Ball and Burmester points in Lorentzian sphere kinematics

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ABSTRACT

In this work, we study Lorentzian spherical motion of rigid bodies by using instantaneous invariants and define Lorentzian inflection curve, Lorentzian circling points curve and Lorentzian cubic of twice stationary curve, which are the loci of points having the same properties during Lorentzian spherical motion of rigid bodies. Also, the intersection points of these curves are called Ball points and Burmester points. We define Lorentzian Ball and Burmester points on Lorentzian sphere.

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INTRODUCTION

The basic concepts of planar mechanisms with the aid of instantaneous invariants are widely studied in the literature (Gupta 1978; Roth & Yang 1977; Ting & Wang 1991). The analysis of trajectory curvature with respect to instantaneous invariants has been extended to three dimensional mechanisms by Veldkamp (1967). Also, the instantaneous angular velocity vector and its derivatives in spherical kinematics are defined by Kamphuis (1969). In spherical kinematics, Ball points are the intersection points of the inflection curves and circling point curves, while Burmester points are the intersection points of circling point curves and cubic stationary curvature curves, as given by Kamphuis (1969) and Roth & Yang (1973). The parametrical formulations of these points in spherical kinematics have been obtained by Chiang (1992); Özçelik (2008) and Özçelik & Şaka (2010).

The kinematics for rolling a Lorentzian sphere (a one-sheet hyperboloid) is studied and the equation of motion of the Lorentzian sphere is given by Korolko & Leite

(2011). In dual Lorentzian space kinematics, the concepts of canonical systems and instantaneous invariants are studied and the instantaneous invariants are derived with respect to the line coordinates (Ayyıldız & Yalçın, 2010). Also, kinematically generated surfaces corresponding to the vectors of dual Lorentzian and hyperbolic spheres have been investigated by Karadağ *et al*. (2014).

In this study, Lorentzian motion is taken into consideration as a three dimensional motion of a rigid body around a fixed point on Lorentzian sphere S_1^2. The Lorentzian Ball and Burmester points are investigated and the parametrical representation based on invariants of the Lorentzian spherical motion is obtained.

PRELIMINARIES

Lorentzian 3-space, R_1^3 is a pseudo-Euclidean space with index 1 endowed with the indefinite inner product given by

$$g(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 - x_3 y_3$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. An arbitrary vector $\mathbf{x} \in R_1^3$ is called timelike, if $g(\mathbf{x}, \mathbf{x}) < 0$, spacelike if $g(\mathbf{x}, \mathbf{x}) > 0$ or $\mathbf{x} = \mathbf{0}$ and null, if $g(\mathbf{x}, \mathbf{x}) = 0$ and $\mathbf{x} \neq \mathbf{0}$. Similarly, a curve $C : I \subset \mathbb{R} \to \mathbb{R}_1^3$ is said to be timelike, spacelike or null, if the velocity vector $\dot{\mathbf{c}}(t)$ is timelike, spacelike or null, respectively. The norm of $\mathbf{x} \in R_1^3$ is defined as $\|\mathbf{x}\| = \sqrt{|g(\mathbf{x}, \mathbf{x})|}$. If $\|\dot{\mathbf{c}}(t)\| \neq 0$ for all $t \in I$, then C is a regular curve. A non-null curve C is parameterized by arc-length parameter t, then the tangent vector $\dot{\mathbf{c}}(t)$ along C has a unit length, i.e., $\|\dot{\mathbf{c}}(t)\| = 1$ for all $t \in I$. Furthermore, any vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are said to be orthogonal if $g(\mathbf{x}, \mathbf{y}) = 0$ andthe Lorentzian product $\mathbf{x} \wedge \mathbf{y}$ is defined as

$$\mathbf{x} \wedge \mathbf{y} = (x_3 y_2 - x_2 y_3,\ x_1 y_3 - x_3 y_1,\ x_1 y_2 - x_2 y_1).$$

From now on curves are considered non-null and unit speed curves in R_1^3.

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be a smooth non-null frame along C called a Frenet frame, where $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are non-null unit tangent, principal normal and binormal vectors, respectively. For smooth non-null unit speed curve, the Frenet formulas

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{N}} \\ \dot{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \kappa & 0 \\ \varepsilon_1 \kappa & 0 & \varepsilon_3 \tau \\ 0 & \varepsilon_2 \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

are satisfied where smooth functions κ and τ are called curvature and torsion of C. The

Frenet vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal vectors and, due to their casual characters, there is

$$g(\mathbf{T},\mathbf{T}) = \varepsilon_1, \; g(\mathbf{N},\mathbf{N}) = \varepsilon_2, \; g(\mathbf{B},\mathbf{B}) = \varepsilon_3 \tag{1}$$

where $\varepsilon_i = \pm 1, 1 \leq i \leq 3$. If C is a non-null unit speed curve and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a Frenet frame along C satisfying (1), then the torsion of C holds

$$\varepsilon_1 \tau = \frac{g(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}})}{g(\dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \ddot{\mathbf{c}})} \tag{2}$$

(O'Neill, 1983).

If M is an oriented surface and a C curve lying on M, then there exists another frame of C, different from Frenet frame. This is called a Darboux frame and is denoted by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$, where $\mathbf{T}$ is the unit tangent vector of C, $\mathbf{n}$ is the unit normal vector of M and $\mathbf{g}$ is the geodesic normal vector given by $\mathbf{g} = \mathbf{n} \wedge \mathbf{T}$.

Let M be a timelike surface and C be a non-null curve lying on M in a Lorentzian space $\mathbb{R}_1^3$. Thus, the derivative formula of a Darboux frame is given by

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{g}} \\ \dot{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon_1 k_n \\ -k_g & 0 & \varepsilon_1 \tau_r \\ k_n & \tau_r & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix}$$

where $g(\mathbf{T},\mathbf{T}) = \varepsilon_1$. In this formula k_g, k_n and τ_r are called geodesic curvature, normal curvature and geodesic torsion of the curve, respectively. The geodesic curvature of any curve C on timelike surface M is

$$-\varepsilon_1 k_g = g(\ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \mathbf{c}) \tag{3}$$

(Uğurlu & Topal, 1996).

LORENTZIAN SPHERICAL MOTION AND INSTANTANEOUS INVARIANTS

The Lorentzian sphere (de Sitter space) is known as the hyperboloid of one sheet and is denoted as $S_1^2 = \{\mathbf{x} \in \mathbb{R}_1^3 \, | \, g(\mathbf{x},\mathbf{x}) = 1\}$, while the Lorentzian spherical motion is defined as the three dimensional motion of a rigid body around a fixed point in Lorentzian space. We will investigate the kinematics of this spherical motion with the aid of the geodesic curvature and the torsion.

Let the position vector $\mathbf{r}_A$ of any point A, moving spherically, be a unit vector with direction from the origin O of sphere to the point A. Thus, $\mathbf{r}_A$ is defined as

$$\overrightarrow{OA} = \mathbf{r}_A = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors of the Cartesian coordinate system of R_1^3. Also, $r_x = \cosh\phi\cos\theta$, $r_y = \cosh\phi\sin\theta$, $r_z = \sinh\phi$ where r_A, θ and ϕ are the elements of a Lorentzian spherical coordinate system of R_1^3.

The Lorentzian spherical motion of a moving sphere with respect to a fixed sphere will be considered. The orthonormal coordinate systems $\{\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\delta}\}$ and $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ being on Lorentzian moving and fixed spheres, respectively, will be investigated with respect to each other. So, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\delta}$ are defined as

$$\boldsymbol{\mu} = \mathbf{r}, \ \ \boldsymbol{\sigma} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \ \ \boldsymbol{\delta} = \sinh\phi\cos\theta\,\mathbf{i} + \sinh\phi\sin\theta\,\mathbf{j} + \cosh\phi\,\mathbf{k}.$$

Moreover, the derivatives of the unit vectors of the moving frame are

$$\frac{d\boldsymbol{\mu}}{dt} = \dot{\theta}\cosh\phi\,\boldsymbol{\sigma} + \dot{\phi}\,\boldsymbol{\delta},$$

$$\frac{d\boldsymbol{\sigma}}{dt} = \dot{\theta}\left(-\cosh\phi\,\boldsymbol{\mu} + \sinh\phi\,\boldsymbol{\delta}\right), \tag{4}$$

$$\frac{d\boldsymbol{\delta}}{dt} = \dot{\theta}\sinh\phi\,\boldsymbol{\sigma} + \dot{\phi}\,\boldsymbol{\mu}.$$

Angular velocity is the rate of change of angular displacement and can be described by the relationship

$$w = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}.$$

By the angular velocity vector $\mathbf{w} = w\mathbf{k}$ and the position vector $\mathbf{r}_A$, the velocity vector of a rigid body at the point A is given by

$$\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = \mathbf{w} \wedge \mathbf{r}_A. \tag{5}$$

We assume that the moving and fixed frames of a Lorentzian sphere are coincident as $\mathbf{x} = \boldsymbol{\mu}$, $\mathbf{y} = \boldsymbol{\sigma}$ and $\mathbf{z} = \boldsymbol{\delta}$ at the initial time $t = 0$. Let us take the angular velocity vector $\mathbf{w}$ as a unit vector. Since the angular velocity vector $\mathbf{w}$ has direction towards the axis $\mathbf{z}$ at this moment, $w_{x0} = w_{y0} = 0, w_{z0} > 0$ where $\mathbf{w} = \left(w_{x0}, w_{y0}, w_{z0}\right)$ for the right hand coordinate system. Also, we will use the notation for derivatives of $\mathbf{w}$ as follows

$$\dot{\mathbf{w}} = \left(w_{x1}, w_{y1}, w_{z1}\right), \ \ddot{\mathbf{w}} = \left(w_{x2}, w_{y2}, w_{z2}\right), \ \dddot{\mathbf{w}} = \left(w_{x3}, w_{y3}, w_{z3}\right).$$

All kinematical properties can be described with $w_{x\ell}$, $w_{y\ell}$ and $w_{z\ell}$ where $\ell = 0,1,2,3$. So, we may call these parameters instantaneous invariants of a Lorentzian spherical motion.

If there is a constant angle ϕ between $\mathbf{w}$ and the axis $\mathbf{z}$, then $\dot{\phi} = \ddot{\phi} = \dddot{\phi} = 0$ is obtained. For the special selection coordinate system as above, by using the equation (4) we get

$$\frac{d\mathbf{w}}{dt} = \dot{\mathbf{w}} = \dot{\theta}\sinh\phi\,\boldsymbol{\sigma}\,. \tag{6}$$

It is easily seen that $w_{x1} = w_{z1} = 0$, $w_{y1} \neq 0$ such that we get the first derivative of the angular velocity vector $\mathbf{w}$, which is called the angular acceleration vector, towards the axis $\mathbf{y}$.

By considering the derivative of the equation (6) with respect to t and the equation (4), we can get the Lorentzian spherical motion instantaneous invariants such that

$$\frac{d^2\mathbf{w}}{dt} = \ddot{\mathbf{w}} = -\left(\cosh\phi\sinh\phi\left(\dot{\theta}\right)^2\right)\boldsymbol{\mu} + \left(\sinh\phi\,\ddot{\theta}\right)\boldsymbol{\sigma} + \left(\sinh^2\phi\left(\dot{\theta}\right)^2\right)\boldsymbol{\delta},$$

$$\frac{d^3\mathbf{w}}{dt} = \dddot{\mathbf{w}} = -\left(3\cosh\phi\sinh\phi\,\dot{\theta}\,\ddot{\theta}\right)\boldsymbol{\mu} + \left(\sinh\phi\left(-\left(\dot{\theta}\right)^3 + \theta^3\right)\right)\boldsymbol{\sigma} + \left(3\sinh\phi\,\dot{\theta}\,\ddot{\theta}\right)\boldsymbol{\delta} \tag{7}$$

where

$$w_{x2} = -\cosh\phi\sinh\phi\left(\dot{\theta}\right)^2, \quad w_{y2} = \sinh\phi\,\ddot{\theta}, \quad w_{z2} = \sinh^2\phi\left(\dot{\theta}\right)^2,$$

$$w_{x3} = -3\cosh\phi\sinh\phi\,\dot{\theta}\,\ddot{\theta}, \quad w_{y3} = \sinh\phi, \quad w_{z3} = 3\sinh\phi\,\dot{\theta}\,\ddot{\theta}.$$

LORENTZIAN INFLECTION CURVE, CIRCLING POINTS CURVE AND CUBIC OF TWICE STATIONARY CURVE

The locus of points whose geodesic curvature is zero is called an inflection curve (Kamphuis, 1969). The curves whose geodesic curvature is zero on Lorentzian sphere are Euclidean or Lorentzian circles. Therefore, Lorentzian inflection curve is the locus of points which trace a path in the form of Euclidean or Lorentzian circle arcs. If $k_g = 0$, from the equation (3) we know

$$g\left(\ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \mathbf{c}\right) = 0\,. \tag{8}$$

If we differentiate C with respect to the t and consider the equation (5), we have

$$\dot{\mathbf{c}} = \mathbf{w} \wedge \mathbf{c} = y\mathbf{i} - x\mathbf{k} \tag{9}$$

where $\mathbf{c}$ is the position vector of the curve C. By taking the derivative of the equation (9) with respect to t and using the equation (5), we have

$$\ddot{\mathbf{c}} = \dot{\mathbf{w}} \wedge \mathbf{c} + \mathbf{w} \wedge \dot{\mathbf{c}} = -\left(x + zw_{y1}\right)\mathbf{i} - y\mathbf{j} - xw_{y1}\mathbf{k} . \tag{10}$$

From the equations (8), (9) and (10), the equation of the Lorentzian inflection curve becomes

$$\left(x^2 + y^2\right)z - xw_{y1} = 0. \tag{11}$$

We can also parameterize the Lorentzian inflection curve. By the Lorentzian spherical coordinates, the last equation gives us
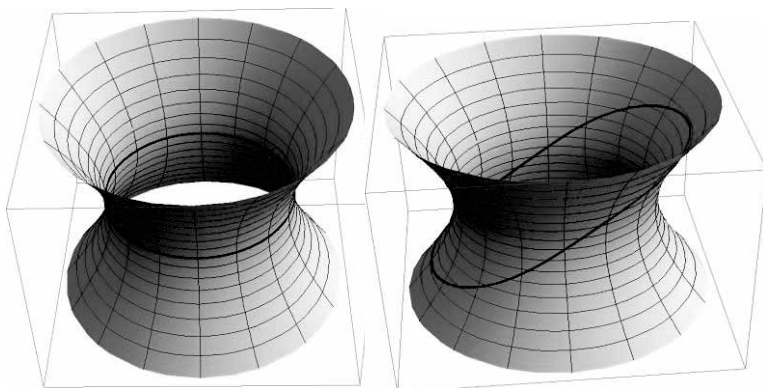
$$-2w_{y1}\cos\theta + \sinh 2\phi = 0. \tag{12}$$

If the equation (12) is solved with respect to θ, we get

$$\theta = \arccos\left(\frac{\sinh 2\phi}{2w_{y1}}\right).$$

With the Lorentzian spherical coordinates, we have

$$C(\phi) = \left(\cosh\phi\frac{\sinh 2\phi}{2w_{y1}}, \cosh\phi\sqrt{1 - \frac{\sinh^2 2\phi}{4w_{y1}^2}}, \sinh\phi\right) \tag{13}$$

where $\phi \in I \subset \mathbb{R}$.

Fig. 1. Lorentzian inflection curves on S_1^2 for $w_{y1} = 0$ and $w_{y1} = 0.4$, resp.

In a similar way, by solving the equation (12) with respect to ϕ, the parametric representation of inflection curve is

$$C(\phi) = \left(\cos\theta\sqrt{1 + 4\cos^2\theta w_{y1}^2}, \sin\theta\sqrt{1 + 4\cos^2\theta w_{y1}^2}, 2\cos\theta w_{y1} \right) \quad (14)$$
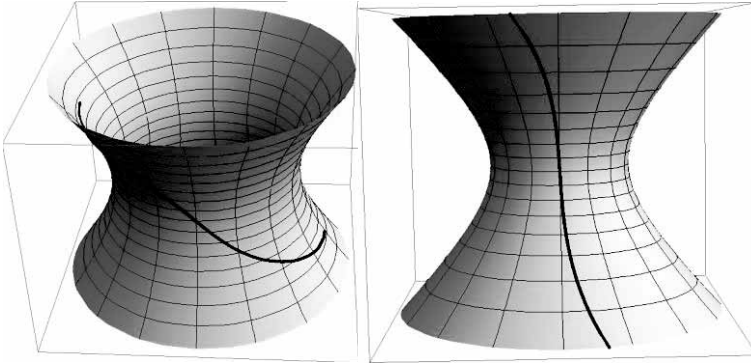
where $\theta \in (0, 2\pi)$.

Fig. 2. Lorentzian inflection curves on S_1^2 for $w_{y1} = 1$ and $w_{y1} = 20$, resp.

The locus of points, whose torsion is zero, is called a circling point curve (Kamphuis, 1969). The curves whose torsions are vanished on Lorentzian sphere are Euclidean or Lorentzian circles. So a Lorentzian circling point curve is the locus of points that trace a circular path on the Lorentzian sphere. With the help of the definition of circling pointcurve and the equation (2),

$$g(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}) = 0 \quad (15)$$

is obtained. The derivative of the equation (10) with respect to t gives us

$$\dddot{\mathbf{c}} = \dddot{\mathbf{w}} \wedge \mathbf{c} + 2(\dot{\mathbf{w}} \wedge \dot{\mathbf{c}}) + \mathbf{w} \wedge \ddot{\mathbf{c}}. \quad (16)$$

By the equations (6), (7), (9) and (10), the last equation is equal to

$$\dddot{\mathbf{c}} = \left(y(w_{z2} - 1) - zw_{y2} \right)\mathbf{i} + \left(x(1 - w_{z2}) + z(w_{x2} + w_{y1}) \right)\mathbf{j} + \left(-xw_{y2} + y(w_{x2} - 2w_{y1}) \right)\mathbf{k}. \quad (17)$$

By the equations (9), (10), (15) and (17), the equation of the Lorentzian circling point curve is

$$(x^2 + y^2)(ax + by) - 3xyz = 0 \quad (18)$$

where $w_{y1} \neq 0$, $a = \dfrac{-w_{y2}}{w_{y1}^2}$ and $b = \dfrac{(w_{x2} - 2w_{y1})}{w_{y1}^2}$.

Similarly, the Lorentzian circling point curve can be parameterized with θ. By the Lorentzian spherical coordinate, the equation (18) is equal to
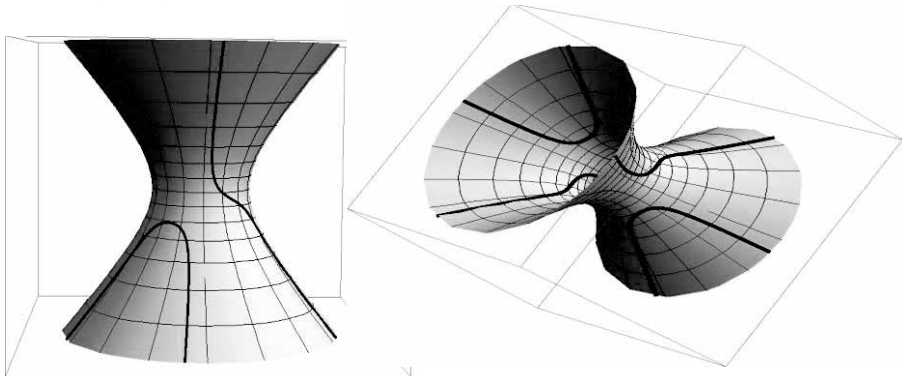
$$-\tanh\phi + a/\sin\theta + b/\cos\theta = 0.$$ (19)

If the equation (19) is solved with respect to the angle ϕ, then

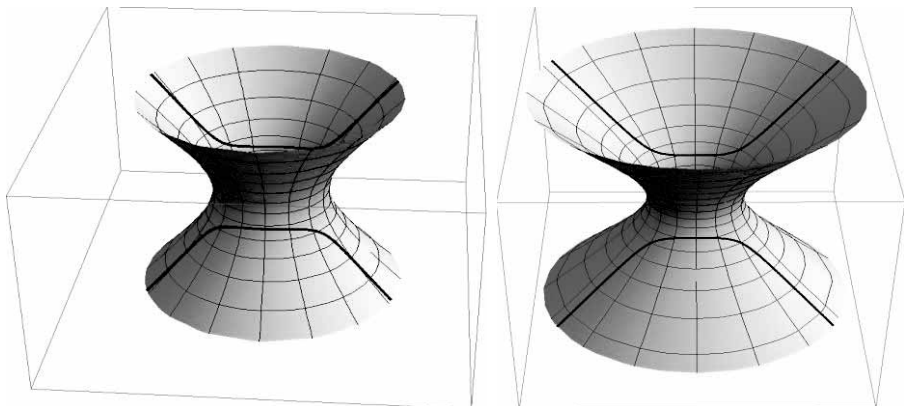$$\phi = \operatorname{arctanh}\left(a/\sin\theta + b/\cos\theta\right)$$ (20)

is obtained. So, the Lorentzian circling point curve is parameterized by θ as

$$C(\theta) = \frac{1}{\sqrt{1 - \left(a\csc\theta + b\sec\theta\right)^2}}\left(\cos\theta, \sin\theta, a\csc\theta + b\sec\theta\right)$$ (21)

where $\theta \in (0, 2\pi)$.

Fig. 3. Lorentzian circling point curves on S_1^2 for $a = 0.25, \ b = 0.1$.

Fig. 4. Lorentzian circling point curves on S_1^2 for $a = 0 \ \ b = 0.3$ and $a = 0.4, b = 0$, resp.

The locus of points whose torsion vanishes, as well as, the change with respect to time vanishes, is called a cubic of twice stationary curvature curve (Kamphuis, 1969). From this definition, we have seen that $\tau = 0$ and $\dfrac{d\tau}{dt} = 0$ is satisfied for a cubic of twice stationary curvature curve. With the help of the equation (2), we have $g\left(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right) = 0$ and

$$\frac{d}{dt}\left(\frac{g\left(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right)}{g\left(\dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}\right)}\right) = -\frac{\left(g\left(\ddot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right) + g\left(\dot{\mathbf{c}}, \dddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right) + g\left(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \ddddot{\mathbf{c}}\right)\right)}{g\left(\dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}\right)} + \frac{g\left(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right)\dfrac{d}{dt}\left(g\left(\dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}\right)\right)}{\left(g\left(\dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}, \dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}\right)\right)^2} = 0.$$

Since $g\left(\ddot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right) = g\left(\dot{\mathbf{c}}, \dddot{\mathbf{c}} \wedge \dddot{\mathbf{c}}\right) = 0$ from the last equation, we see

$$g\left(\dot{\mathbf{c}}, \ddot{\mathbf{c}} \wedge \ddddot{\mathbf{c}}\right) = 0. \tag{22}$$

On the other hand, by the derivative of the equation (16) with respect to t, we get

$$\ddddot{\mathbf{c}} = \dddot{\mathbf{w}} \wedge \mathbf{c} + 3\left(\ddot{\mathbf{w}} \wedge \dot{\mathbf{c}}\right) + 3\left(\dot{\mathbf{w}} \wedge \ddot{\mathbf{c}}\right) + \mathbf{w} \wedge \dddot{\mathbf{c}}. \tag{23}$$

From the equations (6), (7), (9), (10) and (17), we find

$$\ddddot{\mathbf{c}} = \left(x\left(1 + 3w_{y1}^2 - 4w_{z2}\right) + yw_{z3} + z(w_{x2} + w_{y1} - w_{y3})\right)\mathbf{i}$$
$$+ \left(-xw_{z3} + y\left(1 - 4w_{z2}\right) + z\left(w_{x3} + w_{y2}\right)\right)\mathbf{j} \tag{24}$$
$$+ \left(x\left(3w_{y1} - 3w_{x2} - w_{y3}\right) + y\left(w_{x3} - 3w_{y2}\right) + 3zw_{y1}^2\right)\mathbf{k}$$

By the equations (6), (7), (9), (10), (17) and (24), the Lorentzian cubic of twice stationary curvature curve equation is given by

$$\left(x^2 + y^2\right)\left(-z + ey + fx\right) + xw_{y1}\left(gxz + 3x^2 - z^2 + hyz\right) = 0 \tag{25}$$

where $e = \dfrac{3w_{y2} - w_{x3}}{3w_{y1}^2}$, $f = \dfrac{1 + 3w_{x2} - w_{y1}\left(3 + 4w_{y1}\right) + w_{y3}}{3w_{y1}^2}$, $g = \dfrac{4w_{x2} - 2w_{y1}}{3w_{y1}^2}$ and $h = \dfrac{4w_{y2}}{3w_{y1}^2}$.

The Lorentzian cubic of twice stationary curvature curve can be parameterized with θ or ϕ. By the Lorentzian spherical coordinate, the equation (25) is equal to

$$\cosh\phi\left(\cosh\phi\left(f\cos\theta + e\sin\theta\right) - \sinh\phi\right) + \cos\theta\left(\begin{array}{c} 3\cos^2\theta\cosh^2\phi - \sinh^2\phi \\ + \cosh\phi\left(g\cos\theta + h\sin\theta\right)\sinh\phi \end{array}\right)w_{y1} = 0.$$

If the last equation is solved with respect to the θ or ϕ, we can get the parametric equation of the Lorentzian cubic of twice stationary curvature curve.
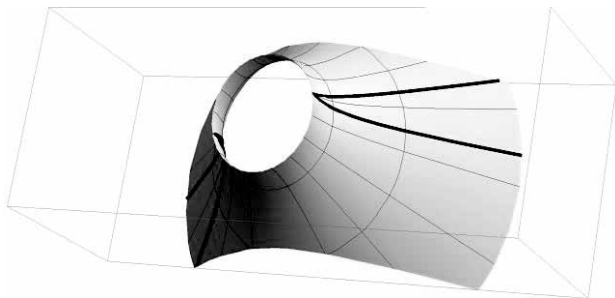
Fig. 5. Lorentzian cubic of twice stationary curvature curve on S_1^2 for $a = 0.25$, $b = 0.1$

LORENTZIAN BALL POINTS AND BURMESTER POINTS

Definition 1. *The intersection points of a Lorentzian inflection curve and a Lorentzian circling point curve in Lorentzian spherical kinematics are called Lorentzian Ball points.*

As can be understood from Definition 1., the common solution of the equation system of the Lorentzian inflection curve and the Lorentzian circling point curve gives us the Lorentzian Ball points. In order to calculate the Lorentzian Ball points, firstly, we will parameterize the Lorentzian inflection curve with parameter t. Thus, we assume that

$$x = ty \tag{26}$$

where $t \neq 0$. From the equation (11), we get

$$\left(y^2 + t^2 y^2 \right) z - ty w_{y1} = 0. \tag{27}$$

If we solve the equation (27) for z, we find

$$z = \frac{t w_{y1}}{\left(1 + t^2 \right) y}, \tag{28}$$

where $\left(1 + t^2 \right) \neq 0$. Substituting the equations (27) and (28) into the Lorentzian circling point curve equation gives us

$$y = \pm \frac{t \sqrt{w_{y1}}}{\sqrt{1 + t^2} \sqrt{(b + at)\left(1 + t^2 \right)}}, \tag{29}$$

where $(b + at) > 0$. Since the Lorentzian Ball points lie on the Lorentzian sphere S_1^2, by the equations (26), (27), (28) and (29), the biquadratic equation of t satisfies

$$c_1 t^4 + c_2 t^3 + c_3 t^2 + c_4 t + c_5 = 0 \tag{30}$$

where

$$c_1 = \frac{a^2 w_{y1}}{(b+at)(1+t^2)}, c_2 = \frac{a(1+2bw_{y1})}{(b+at)(1+t^2)}, c_3 = \frac{(b+w_{y1}(a^2+b^2-1))}{(b+at)(1+t^2)},$$

$$c_4 = \frac{a(1+2bw_{y1})}{(b+at)(1+t^2)}, c_5 = \frac{b(1+bw_{y1})}{(b+at)(1+t^2)}.$$

The number of real roots of the equation (30) gives the number of Lorentzian Ball points. Also, it can be concluded that the maximum number of Lorentzian Ball points is eight. By the equations (26), (28) and (29), Lorentzian Ball points coordinate can be found.
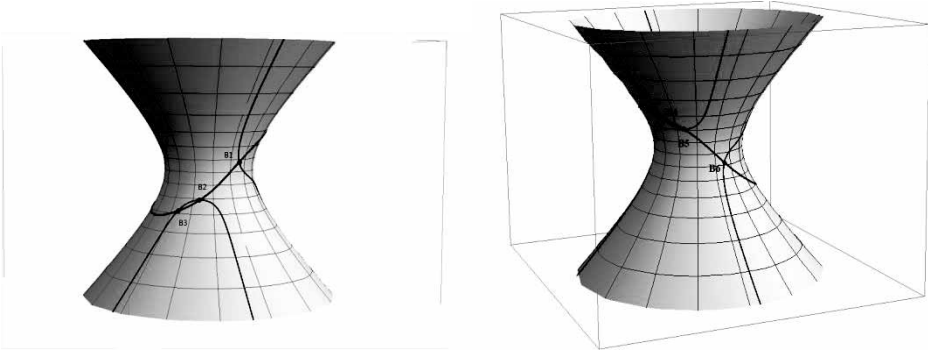

Fig. 6. Lorentzian Ball points for $w_{y1} = 0.45$, $a = 0.25$, $b = 0.1$

Definition 2. *The intersection points of a Lorentzian circling point curve and a Lorentzian cubic of twice stationary curvature in Lorentzian spherical kinematics are called Lorentzian Burmester points.*

Similarly, to find Lorentzian Burmester points, if $x = ty, t \neq 0$ is assumed and calculations are done for the Lorentzian circling point curve, we get

$$x = \frac{t^2}{E}, \; y = \frac{t}{E}, \; z = \frac{(b+at)(1+t^2)}{E} \tag{31}$$

where $\sqrt{(1+t^2)(t^2+(b+at)^2(1+t^2))} = E \neq 0$. If we substitute the equation (31) into the Lorentzian cubic of twice stationary curvature equation, we find

$$d_1 t^7 + d_2 t^6 + d_3 t^5 + d_4 t^4 + d_5 t^3 + d_6 t^2 + d_7 t = 0 \tag{32}$$

where

$$d_1 = \frac{a^2 w_{y1}}{E}, d_2 = \frac{a\left(1 + (2b - g)w_{y1}\right)}{E}, d_3 = \frac{b - f + \left(2a^2 + b^2 - bg - ah - 3\right)w_{y1}}{E},$$

$$d_4 = \frac{2a - e + \left(a(4b - g) - bh\right)w_{y1}}{E}, d_5 = \frac{2b - f + \left(a^2 + b(2b - g) - ah\right)w_{y1}}{E},$$

$$d_6 = \frac{a - e + b(2a - h)w_{y1}}{E}, d_7 = \frac{b\left(1 + bw_{y1}\right)}{E}.$$

The number of the real roots of the equation (32) gives the Lorentzian Burmester points numbers. The maximum number of Lorentzian Burmester points is twelve. Because the obvious solution of the equation (32) is $t = 0$, which is contradiction to $t \neq 0$. If the real value of t is obtained from the equation (32) and substituted into the equation (31), the coordinates of the Lorentzian Burmester points can be found.
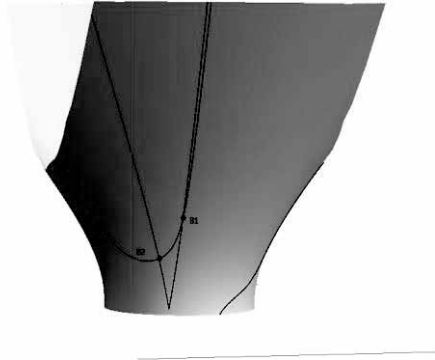

Fig. 7. Lorentzian Burmester points for $a = 0.25$, $b = 0.1$

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REFERENCES

Ayyıldız, N. & Yalçın, Ş. N. 2010. On instantaneous invariants in dual Lorentzian space kinematics, Archives of Mechanics., **62**, 3, 223–238.

Chiang, C. H. 1992. Spherical kinematics in contrast to planar kinematics, Mechanism and Machine Theory, **27**, 243–250.

Gupta, K. C. 1978. A direct method for the evaluation of instantaneous invariants of a given method, Mechanism and Machine Theory, **13**, 567–576.

Kamphuis, H. J. 1969. Application of spherical instantaneous kinematics to the spherical slider-crank mechanism, Journal of Mechanisms, **4**, 43–56.

Karadağ, H. B., Kılıç, E. & Karadağ, M. 2014. On the developable ruled surfaces kinematically generated in Minkowski 3-Space, Kuwait Journal of Science, **41**(1), 21–34.

Korolko, A. & Leite, F. S. 2011. Kinematics for rolling a Lorentzian sphere, 2011 50th IEEE Conference on Decisions and Control and European Control Conference (CDC-ECC), Orlando, FL, USA.

O'Neill, B. 1983. Semi-Riemannian Geometry, Academic Press, New York.

Özçelik, Z. & Şaka, Z. 2010. Ball and Burmester points in spherical kinematics and their special cases, Forschung im Ingenieurwesen, **74**, 111–122.

Özçelik, Z. 2008. Design of spherical mechanisms by using instantaneous invariants, Ph.D. Dissertation, Selcuk University, Mechanical Engineering Department (in Turkish).

Roth, B. & Yang, A. T. 1977. Application of instantaneous invariants to the analysis and synthesis of mechanisms, Journal of Engineering for Industry-Transactions of the ASME, **99**, 97–103.

Roth, B. & Yang, A. T. 1973. Higher–order path curvature in spherical kinematics, Journal of Engineering for Industry ASME, **95**(2), 612–616.

Ting, K. L. & Wang, S. C. 1991. Fourth and fifth order double Burmester points and the highest attainable order of straight lines, Journal of Engineering for Industry ASME, **113**(1), 213–219.

Uğurlu, H. H. & Topal, A. 1996. Relation between Darboux instantaneous rotation vectors of curves on a timelike surface, Mathematical & Computational Applications, **1**(2), 149–157.

Veldkamp, G. R. 1967. Canonical systems and instantaneous invariants in spatial kinematics, Journal of Mechanisms, **2**(3), 329–388.

نقاط بول وبرمستر في حركة كرة لورنتز

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خلاصة

نقوم في هذا البحث بدراسة حركة كرات لورنتز للأجسام الصلبة و ذلك بإستخدام متغيرات لحظية و بتعريف منحنى لورنتز الانعطافي ، منحنى لورنتز الدائرية ، وكذلك منحنى لورنتز التكعيبي ثنائي التوقف، حيث هذه المنحنيات هي المحلات الهندسية التي لها نفس الخصائص خلال حركة لورنتز الكروية للأجسام الصلبة. وتسمى نقاط تقاطع هذه المنحنيات بنقاط بول و نقاط برمستر. كما نقوم في هذا البحث بتعريف نقاط لورنتز بول و نقاط برمستر على كرة لورنتر.