Kuwait J. Sci. **45 (1)** pp 1-6, 2018

The eigenvalues of some anti-tridiagonal Hankel matrices

Carlos M. da Fonseca

Dept. of Mathematics, Kuwait University, Safat 13060, Kuwait
carlos@csi.kuniv.edu.kw

Abstract

We determine the spectra of two families of anti-tridiagonal Hankel matrices of any order. The approach is much stronger and more concise than those particular cases appearing in the literature. At the same time, it simplifies significantly all the known results up to now.

Keywords: Anti-tridiagonal matrices; Hankel matrices; Chebyshev polynomials, eigenvalues

1. Introduction

In general, an *anti-tridiagonal Hankel matrix* of order n, $H_n(b,a,c)$, is a matrix of the form

$$H_n(b,a,c) = \begin{pmatrix} & & b & a \\ & \iddots & \iddots & c \\ b & \iddots & \iddots & \\ a & c & & \end{pmatrix}_{n\times n}.$$

In the recent years anti-tridiagonal Hankel matrices and their eigenvalues revealed to be of significant importance in many areas of pure and applied mathematics (see (Gutiérrez-Gutiérrez, 2011 and 2008; Gutiérrez-Gutiérrez & Zárraga-Rodríguez, 2016; El-Mikkawy & Rahmo, 2008; Rimas, 2013, 2009, and 2008; Wang, 2014; Wu, 2010; Yin, 2008) and the references therein). For different reasons, the main study is commonly diverted to the analysis of the powers of these matrices. Inevitably, finding the eigenvalues of these matrices is always a key step. In particular, quite recently (Rimas; 2013) found the spectra of

$$H_n(b,a,-b) = \begin{pmatrix} & & b & a \\ & \iddots & \iddots & -b \\ b & \iddots & \iddots & \\ a & -b & & \end{pmatrix}_{n\times n},$$

extending some results in his previous paper (Rimas, 2009), through a very intricate method. A persymmetric case was investigated lately in (Akbulak, da Fonseca, & Yılmaz, 2013). The case $H_n(b,a,b)$ can be found, for example, in (Gutiérrez-Gutiérrez, 2011 and 2008; Wang, 2014).

Another type of anti-tridiagonal Hankel matrices which has become of interest is

$$H_n(b,0,c) = \begin{pmatrix} & & b & 0 \\ & \iddots & \iddots & c \\ b & \iddots & \iddots & \\ 0 & c & & \end{pmatrix}_{n\times n}.$$

The powers of $H_n(b,0,c)$ were recently discussed in (da Silva, 2015; Wang, 2014).

Our main aim here is to provide a common frame for the eigenvalues of $H_n(b,a,\pm b)$ and $H_n(b,0,c)$, providing a direct approach and a better understanding to them.

2. The eigenvalues of $H_n(b,a,-b)$

Let us write $H_n(b,a,-b) = aK_n + bH_n(1,0,-1)$, where K_n is the backwards identity of order n. Since $K_n^2 = I_n$ and $K_nH_n(b,0,-b) = -H_n(b,0,-b)K_n$, we have $K_nH_n(b,0,-b)K_n = -H_n(b,0,-b)$, i.e., $H_n(b,0,-b)$ and $-H_n(b,0,-b)$ are similar. Consequently the eigenvalues of $H_n(b,0,-b)$ are coming in pairs, say, $\lambda,-\lambda$.

Lemma 2.1. The eigenvalues of $H_n(1,0,-1)$ are simple.

Proof. Rearranging conveniently the rows and columns of $H_n(1,0,-1)$, this matrix is permutation similar to

$$\begin{pmatrix} 1 & -1 & & & \\ -1 & 0 & 1 & & \\ & 1 & 0 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}_{p\times p} \oplus$$

$$\oplus \begin{pmatrix} -1 & 1 & & & \\ 1 & 0 & -1 & & \\ & -1 & 0 & 1 & \\ & & 1 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}_{p\times p},$$

where $n = 2p$. Otherwise, it is permutation similar to

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & -1 \\ & & & & -1 & 0 \end{pmatrix}_{n\times n}.$$

Hence, in the even case, the eigenvalues of $H_n(1, 0, -1)$ are (cf. (Akbulak, da Fonseca, & Yılmaz, 2013; da Fonseca, 2007)

$$\lambda_k^{\pm} = \pm 2\cos\left(\frac{2k+1}{n+1}\,\pi\right),$$

$$\text{for } k = 0, \ldots, \tfrac{n-2}{2},$$

while in the second case, they are 0 together with

$$\lambda_k^{\pm} = \pm\sqrt{2\left(1 + \cos\left(\frac{2k\pi}{n+1}\right)\right)},$$

$$\text{for } k = 1, \ldots, \tfrac{n-1}{2}.$$

In each case, they are all distinct. □

Remark 2.1. Obviously, the eigenvalues are, respectively, the same as

$$\begin{pmatrix} \pm 1 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}_{n\times n}.$$

Observe that if n is even, $H_n(b, 0, -b)$ is clearly nonsingular.

In general, if u is an eigenvector of $H_n(b, 0, -b)$ associated with λ, then $K_n u$ is an eigenvector of $H_n(b, 0, -b)$ associated with $-\lambda$. If n is odd, then $(1, 0, 1, \ldots, 0, 1)$ is an eigenvector associated with 0.

Taking into account the orthogonality of the eigenvectors of $H_n(b, 0, -b)$, it is possible to construct an orthonormal basis in $\mathbb{R}^n$ such that it is a direct sum of blocks $\operatorname{diag}(b\lambda_k^+, b\lambda_k^-)$. If n is odd, there is an additional 1×1 block with a zero entry. In this basis, K_n is represented by a direct sum of K_2. Again, if n is odd, there is an additional 1×1 block with a sole entry equal to 1. Consequently, $H_n(b, a, -b)$ is similar to

$$\left(\begin{array}{c|cc|cc|c} a & & & & & \\ \hline & b\lambda_1^+ & a & & & \\ & a & b\lambda_1^- & & & \\ \hline & & & b\lambda_2^+ & a & \\ & & & a & b\lambda_2^- & \\ \hline & & & & & \ddots \end{array}\right),$$

if n is odd, and to

$$\left(\begin{array}{cc|cc|c} b\lambda_0^+ & a & & & \\ a & b\lambda_0^- & & & \\ \hline & & b\lambda_1^+ & a & \\ & & a & b\lambda_1^- & \\ \hline & & & & \ddots \end{array}\right),$$

if n is even.

Theorem 2.2. The eigenvalues of $H_n(b, a, -b)$ are a, together with

$$\pm\sqrt{a^2 + 2b^2\left(1 + \cos\left(\frac{2k\,\pi}{n+1}\right)\right)},$$

$$\textit{for } k = 1, \ldots, \tfrac{n-1}{2},$$

if n is odd, and

$$\pm\sqrt{a^2 + 4b^2\cos^2\left(\frac{2k+1}{n+1}\,\pi\right)},$$

$$\textit{for } k = 0, \ldots, \tfrac{n-2}{2},$$

otherwise.

3. The eigenvalues of $H_n(b, a, b)$

We start with the even order. From the previous case, $H_n(1, 0, 1)$ is permutation similar to

$$\begin{pmatrix} 1 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}_{p\times p} \oplus \begin{pmatrix} 1 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}_{p\times p}, \tag{3.1}$$

where $n = 2p$. This means the eigenvalues of $H_n(1, 0, 1)$ are

$$\lambda_k = 2\cos\left(\frac{2k+1}{n+1}\,\pi\right), \quad \text{for } k = 0, \ldots, \tfrac{n-2}{2},$$

each of multiplicity 2. In this case one eigenvector associated with an eigenvalue in $H_n(1, 0, 1)$ corresponds to a certain zero-nonzero pattern, where the nonzero entries are from the eigenvector of the matrix in the decomposition (3.1) associated to the same eigenvalue. The other eigenvector has an opposite pattern. Therefore, $H_n(b,a,b)$ is similar to

$$\left(\begin{array}{cc|cc|c} b\lambda_0 & a & & & \\ a & b\lambda_0 & & & \\ \hline & & b\lambda_1 & a & \\ & & a & b\lambda_1 & \\ \hline & & & & \ddots \end{array}\right).$$

So, we have the following theorem

Theorem 3.1. The eigenvalues of $H_{2p}(b,a,b)$ are

$$\pm\, a + 2b\cos\left(\frac{2k+1}{n+1}\,\pi\right), \quad \textit{for } k = 0,\ldots,p-1.$$

Let us focus now on the odd order, say $n = 2p+1$. This case is very similar to the $H_n(b,a,-b)$, with the same parity. However, here $H_n(b,a,b)$ will be similar to the diagonal matrix

$$\begin{pmatrix} (-)^p a & & & & & \\ & a + b\lambda_1^+ & & & & \\ & & a + b\lambda_1^- & & & \\ & & & -a + b\lambda_2^+ & & \\ & & & & -a + b\lambda_2^- & \\ & & & & & \ddots \end{pmatrix}.$$

In conclusion, we have the following theorem.

Theorem 3.2. The eigenvalues of $H_{2p+1}(b,a,b)$ are $(-1)^p a$, together with

$$(-1)^{k-1}a \pm b\sqrt{2\left(1+\cos\left(\frac{2k\pi}{n+1}\right)\right)},$$

$$\textit{for } k = 1,\ldots,p.$$

4. About the spectra of $H_n(b,0,c)$

In this section, we extend the previous analysis to the matrices of the type $H_n(b,0,c)$. We recall the notion of symmetric tridiagonal 2-Toeplitz matrix (da Fonseca & Petronilho, 2001 and 2005; Gover, 1994)

$$T_n^{(2)} = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ i & b_2 & a_1 & b_1 & \\ & & b_1 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}_{n\times n}$$

and the characteristic polynomials of some perturbations (Akbulak, da Fonseca & Yılmaz, 2013), which will be crucial for the understanding of the spectra of this family of matrices.

Let us define the polynomials

$$\pi_2(x) = (x-a_1)(x-a_2)$$

and

$$P_k(x) = (b_1 b_2)^k\, U_k\left(\frac{x - b_1^2 - b_2^2}{2b_1 b_2}\right),$$

where U_k's are the Chebyshev polynomials of second kind satisfying the three-term recurrence relations

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)\,, \quad \text{for all } k = 1,2,\ldots \tag{4.1}$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. It is well-known that each U_k satisfies

$$U_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi), \tag{4.2}$$

for all $k = 0,1,2\ldots$. Naturally, we assume that $T_n^{(2)}$ is irreducible, i.e., $b_1 b_2 \neq 0$.

The eigenvalues of $T_n^{(2)}$ are the zeros of the (characteristic) polynomial Q_n defined by

$$Q_{2k+1}(x) = (x-a_1)Pi_k(\pi_2(x))$$

and

$$Q_{2k}(x) = P_k(\pi_2(x)) + b_2^2\, P_{k-1}(\pi_2(x))\,,$$

for $k = 1,2,\ldots$.

In the case of $n = 2k+1$, the eigenvalues of $T_n^{(2)}$ are $\lambda_0 = a_1$ and

$$\lambda_\ell^\pm = \frac{a_1+a_2}{2} \pm$$

$$\sqrt{\frac{(a_1-a_2)^2}{4} + b_1^2 + b_2^2 + 2b_1 b_2\cos\left(\frac{\ell\pi}{k+1}\right)}\,,$$

for $\ell = 1,\ldots,k$. If $n = 2k$, the eigenvalues are

$$\lambda_\ell^\pm = \frac{a_1+a_2}{2} \pm$$

$$\sqrt{\frac{(a_1-a_2)^2}{4} + b_1^2 + b_2^2 + 2b_1 b_2\cos\theta_{k\ell}}\,,$$

for $\ell = 1,\ldots,k$, where each $\theta_{k\ell}$ is a solution in the interval $(0,\pi)$ of the trigonometric equation $b_1\sin((k+1)\theta) + b_2\sin(k\theta) = 0$.

Therefore, the characteristic polynomial of the Jacobi matrix

$$\tilde{T}_{2k+1}^{(2)} = \begin{pmatrix} 0 & b & & & & \\ b & 0 & c & & & \\ & c & 0 & b & & \\ & & b & \ddots & \ddots & \\ & & & \ddots & 0 & c \\ & & & & c & b \end{pmatrix}$$

is

$$p(x) = (x-b)P_k(x^2) + bc^2 P_{k-1}(x^2)\,, \qquad (4.3)$$

while of the matrix

$$\tilde{T}_{2k}^{(2)} = \begin{pmatrix} 0 & b & & & \\ b & 0 & c & & \\ & c & \ddots & \ddots & \\ & & \ddots & 0 & b \\ & & & b & c \end{pmatrix}$$

is

$$p(x) = P_k(x^2) - b(x-b)\,P_{k-1}(x^2)\,. \qquad (4.4)$$

In both cases, the determinant is b^n, where n is the order of the tridiagonal matrix. For more details, the reader is referred to (Akbulak, da Fonseca, & Yılmaz, 2013).

Now, if we want to get the eigenvalues of $H_n(b,0,c)$, we only need to find an appropriate permutation matrix such that it will be similar to a convenient Jacobi matrix. Let us start with the odd case.

Set the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n+1 & \cdots \\ 1 & 2n & 3 & 2n-2 & \cdots & n+1 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} \cdots & 2n-2 & 2n-1 & 2n & 2n+1 \\ \cdots & 4 & 2n-1 & 2 & 2n+1 \end{pmatrix}.$$

Clearly, the permutation matrix P_σ is an involution and $P_\sigma H_{2n+1}(b,0,c) P_\sigma$ is equal to the $(2n+1)\times(2n+1)$ 2-Toeplitz tridiagonal matrix

$$\begin{pmatrix} 0 & b & & & & \\ b & 0 & c & & & \\ & c & 0 & b & & \\ & & b & \ddots & \ddots & \\ & & & \ddots & \ddots & c \\ & & & & c & 0 \end{pmatrix}.$$

Thus the eigenvalues of $H_{2n+1}(b,0,c)$ are 0 and

$$\lambda_\ell^{\pm} = \pm\sqrt{b^2 + c^2 + 2bc\cos\left(\frac{\ell\pi}{n+1}\right)}\,,$$

for $\ell = 1, \ldots, n$.

In conclusion, since the powers of this tridiagonal matrix are known, even in a more general form (cf. Theorem 3.2 in (Rimas, 2012)), if we want to find the powers of $H_{2n+1}(b,0,c)$, one just need to rearrange the rows and columns via the permutation σ and we get Theorem 3 of (da Silva, 2015).

For the even case, in general, we cannot obtain explicit expressions of the eigenvalues. Nevertheless, the formulas arise in a natural way. For, if we consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & n+2 & 3 & n+4 & \cdots & n \end{pmatrix}$$

$$\begin{pmatrix} n+1 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ 2n & \cdots & 4 & n+3 & 2 & n+1 \end{pmatrix},$$

then $H_{2n}(b,0,c)$ becomes permutation similar to

$$\begin{pmatrix} 0 & b & & \\ b & \ddots & \ddots & \\ & \ddots & 0 & c \\ & & c & b \end{pmatrix} \oplus \begin{pmatrix} 0 & c & & \\ c & \ddots & \ddots & \\ & \ddots & 0 & b \\ & & b & c \end{pmatrix},$$

if n is odd, and to

$$\begin{pmatrix} 0 & b & & & \\ b & 0 & c & & \\ & c & \ddots & \ddots & \\ & & \ddots & 0 & b \\ & & & b & c \end{pmatrix} \oplus \begin{pmatrix} 0 & c & & & \\ c & 0 & b & & \\ & b & \ddots & \ddots & \\ & & \ddots & 0 & c \\ & & & c & b \end{pmatrix},$$

otherwise. Thus, for p odd, from (4.3), the eigenvalues of $H_{2p}(b,0,c)$ are the solutions of the equations

$$(x-b)P_p(x^2) + bc^2 P_{p-1}(x^2) = 0$$

and

$$(x-c)P_n(x^2) + bc^2 P_{p-1}(x^2) = 0\,,$$

while, for p even, from (4.4), the eigenvalues are the solutions of the equations

$$P_p(x^2) - c(x-c)\,P_{p-1}(x^2) = 0$$

and

$$P_p(x^2) - b(x-b)\,P_{n-1}(x^2) = 0\,.$$

In any case, the determinant is always equal to $(bc)^n$.

5. Conclusion

We believe that this strategy of rearranging indexes is convenient and more powerful than ones that can be found in the recent literature to calculate the powers of different types of matrices. Unfortunately, this simple approach has not been appropriately explored.

For example, in (da Silva, 2015) it is not clear what are the eigenvalues of $H_n(b,0,c)$ and where they come from. To compute the determinant of $H_n(b,0,c)$, we can use the direct sum above or use the Laplace expansion. Still in this paper, it is worth mentioning that the proof of Lemma 2 in (da Silva, 2015) is not complete, since we have a three terms recurrence relation. Moreover, the result is standard because $P_n(x)$ is the characteristic polynomial of

$$\begin{pmatrix} 0 & 1 & & \\ 1 & \alpha & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \alpha \end{pmatrix}.$$

As a last remark, a "negative" power, say $-n$, of a nonsingular matrix A is simply the inverse of A^n. The rest is elementary linear algebra. Therefore, it is unnecessary to state and prove results involving "negative" powers as was done, for example, in (da Silva, 2015; Wang, 2014).

References

Akbulak, M. , da Fonseca, C.M. & Yılmaz, F. (2013). The eigenvalues of a family of persymmetric anti-tridiagonal 2-Hankel matrices, Applied Mathematics and Computation **225**:352-357. DOI: 10.1016/j.amc.2013.09.014

da Fonseca, C.M. (2007). On the eigenvalues of some tridiagonal matrices, Journal of Computational and Applied Mathematics **200**(1):283-286. DOI: 10.1016/j.cam.2005.08.047

da Fonseca, C.M. & Petronilho, J. (2005). Explicit inverse of a tridiagonal k-Toeplitz matrix, Numerische Mathematik **100**(3): 457-482. DOI: 10.1007/s00211-005-0596-3

da Fonseca, C.M. & Petronilho, J. (2001). Explicit inverses of some tridiagonal matrices, Linear Algebra and its Applications **325**(1-3):7-21. DOI: 10.1016/S0024-3795(00)00289-5

Gover, M.J.C. (1994). The eigenproblem of a tridiagonal 2-Toeplitz matrix, Linear Algebra and its Applications **197/198**:63-78. DOI: 10.1016/0024-3795(94)90481-2

Gutiérrez-Gutiérrez, J. (2011). Powers of complex persymmetric or skew-persymmetric anti-tridiagonal matrices with constant anti-diagonals, Applied Mathematics and Computation **217**(13):6125-6132. DOI: 10.1016/j.amc.2010.12.091

Gutiérrez-Gutiérrez, J. (2008). Powers of real persymmetric anti-tridiagonal matrices with constant anti-diagonals, Applied Mathematics and Computation **206**(2):919-924. DOI: 10.1016/j.amc.2008.10.003

Gutiérrez-Gutiérrez, J. & Zárraga-Rodríguez, M. (2016). Orthogonal diagonalization for complex skew-persymmetric anti-tridiagonal Hankel matrices, Special Matrices **4**:73-79. DOI: 10.1515/spma-2016-0008

El-Mikkawy, M. & Rahmo, E-L. (2008). A new recursive algorithm for inverting general tridiagonal and anti-tridiagonal matrices, Applied Mathematics and Computation **204**(1):368-372. DOI: 10.1016/j.amc.2008.06.053

Rimas, J. (2013). Integer powers of real even order anti-tridiagonal Hankel matrices of the form $\text{antitridiag}_n(a, c, -a)$, Applied Mathematics and Computation **225**:204-215. 10.1016/j.amc.2013.09.029

Rimas, J. (2013). Integer powers of real odd order skew-persymmetric anti-tridiagonal matrices with constant anti-diagonals ($\text{antitridiag}_n(a, c, -a), a \in R \setminus \{0\}, c \in R$), Applied Mathematics and Computation **219**(12):7075-7088. DOI: 10.1016/j.amc.2012.12.051

Rimas, J. (2012). Explicit expression for powers of tridiagonal 2-Toeplitz matrix of odd order, Linear Algebra and its Applications **436**(9):3493-3506. DOI: 10.1016/j.laa.2011.12.025

Rimas, J. (2009). On computing of arbitrary integer powers of odd oirder anti-tridiagonal matrices with zeros in main skew diagonal and elements $1, 1, 1, \ldots, 1; -1, -1, -1, \ldots, -1$ in neighbouring diagonals, Applied Mathematics and Computation **210**(1):64-71. DOI: 10.1016/j.amc.2008.11.001

Rimas, J. (2008). On computing of arbitrary integer powers of even order anti-tridiagonal matrices with zeros in main skew diagonal and elements $1, 1, 1, \ldots, 1; -1, -1, -1, \ldots, -1$ in neighbouring diagonals, Applied Mathematics and Computation **204**(2):754-763. DOI: /10.1016/j.amc.2008.07.021

Rimas, J. (2008). On computing of arbitrary positive integer powers for one type of symmetric anti-tridiagonal matrices of odd order, Applied Mathematics and Computation **203**(2):573-581. DOI: 10.1016/j.amc.2008.05.008

Rimas, J. (2008). On computing of arbitrary integer powers for one type of symmetric anti-tridiagonal matrices of even order, Applied Mathematics and Computation **204**(1):288-298. DOI: 10.1016/j.amc.2008.06.044

da Silva, J.L. (2015). Integer powers of anti-tridiagonal matrices of the for $\text{antitridiag}_n(a, c, -a), a \in \mathbb{R}$, Computers & Mathematics with Applications **69**:1313-1328. DOI: 10.1016/j.camwa.2015.03.016

Wang, H. (2014). Powers of complex persymmetric antitridiagonal matrices with constant antidiagonals, ISRN Computational Mathematics **2014**: Art. 451270. DOI: 10.1155/2014/451270

Wu, H. (2010). On computing of arbitrary positive powers for one type of anti-tridiagonal matrices of even order, Applied Mathematics and Computation **217**(6):2750-2756. DOI: 10.1016/j.amc.2010.08.010

Yin, Q. (2008). On computing of arbitrary positive powers for anti-tridiagonal matrices of even order, Applied Mathematics and Computation **203**(1):252-257. DOI: 10.1016/j.amc.2008.04.031

Submitted: 27/02/2016
Revised : 12/04/2016
Accepted : 13/04/2016

القيم الذاتية لبعض مصفوفات
هانكل المضادة للقطر الثالث

كارلوس م. دا فونسيكا
قسم الرياضيات، جامعة الكويت، الصفاة 13060، الكويت
carlos@csi.kuniv.edu.kw

خـلاصـة

نقوم بتحديد أطياف عائلتين من مصفوفات هانكل المضادة للقطر الثالث بأي ترتيب. وهذا النهج أقوى وأكثر إيجازاً من بعض الحالات الخاصة المنشورة. وفي الوقت نفسه، فإنه إلى حد كبير يُبسط جميع النتائج المعروفة حتى الآن.