Kuwait J. Sci. **42 (3)** pp. 28-35, 2015

Extension of Mazhar's theorem on summability factors

MEHMET ALI SARIGOL*

**Department of Mathematics, University of Pamukkale, Denizli, TURKEY*

**msarigol@pau.edu.tr*

ABSTRACT

By (X, Y) we denote the set of all sequences ϵ such that $\sum a_n \epsilon_n$ is summable Y whenever $\sum a_n$ is summable X, where X and Y are summability methods. In this paper we characterize the set $(|C,\alpha|_k, |\overline{N}, p_n|)$ for $k > 1$, $\alpha > -1$ and arbitrary positive sequences (p_n) using functional analytic techniques, and so extend some known results.

Mathematics Subject Classification (2010): 40D09, 40F05, 46B45

Keywords: Absolute Cesàro summability; absolute Riesz summability; bounded linear operators; matrix transformations; summability factors.

INTRODUCTION

Let $\sum a_n$ be an infinite series with partial sums s_n. We denote by (σ_n^α) and (t_n^α) the n-th Cesàro means (C, α) of the sequences (s_n) and (na_n), respectively. The concept of absolute summability of order k was first introduced by Flett (1957) as follows. A series $\sum a_n$ is summable $|C,\alpha|_k$, $k \geq 1$, $\alpha > -1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \tag{1}$$

The method $|C,\alpha|_1$ is reduced to $|C,\alpha|$. On the other hand, in view of the well known identity $t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$, the condition (1) can be stated by

$$\sum_{n=1}^{\infty} \frac{|t_n^\alpha|^k}{n} < \infty. \tag{2}$$

Let (p_n) be a sequence of positive numbers with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$. A series $\sum a_n$ is said to be summable $|\overline{N}, p_n|$ if

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty \tag{3}$$

where

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v s_v \tag{4}$$

For any real α and integers $n \geq 0$, we define $\Delta^\alpha U_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1} U_v$, whenever the series is convergent. Let X and Y summability methods. A type of summability factors (X, Y) defined by

$$(X,Y) = \{\epsilon = (\epsilon_n) : \sum \epsilon_\nu \, a_\nu \text{ is summable } Y \text{ whenever } \sum a_\nu \text{ is summable } X\}$$

were investigated by many authors, see (Bor & Kuttner, 1989; Bosanquet & Das, 1979; Chow, 1954; Flett, 1957; Mazhar, 1971; Rhoades & Savas, 2004; Sarigol, 1993b; Sarigol, 2011a,b; Sulaiman, 1992). It is known that the summability $|\overline{N}, p_n|$ and the summability $|C,\alpha|_k$ are, in general, independent of each other. It is therefore natural to find out suitable summability factors of type $(|C,\alpha|_k, |\overline{N}, p_n|)$. In this direction the following theorems are well known.

Theorem 1.1. (Mehdi, 1960). The necessary and sufficient conditions for $\sum \epsilon_n \, a_n$ to be summable $|C,1|$ whenever $\sum a_n$ is summable $|C,\alpha|_k$, $\alpha \geq 0, k > 1$, are

$$\left\{n^{\alpha+1-\frac{1}{k'}} \Delta^\alpha \left(\frac{\epsilon_n}{n}\right)\right\} \in l_{k'} , \frac{1}{k} + \frac{1}{k'} = 1 , \tag{5}$$

$$\sum_{m=1}^{\infty} \frac{|\epsilon_m|^{k'}}{m} < \infty , \alpha \leq 1 ,$$

$$\sum_{m=1}^{\infty} m^{\alpha k' - k' - 1} |\epsilon_m|^{k'} < \infty , \ \alpha > 1.$$

This result is contained in the following.

Theorem 1.2. (Mazhar, 1971). The necessary and sufficient conditions for $\sum \epsilon_n \, a_n$ to be summable $|\overline{N}, p_n|$ whenever $\sum a_n$ is summable $|C,\alpha|_k$, $\alpha \geq 0, k \geq 1$, are (5) and

$$\left\{n^{-\frac{1}{k'}} \epsilon_n\right\} \in l_{k'}, \ 0 \leq \alpha \leq 1, \tag{6}$$

$$\left\{n^{\alpha-\frac{1}{k'}} \left(\frac{p_n}{P_n}\right) \epsilon_n\right\} \in l_{k'} , \ \alpha > 1, \tag{7}$$

where

$$(a) \ \frac{p_n}{p_{n+1}} = O(1), \ (b) \ (n+1)\frac{p_n}{P_n} = O(1) \ \text{ and } \ (c) \ \frac{P_n}{n^\alpha p_n} = O(1), (\alpha > 1). \tag{8}$$

Theorem 1.3. (Sarigol, 1993a and Sarigol & Bor, 1995). The necessary and sufficient conditions for $\epsilon \in (|C,1|_k, |\overline{N}, p_n|), \ k > 1,$ are

$$\sum_{m=1}^{\infty} m^{k'-1} \left(\frac{p_m}{P_m} |\epsilon_m|\right)^{k'} < \infty,$$

$$\sum_{m=1}^{\infty} m^{k'-1} \left|\Delta \epsilon_m + \frac{\epsilon_{m+1}}{m+1}\right|^{k'} < \infty.$$

MAIN RESULTS

Theorem 1.2 does not include the case $-1 < \alpha < 0$ and arbitrary positive sequence (p_n). So, motivated by this theorem, a natural problem is that, what are the necessary and sufficient conditions in order that these results should be satisfied for $\alpha > -1$ and arbitrary positive sequence (p_n) . The aim of this paper is to answer the problem by establishing the following theorem, giving also a new characterization of the matrix $C: l_k \to l$, and deduce some known results.

Theorem 2.1. Let $\alpha > -1$ and $k > 1$. Then the necessary and sufficient conditions for $\epsilon \in (|C,\alpha|_k, |\overline{N}, p_n|)$ are

$$\sum_{m=1}^{\infty} m^{\alpha k'+k'-1} \left(\sum_{n=m}^{\infty} \frac{p_n}{P_n P_{n-1}} \left|\sum_{r=m}^{\infty} A_{r-m}^{-\alpha-1} \frac{\epsilon_r}{r} P_{r-1}\right|\right)^{k'} < \infty. \qquad (9)$$

It may be remarked that, in the case when $\alpha \geq 0$ and (p_n) is a sequence satisfying the condition (8), Theorem 2.1 is reduced to Theorem 1.2. In fact, now $\epsilon \in (|C,\alpha|_k, |\overline{N}, p_n|) \subset (|C,0|_k, |C,\alpha|_k)$, since $1 \in (|C,0|_k, |C,\alpha|_k)$, see (Flett, 1957) and Therefore, if $\epsilon \in (|C,\alpha|_k, |\overline{N}, p_n|)$, then $\epsilon \in (|C,0|_k, |\overline{N}, p_n|)$. By applying Theorem 2.1 with $\alpha = 0$, we have $(m^{-1/k'} \epsilon_m) \in l_{k'}$ which implies $\epsilon_m = O(m)$. Thus, considering that

$$\sum_{n=m}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{r=m}^{\infty} |A_{r-m}^{-\alpha-1}| \frac{|\epsilon_r|}{r} P_{r-1}$$
$$= \sum_{r=m}^{\infty} |A_{r-m}^{-\alpha-1}| \frac{|\epsilon_r|}{r} P_{r-1} \sum_{n=r}^{n} \frac{p_n}{P_n P_{n-1}}$$
$$= \sum_{r=m}^{\infty} |A_{r-m}^{-\alpha-1}| \frac{|\epsilon_r|}{r} P_{r-1} = O(1) \sum_{r=m}^{\infty} |A_{r-m}^{-\alpha-1}| < \infty,$$

it follows that

$$\sum_{m=1}^{\infty} m^{\alpha k'+k'-1} \left(\sum_{n=m}^{\infty} \frac{p_n}{P_n P_{n-1}} \left|\sum_{r=m}^{\infty} A_{r-m}^{-\alpha-1} \frac{\epsilon_r}{r} P_{r-1}\right|\right)^{k'}$$
$$\geq \sum_{m=1}^{\infty} m^{\alpha k'+k'-1} \left|\sum_{r=m}^{\infty} A_{r-m}^{-\alpha-1} \frac{\epsilon_r}{r} P_{r-1} \sum_{n=r}^{\infty} \frac{p_n}{P_n P_{n-1}}\right|^{k'}$$
$$= \sum_{m=1}^{\infty} m^{\alpha k'+k'-1} \left|\Delta^{\alpha}\left(\frac{\epsilon_m}{m}\right)\right|^{k'}$$

which gives that (9) implies (5), and clearly (7). The sufficiency can be shown as in the result of Mazhar (1971). So Theorem 2.1 includes Theorem 1.2.

For $p_n = 1$, Theorem 2.1 is extended Theorem 1.1 of Mehdi (1960) to $\alpha > -1$.

Also, if we take $\alpha = 1$, we get Theorem 1.3 contained in Theorem 2.2 in (Sarigol, 1993a; Sarigol & Bor, 1995).

On other hand, if $-1 < \alpha < 0$, then

$$\Delta^{\alpha}\left(\frac{1}{m}\right) = \sum_{n=m}^{\infty} \frac{A_{n-m}^{-\alpha-1}}{n} = \frac{1}{mA_m^{\alpha}} \sim \frac{\Gamma(n+1)}{m^{\alpha+1}},$$

see (Chow, 1954 and Peyerimhoff, 1954). Hence, by considering the above comment for $\alpha \geq 0$ the condition (9) is reduced to

$$\sum_{m=1}^{\infty} \frac{1}{m} < \infty \tag{10}$$

for > -1, which is impossible. So we have the interesting following result.

Corollary 2.2. If $k > 1$, then $1 \notin (|C,\alpha|_k, |\overline{N},p_n|)$ for all $\alpha > -1$ and positive sequence (p_n), i.e., there is no a series summable $|\overline{N},p_n|$, whenever it is summable by $|C,\alpha|_k$.

For the proof of the Theorem, we need the characterization of the matrices $C: l_k \to l$, which established in (Stieglitz & Tietz, 1977). However, it exposes a rather difficult condition to apply in applications. Therefore we need a new characterization of the class of these matrices with a simpler condition as follows.

Lemma 2.3. Let $1 < k < \infty$. Then, the necessary and sufficient conditions for an infinite matrix $C: l_k \to l$ are

$$U(C) = \left\{\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}|c_{nm}|\right)^{k'}\right\}^{\frac{1}{k'}} < \infty. \tag{11}$$

Proof. We first note that $\|C\|_{(l^k,l)} = \|C^t\|_{\left(l_\infty,l^{k'}\right)} < \infty$, i.e., $C: l_k \to l$ if and only if $C^t: l_\infty \to l_{k'}$, see (Jakimovsky & Russel, 1972). Now if the condition (11) is satisfied, then it is seen that

$$\|C^t(x)\|_{l^{k'}} = \left\{\sum_{m=0}^{\infty}\left|\sum_{n=0}^{\infty}|c_{nm}x_n|\right|^{k'}\right\}^{\frac{1}{k'}}$$

$$\leq \|x\|_\infty \left\{\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}|c_{nm}|\right)^{k'}\right\}^{\frac{1}{k'}}$$

$$= U(C)\|x\|_\infty$$

for all $x \in l_\infty$, which implies $C: l_k \to l$. Hence the condition (11) is sufficient.

Conversely, if $C: l_k \to l$, then $C^t: l_\infty \to l_{k'}$. Because of the fact that l_∞ is bk space, the mapping C^t is continuous and so there exists a constant M such that

$$\|C^t(x)\|_{l^{k'}} \le M\|x\|_\infty \tag{12}$$

for all $x \in l_\infty$. Let N be any finite subset of all nonnegative integers. Define the sequence x as $x_n = 1$ for $n \in N$, and zero otherwise. Then it follows from (12) that

$$\sum_{m=0}^{\infty}\left|\sum_{n\in N} c_{nm}\right|^{k'} \le M^{k'}.$$

If c_{nv} $(n, v = 0,1,\dots)$ are real numbers, then, by Minkowsky's inequality, we have

$$U(C) = \left\{\sum_{m=0}^{\infty}\left(\sum_{n\in N}|c_{nm}|\right)^{k'}\right\}^{\frac{1}{k'}}$$

$$= \left\{\sum_{m=0}^{\infty}\left(\sum_{n\in N_+} c_{nm} + \sum_{n\in N_-}(-c_{nm})\right)^{k'}\right\}^{\frac{1}{k'}}$$

$$\le \left\{\sum_{m=0}^{\infty}\left(\sum_{n\in N_+} c_{nm}\right)^{k'}\right\}^{\frac{1}{k'}} + \left\{\sum_{m=0}^{\infty}\left(\sum_{n\in N_-}(-c_{nm})\right)^{k'}\right\}^{\frac{1}{k'}}$$

$$\le 2M\,,$$

where $N_+ = \{n \in N : c_{nm} \ge 0\}$ and $N_- = \{n \in N : c_{nm} < 0\}$. So, if c_{nv} is complex number, $c_{nv} = c_{nv}^{(1)} + ic_{nv}^{(2)}$ say, then, since $U(C_1) \le U(C)$ and $U(C_2) \le U(C)$, it follows that $U(C) \le 4M < \infty$. This shows that the condition (11) is necessary, completing the proof.

Proof of Theorem 2.1. By the definition of t_n^α and T_n, we can write $t_0^\alpha = a_0$,

$$t_n^\alpha = \frac{1}{n^{1/k}A_n^\alpha}\sum_{r=1}^{\infty} A_{n-r}^{\alpha-1}\, v a_v\,, n \ge 1\,,$$

and

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu\epsilon_\nu \ , P_{-1} = 0$$

which implies

$$y_n = T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n} P_{r-1}\, a_r\epsilon_r \, , n \geq 1, y_0 = a_0\epsilon_0.$$

Hence we have

$$y_n = \frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n} P_{r-1}\, a_r\epsilon_r$$

$$= \frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n}\frac{\epsilon_r}{r}P_{r-1}\sum_{m=1}^{r} A_{r-m}^{-\alpha-1}\, m^{1/k}A_m^\alpha t_m^\alpha$$

$$= \frac{p_n}{P_nP_{n-1}}\sum_{m=1}^{n} m^{1/k}A_m^\alpha\sum_{r=m}^{n} A_{r-m}^{-\alpha-1}\frac{\epsilon_r}{r}P_{r-1}t_m^\alpha$$

$$= \sum_{m=1}^{n} C_{nm}t_m^\alpha$$

where

$$C_{nm} = \begin{cases} \dfrac{p_n}{P_nP_{n-1}} m^{1/k}A_m^\alpha \displaystyle\sum_{r=m}^{n} A_{r-m}^{-\alpha-1}\frac{\epsilon_r}{r}P_{r-1} & , m \leq n \\ 0 & , m > n. \end{cases}$$

Then, $\sum \epsilon_n\, a_n$ is summable $|\overline{N}, p_n|$ whenever $\sum a_n$ is summable $|C, \alpha|_k$ if and only if $y \in l$ whenever $t^\alpha \in l_{k'}$, or, equivalently,

$$\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty}|c_{nm}|\right)^{k'} < \infty \, , \tag{13}$$

by Lemma 2.3. Since $A_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$ for $\alpha > -1$, see (Flett, 1957), it follows from the definition C that

$$\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty}|c_{nm}|\right)^{k'}$$

$$= \sum_{m=1}^{\infty} m^{\alpha k'+k'-1}\left(\sum_{n=m}^{\infty}\left|\frac{p_n}{P_nP_{n-1}}\sum_{r=m}^{n} A_{r-m}^{-\alpha-1}\frac{\epsilon_r}{r}P_{r-1}\right|\right)^{k'},$$

which completes the proof with (13).

REFERENCES

Bor, H. & B. Kuttner, B. 1989. On the necessary condition for absolute weighted arithmetic mean summability factors. Acta Mathematica Hungarica **54**: 57-61.

Bosanquet, L. S. & Das, G. 1979. Absolutely summability factors for Nörlund means. Proceedings London Mathematical Society **838**: 1-52.

Chow, H. C. 1954. Note on convergence and summability factors. Journal of the London Mathematical Society **29**: 459-476.

Flett, T. M. 1957. On an extension of absolute summability and some theorems of Littlewood and Paley. Proceedings London Mathematical Society **7**: 113-141.

Jakimovsky, A. & Russel, D. C. 1972. Matrix mapping between BK-spaces, Bulletin London Mathematical Society **4**: 345-353.

Mazhar, S. M. 1971. On the Absolute summability factors of infinite series. Tohoku Mathematical Journal **23**: 433-451.

Mehdi, M. R. 1960. Summability factors for generalized absolute summability I. Proceedings London Mathematical Society. **10**: 180-200.

Peyerimhoff, A. 1954. Summierbarkeitsfactoren für absolut Cesàro summierbareReihen. Mathematische Zeitschrift **59**: 417-424.

Rhoades, B. E. & Savas, E. 2004. A characterization of absolute summability factors. Taiwanese Journal of Mathematics **8**: 453-465.

Sarigol, M. A. 1993a. On two absolute Riesz summability factors of infinite series. Proceedings of the American Mathematical Society **118**: 485-488.

Sarigol, M. A. 1993b. A note on summability. Studia Scientiarum Mathematicarum Hungarica **28**; 395-400.

Sarigol, M. A. & Bor, H. 1995. Characterization of absolute summability factors. Journal of Mathematical Analysis and Applications **195**: 537-545.

Sarigol, M. A. 2011a. Matrix transformations on fields of absolute weighted mean summability. Studia Scientiarum Mathematicarum Hungarica **48**; 331-341.

Sarigol, M. A. 2011b. Characterization of general summability factors and applications, Computers and Mathematics with Applications **62**: 2665-2670.

Stieglitz, M. & Tietz, H. 1977. Matrixtransformationen von Folgenraumen Eine Ergebnisübersicht. Mathematische Zeitschrift **154**: 1-16.

Sulaiman, W. T. 1992. On summability factors of infinite series. Proceedings of the American Mathematical Society **115**: 313-317.

Submitted **:** 19/01/2014

Revised **:** 13/03/2014

Accepted **:** 19/03/2014

تطوير نظرية مظهر حول عوامل التجميعية

محمد علي سرجول

قسم الرياضيات- جامعة باموكالي- دنيزلي- تركيا

خلاصة

نرمز بالرمز (X,Y) لمجموعة كل المتتاليات E بحيث قابلة للجمع Y عندما تكون قابلة للجمع X حيث Y,X هي طرق تجميعية . نقوم في هذا البحث بالحصول على خصائص المجموعة $(| c,a |_k , \ulcorner n,p_n |)$ و حيث $k>1$، $a>-1$ (p_n) متتاليات موجبة إختيارية ، وذلك بإستخدام تقنيات دالية تحليلية. ونتيجة لدراستنا هذه نقوم بتطوير بعض النتائج المعروقة.