konca et al. 2015b).

\[
\| x, z \|_{2,v} := \left[ \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} \frac{x_{k_1} x_{k_2} z_{k_1} z_{k_2}}{z_{k_1} z_{k_2}} \right]^{\frac{1}{2}}.
\]

For a deeper understanding of these concepts, we also recommend the papers related to the space \( \ell^p \) given in Konca et al. (2014 and 2016).

Recall that a topological space is called separable if it contains a countable, dense subset. In other words, there exists a sequence \( (x_n)_{n=1}^\infty \) of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence. If we refer to a countable set, we mean that the set is in one-to-one correspondence with the set of natural numbers. For further research on this, see Roy (1970) and Bella and Constantini (2015).

Let \( (X,d) \) and \( (\tilde{X},\tilde{d}) \) be metric spaces.

Then a mapping \( T \) of \( X \) into \( \tilde{X} \) is said to be an isometry if \( T \) preserves distances. This is if for all \( x,y \in X \), \( \tilde{d}(Tx,Ty) = d(x,y) \), where \( Tx \) and \( Ty \) are the images of \( x \) and \( y \), respectively. The space \( X \) is said to be isometric with the space \( \tilde{X} \) if there exists a bijective isometry of \( X \) onto \( \tilde{X} \). The spaces \( X \) and \( \tilde{X} \) are then called isometric spaces. For a metric space \( (X,d) \) there exists a complete metric space \( (\hat{X},\hat{d}) \) which has a subspace \( W \) that is isometric with \( X \) and is dense in \( \hat{X} \). This space \( \hat{X} \) is unique up to isometry, that is, if \( \tilde{X} \) is any complete metric space having a dense
subspace $\tilde{W}$ isometric with $X$, then $\tilde{X}$ and $\hat{X}$ are isometric. The space $\hat{X}$ is called the completion of the given space $X$.

Let $(X, \|\|)$ be a normed space, then we can extend the norm to $\hat{X}$ by setting $\|\hat{x}\| := \hat{d}(0, \hat{x})$ for every $\hat{x} \in \hat{X}$ (Kreyszig, 1989, pp. 41-45, 59).

Now, we need the following known lemma to prove our main results.

Lemma 1.1 (Kreyszig, 1989, pp. 21-23) Let $(X, d)$ be a metric space and $Y \subseteq X$. $\bar{Y} = X$ if and only if for every $x \in X$ and for every $\varepsilon > 0$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$.

In this paper, we show that $\ell^p$ is dense in $\ell^2_v$, which is defined as a normed space with respect to the new norm $\|\|_2$ (Konca et al. 2015a) and as a 2-normed space with respect to the new 2-norm $\|\|_2$ defined on it (Konca et al. 2015b). Further, we show that $\ell^2_v$ is separable. Thus, we can conclude that $\ell^2_v$ is isometric to the completion of $\ell^p$.

Throughout the paper, by $\bar{A}$ and $\|x_1 y_1\| := \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ we mean the closure of the set $A$ and the absolute value of the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$, respectively.

2. Main results

As we have seen in Konca et al. (2015a), $\ell^p \subset \ell^2_v$ and $(\ell^2_v, \|\|_2)$ is complete. We also know that as a set in $(\ell^2_v, \|\|_2)$, $\ell^p$ is not closed.

Theorem 2.1. $(\ell^2_v, \|\|_2)$ is separable and $\ell^p$ is dense in $(\ell^2_v, \|\|_2)$.

Proof. Let $\tilde{c}_{00}$ be the space of all sequences such that $(y_1, y_2, ..., y_n, 0, ...)$, where $y_1, y_2, ..., y_n$ are any rational numbers and $n \in N$. Then $\tilde{c}_{00}$ is countable. To show that $\tilde{c}_{00}$ is dense in $(\ell^2_v, \|\|_2)$, let $(x_1, x_2, ..., x_n, ...) \in \ell^2_v$. Since $\sum_{k=1}^{\infty} v_k^{p-2} |x_k|^2$ is convergent series, then for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for every $n_0 \geq n_e$ we have $\sum_{k=n_0+1}^{\infty} v_k^{p-2} |x_k|^2 < \frac{\varepsilon^2}{2}$.

Since $Q$ is dense in $R$, for every $x_k \in R$ ($k = 1, 2, ..., n_0$) there exists $y_k \in Q$ such that $|x_k - y_k| < \frac{\varepsilon}{2 \cdot n_0}$, then $v_k^{p-2} |x_k - y_k|^2 < \frac{\varepsilon^2}{2}$. Let $(y_1, y_2, ..., y_n, 0, ...) \in \tilde{c}_{00}$. Hence,
Some notes on the space of p-summable sequences

\[ \|y - x\|_{2,v}^2 = \sum_{k=1}^{\infty} v_k^{p-2} |y_k - x_k|^2 \]
\[ = \sum_{k=1}^{n_0} v_k^{p-2} |y_k - x_k|^2 + \sum_{k=n_0+1}^{\infty} v_k^{p-2} |x_k|^2 \]
\[ \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \]

This shows us that \( c_{00} \) is dense in \( \ell_v^2 \), that is, \( c_{00} = \ell_v^2 \). Since \( c_{00} \subset \ell^p \subset \ell_v^2 \), we obtain \( \bar{c}_{00} = \ell_v^2 \subset \ell^p \subset \ell_v^2 = \ell_v^2 \) and then \( \ell^p = \ell_v^2 \).

Thus \( \ell^p \) is dense in \( \ell_v^2 \). This completes the proof.

Note that by \( d_{2,v}(y,x) = \|y - x\|_{2,v} \), a metric can be defined on \( \ell_v^2 \).

Corollary 2.2. Let \( X \) be the completion of \( \ell^p \). Then the spaces \( \bar{X} \) and \( \left( \ell_v^2, \|\cdot\|_{2,v} \right) \) are isometric.

Now, we obtain the following result for \( \ell_v^2 \) as a 2-normed space when \( 2 < p < \infty \).

Theorem 2.3. \( \bar{c}_{00} \) is dense in \( \left( \ell_v^2, \|\cdot\|_{2,v} \right) \) and \( \ell^p \) is dense in \( \left( \ell_v^2, \|\cdot\|_{2,v} \right) \).

Proof. To show that \( \bar{c}_{00} \) is dense in \( \left( \ell_v^2, \|\cdot\|_{2,v} \right) \), let \((x_1, x_2, \ldots, x_n, \ldots)\) be an arbitrary sequence in \( \ell_v^2 \) and \( \{z_1, z_2\} \) be a linearly independent set in \( \ell_v^2 \) such that \( z_1 = (1,0,\ldots) \) and \( z_2 = (0,1,0,\ldots) \). Since \( \sum_{k=1}^{n_0} v_k^{p-2} |x_k|^2 \) is a convergent series, then for every \( \varepsilon > 0 \) there exists \( n_{\varepsilon} \in \mathbb{N} \) such that for every \( n_0 \geq n_{\varepsilon} \) we have

\[ \sum_{k=1}^{n_0} v_k^{p-2} |x_k|^2 < \frac{\varepsilon^2}{2|v_i|^{p-2}}. \]

Since \( Q \) is dense in \( R \), for every \( x_k \in R \) \((k = 1, 2, \ldots, n_0)\) there exists \( y_k \in Q \) such that \( |x_k - y_k| < \frac{\varepsilon}{2n_0|v_i|^{p-2}} \) for each \( i = 1, 2 \).

Hence, we obtain

\[ v_k^{p-2} |x_k - y_k|^2 < \frac{\varepsilon^2}{2|v_i|^{p-2}}. \]

Let \((y_1, y_2, \ldots, y_n, 0, \ldots) \in \bar{c}_{00} \). Thus, for each \( i = 1, 2 \) we have

\[ \|y - x, z_i\|_{2,v}^2 = \frac{1}{2} \sum_{k=1}^{n_0} \sum_{k \neq i} v_k v_{k,i}^{p-2} \left| y_k - x_k \right|^2 \]
\[ = \frac{1}{2} \sum_{k=1}^{n_0} |v_k v_{k,i}^{p-2} \left| y_k - x_k \right|^2 \]
\[ + \frac{1}{2} \sum_{k=1}^{n_0} |v_k v_{k,i}^{p-2} \left| y_k - x_k \right|^2 \]
\[ \leq \sum_{k=1}^{n_0} |v_k v_{k,i}^{p-2} \left| y_k - x_k \right|^2 = |v_i|^{p-2} \sum_{k=1}^{n_0} v_k^{p-2} |y_k - x_k|^2 \]
\[ = |v_i|^{p-2} \left[ \sum_{k=1}^{n_0} v_k^{p-2} |y_k - x_k|^2 + \sum_{k=n_0+1}^{\infty} v_k^{p-2} |x_k|^2 \right] \]
\[ < |v_i|^{p-2} \left( \frac{\varepsilon^2}{2|v_i|^{p-2}} + \frac{\varepsilon^2}{2|v_i|^{p-2}} \right) = \varepsilon^2. \]

Hence, \( \|y - x, z_i\|_{2,v} < \varepsilon \). This shows that \( \bar{c}_{00} \) is dense in \( \ell_v^2 \). That is, \( \bar{c}_{00} = \ell_v^2 \). Since \( \bar{c}_{00} \subset \ell^p \subset \ell_v^2 \), then \( \bar{c}_{00} = \ell_v^2 \subset \ell^p \subset \ell_v^2 = \ell_v^2 \) and hence \( \ell^p = \ell_v^2 \). Thus \( \ell^p \) is dense in \( \ell_v^2 \) and this completes the proof.
Corollary 2.4. \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \) is separable.

Proof. Let \( \tilde{c}_{00} \) be the space of all sequences such that \( (y_1, y_2, ..., y_n, 0, ...) \), where \( y_1, y_2, ..., y_n \) are any rational numbers and \( n \in N \). Then \( \tilde{c}_{00} \) is countable. From Theorem 2.3, we know that \( \tilde{c}_{00} \) is dense in \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \). Thus \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \) is separable.

Corollary 2.5. Let \( \tilde{X} \) be the completion of \( \ell^p \). Then the spaces \( \tilde{X} \) and \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \) are isometric.

We have the following result for \( 1 \leq p < 2 \).

Theorem 2.6. \( \tilde{c}_{00} \) is dense in \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \).

Proof. To show that \( \tilde{c}_{00} \) is dense in \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \), where \( \| \cdot \|_{\ell^2} \) is the usual 2-norm on \( \ell^2 \) such that

\[
\| x, z \|_2 := \left[ \frac{1}{2} \sum_{k=1}^{\infty} \sum_{\xi=1}^{2} \left| x_{k_\xi} z_{k_\xi} \right|^2 \right]^{1/2},
\]

\((x_1, x_2, ..., x_n, ...)\) be an arbitrary sequence in \( \ell^2 \) and \( \{ z_1, z_2 \} \) be a linearly independent set in \( \ell^2 \) such that \( z_1 = (1, 0, 0, ...) \) and \( z_2 = (0, 1, 0, ...) \). Since \( \sum_{k=1}^{\infty} \left| x_k \right|^2 \) is a convergent series, then for every \( \varepsilon > 0 \) there exists \( n_\varepsilon \in N \) such that for every \( n_0 \geq n_\varepsilon \) we have

\[
\sum_{k=n_0+1}^{\infty} \left| x_k \right|^2 < \frac{\varepsilon^2}{2}.
\]

Since \( Q \) is dense in \( R \), for every \( x_k \in R \) \((k = 1, 2, ..., n_0)\) there exists \( y_k \in Q \) such that

\[
\left| x_k - y_k \right| < \frac{\varepsilon}{1 + n_0^2} \Rightarrow \left| x_k - y_k \right|^2 < \frac{\varepsilon^2}{2n_0^2}.
\]

Then we have \( \sum_{k=1}^{n_0} \left| x_k - y_k \right|^2 < \frac{\varepsilon^2}{2} \). Let \( (y_1, y_2, ..., y_{n_0}, 0, ...) \in \tilde{c}_{00} \). Thus, for each \( i = 1, 2 \) we have

\[
\| y - x, z_i \|_2 = \frac{1}{2} \sum_{k=1}^{n_0} \left| y_k - x_k \right|^2 \| z_{i_k} \|_2
\]

\[
= \frac{1}{2} \sum_{k=1}^{n_0} \left| y_k - x_k \right|^2 \| z_{i_k} \|_2
\]

\[
= \frac{1}{2} \sum_{k=1}^{n_0} \left| y_k - x_k \right|^2 \| z_{i_k} \|_2
\]

\[
\leq \| y - x \|_2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.
\]

Hence, \( \| y - x, z_i \|_2 < \varepsilon \). This shows that \( \tilde{c}_{00} \) is dense in \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \).

Corollary 2.7. \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \) is separable.

Proof. Let \( \tilde{c}_{00} \) be the space of all sequences such that \( (y_1, y_2, ..., y_n, 0, ...) \), where \( y_1, y_2, ..., y_n \) are any rational numbers and \( n \in N \). Then \( \tilde{c}_{00} \) is countable. From Theorem 2.9, we know that \( \tilde{c}_{00} \) is dense in

\( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \), i.e. \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \) is separable.

Corollary 2.8. Let \( \tilde{X} \) be the completion of \( \ell^p \). Then the spaces \( \tilde{X} \) and \( \left( \ell^2, \| \cdot \|_{\ell^2} \right) \) are isometric.
3. Conclusions

It has been shown that the space $\ell^p$ can be equipped with a weighted inner product and its induced norm (Konca et al. 2015b). Using the inner product, one may define orthogonality on $\ell^p$, carry out the Gram-Schmidt process to get an orthogonal set, then define the volume of an n-dimensional parallelepiped on $\ell^p$ (Gunawan et al. 2005), and so on. In this work, we show with the details that $\ell^p$ is dense in $\ell^2_v$ as a normed space and as a 2-normed space for $2 < p < \infty$. In addition, we explain that $\ell^p$ is dense in $\ell^2$ for $1 \leq p \leq 2$. Therefore $\ell^2$ and $\ell^2_v$ are separable. We conclude that $\ell^2$ for $1 \leq p \leq 2$ and $\ell^2_v$ for $2 < p < \infty$ are isometric to the completion of $\ell^p$.

Separability is especially important in numerical analysis and constructive mathematics, since many theorems that can be proved for nonseparable spaces have constructive proofs only for separable spaces. Such constructive proofs can be turned into algorithms for use in numerical analysis. Furthermore, they are the only sorts of proofs acceptable in constructive analysis. A famous example of a theorem of this sort is the Hahn-Banach Theorem.

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5. References


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الملخص

قام كونكا وآخرون، مؤخراً بإعادة النظر في فضاء المتواليات اللامتناهية $\ell^p$ وبناءً على هذا المنهج الحقيقي، قاموا بدراسة فضاء ضرب داخلي يُسمى $\ell^2$. وفي بحث آخر للهم، قاموا بالبحث في الفضاء $\ell^p$ لتبائل وجوده كذلك في فضاء ضرب داخلي $\ell^2$. وفي هذا البحث، نوضح أن $\ell^p$ كثيفة في $\ell^2$ كفضاء معياري وكفضاء معابر 2، وبالتالي $\ell^2$ قابل للانفصال. نستنتج من ذلك أن $\ell^p$ متناصر مكمله.