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The Lehmer matrix with recursive factorial entries

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ABSTRACT

A generalized Lehmer matrix with recursive entries from Kılıç *et al.* (2010b) is further generalized, introducing three additional parameters and taking recursive factorials instead of a term. Certain formulae are derived for the *LU* and *Cholesky* factorizations and their inverses, as well as the determinants. Then we precisely compute the elements of the inverse of the generalized Lehmer matrix.

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INTRODUCTION

The generalized Fibonacci sequence $\{U_n(p,q)\}$ is defined by

$$U_n(p,q) = pU_{n-1}(p,q) - qU_{n-2}(p,q),$$

with the initial conditions $U_0(p,q) = 0$ and $U_1(p,q) = 1$ for $n > 1$.

When $p = 2$ and $q = 1$, the sequence $\{U_n(2,1)\}$ is reduced to the sequence of natural numbers. When $p = 1$ and $q = -1$, the sequence $\{U_n(1, -1)\}$ is reduced to the well known Fibonacci sequence $\{F_n\}$. Throughout this paper, we consider the case $q = -1$ and we denote $U_n(p,-1)$ with u_n.

An $n \times n$ generalized Lehmer matrix $\mathcal{F}_n = (g_{ij})_{1 \le i, j \le n}$ is defined by

$$g_{ij} = \frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}} = \begin{cases} \dfrac{u_{i+1}}{u_{j+1}} & \text{if } j \ge i, \\[2ex] \dfrac{u_{j+1}}{u_{i+1}} & \text{if } i > j, \end{cases}$$

as a recursive analogue of the Lehmer matrix where u_n is the nth term of the sequence $\{u_n\}$, Kılıç *et al*. (2010b) defined the recursive analogue of the Lehmer matrix and derived its algebraic properties. The authors also gave its *LU* and Cholesky factorizations and so derived explicit formulae for the determinant and inverse.

The Lehmer matrix and its recursive analogue are known as special matrices with known inverses, determinants, etc. For other known special matrices and their properties (inverses, determinants, etc.), we refer to Kılıç (2010), Kılıç *et al*. (2010a), Kılıç *et al*. (2013), Marcus (1960), Newman *et al*. (1958), Stanica (2005).

The purpose of this paper is to introduce a new kind further generalization of the Lehmer matrix. Our approach to the subject will be as follows:

- We consider product of consecutive k terms of the sequence $\{u_n\}$ and briefly denote this product with three parameters by X_n,
- We define the generalized Lehmer matrix, namely, $\mathcal{F}_n$ in terms of X_n's,
- We give *LU* factorization of $\mathcal{F}_n$ as $\mathcal{F}_n = PL_nU_n$, where p is an $n \times n$ unit matrix,
- We derive explicit formula for the determinant,
- We give Cholesky factorization $\mathcal{F}_n = C_nC_n^T$,
- We determine the inverse matrix $\mathcal{F}_n^{-1}$.

A GENERALIZED LEHMER MATRIX WITH 3 ADDITIONAL PARAMETERS

Definition 1. For any integer parameters $\lambda \geq 1$, $r \geq 0$ and $k \geq 1$,

$$X_n := \prod_{s=1}^{k} u_{\lambda(n+s-1)+r}, n > 1,$$

where u_n *is defined as before.*

For $r = 0$ and $\lambda = k = 1$, this definition is reduced to the usual recursive coefficients denoted by u_n.

Definition 2. *An* $n \times n$ *recursive generalized Lehmer matrix, say* $\mathcal{F}_n = (g_{ij})_{1 \leq i, j \leq n}$, *has the following entries:*

$$g_{ij} = \frac{\min\{X_{i+1}, X_{j+1}\}}{\max\{X_{i+1}, X_{j+1}\}} = \begin{cases} \dfrac{X_{i+1}}{X_{j+1}} & \text{if } j \geq i, \\ \dfrac{X_{j+1}}{X_{i+1}} & \text{if } i > j, \end{cases}$$

where u_n *is the* n *th term of the sequence* $\{u_n\}$ *and* $\lambda \geq 1$, $r \geq 0$ *and* $k \geq 1$ *are integer parameters.*

Here we note that the case $r = 0$ and $\lambda = k = 1$ was given in Kılıç *et al.* (2010b) so that we shall study the case $r > 0$ and $\lambda, k > 1$ throughout this paper.

In order to give the *LU* factorization of the matrix $\mathcal{F}_n$ we define two triangular matrices. First, define the $n \times n$ unit lower triangular matrix $L_n = (\ell_{ij})$ with

$$\ell_{ij} = \begin{cases} \dfrac{X_{j+1}}{X_{i+1}} & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

For example, when $n = 4$, the matrix has the following form:

$$L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dfrac{X_2}{X_3} & 1 & 0 & 0 \\ \dfrac{X_2}{X_4} & \dfrac{X_3}{X_4} & 1 & 0 \\ \dfrac{X_2}{X_5} & \dfrac{X_3}{X_5} & \dfrac{X_4}{X_5} & 1 \end{bmatrix}.$$

Before defining an upper triangular matrix for the *LU* factorization of the matrix $\mathcal{F}_n$, we need to introduce another new sequence.

Definition 3. *The sequence* $\{Y_n\}$ *is defined as follows*

$$Y_n := X_{n+1} - X_n, n > 1,$$

where $\lambda \geq 1$, $r \geq 0$ *and* $k \geq 1$ *are integers and* u_n *is defined as before.*

When $r = 0$ and $\lambda = k = 1$, this definition is reduced to the sequence $\{t_n\}$ defined in Kılıç *et al.* (2010b).

Secondly, now define the $n \times n$ upper triangular matrix $U_n = (\upsilon_{ij})$ with $\upsilon_{1j} = \dfrac{X_2}{X_{j+1}}$ for $1 \leq j \leq n$, and

$$\upsilon_{ij}=\begin{cases}\dfrac{(X_i+X_{i+1})Y_i}{X_{i+1}X_{j+1}} & \text{if} 1<i\le j\le n,\\ 0 & \text{if } i>j.\end{cases}$$

From the definition of the sequence $\{Y_n\}$, we restate the matrix U_n with $\upsilon_{1j}=\dfrac{X_2}{X_{j+1}}$ for $1\le j\le n$, and

$$\upsilon_{ij}=\begin{cases}\dfrac{X_{i+1}^2-X_i^2}{X_{i+1}X_{j+1}} & \text{if} 1<i\le j\le n,\\ 0 & \text{if } i>j.\end{cases}$$

For example, when $n = 4$, the matrix has the following form:

$$U_4=\begin{bmatrix}1 & \dfrac{X_2}{X_3} & \dfrac{X_2}{X_4} & \dfrac{X_2}{X_5}\\ 0 & \dfrac{X_3^2-X_2^2}{X_3^2} & \dfrac{X_3^2-X_2^2}{X_3X_4} & \dfrac{X_3^2-X_2^2}{X_3X_5}\\ 0 & 0 & \dfrac{X_4^2-X_3^2}{X_4^2} & \dfrac{X_4^2-X_3^2}{X_4X_5}\\ 0 & 0 & 0 & \dfrac{X_5^2-X_4^2}{X_5^2}\end{bmatrix}.$$

Theorem 4. The PLU factorization of generalized Lehmer matrix is given by

$$\mathcal{F}_n=PL_nU_n$$

where $n > 0$, P , L_n and U_n as defined earlier.

Proof. Assume that $L_nU_n=(h_{ij})$. We consider two cases, $i>j$ and $i\le j.$ For the first situation,

$$\begin{aligned}h_{ij}&=\sum_{m=1}^{n}\ell_{im}\upsilon_{mj}=\sum_{m=1}^{j}\ell_{im}\upsilon_{mj}\\&=\ell_{i1}\upsilon_{1j}+\sum_{m=2}^{j}\left(\frac{X_{m+1}}{X_{i+1}}\frac{\left(X_{m+1}^2-X_m^2\right)}{X_{m+1}X_{j+1}}\right)\\&=\frac{X_2^2}{X_{i+1}X_{j+1}}+\frac{1}{X_{i+1}X_{j+1}}\sum_{m=2}^{j}\left(X_{m+1}^2-X_m^2\right)\\&=\frac{X_2^2}{X_{i+1}X_{j+1}}+\frac{1}{X_{i+1}X_{j+1}}\left(X_{j+1}^2-X_2^2\right)\\&=\frac{X_{j+1}}{X_{i+1}}=g_{ij}\end{aligned}$$

Secondly, for $i \leq j$,

$$h_{ij} = \sum_{m=1}^{n} \ell_{im} \upsilon_{mj} = \sum_{m=1}^{i} \ell_{im} \upsilon_{mj}$$
$$= \ell_{i1}\upsilon_{1j} + \sum_{m=2}^{i} \left(\frac{X_{m+1}}{X_{i+1}} \frac{\left(X_{m+1}^2 - X_m^2\right)}{X_{m+1}X_{j+1}} \right)$$
$$= \frac{X_2^2}{X_{i+1}X_{j+1}} + \frac{1}{X_{i+1}X_{j+1}} \sum_{m=2}^{i} \left(X_{m+1}^2 - X_m^2\right)$$
$$= \frac{X_{i+1}}{X_{j+1}} = g_{ij}$$

Corollary 5. For any integers $n > 0$, $\lambda \geq 1$, $r \geq 0, k \geq 1$, then

$$\det(\mathcal{F}_n) = \prod_{i=2}^{n} \left(\frac{X_{i+1}^2 - X_i^2}{X_{i+1}^2} \right).$$

Now, we give the Cholesky factorization of the generalized Lehmer matrix $\mathcal{F}_n$. For this we define a lower triangular matrix $C_n = (c_{ij})$ with $c_{i1} = \frac{X_2}{X_{i+1}}$ for $1 \leq i \leq n$, and

$$c_{ij} = \begin{cases} \frac{1}{X_{i+1}} \sqrt{X_{j+1}^2 - X_j^2} & \text{if } 1 < j \leq i \leq n, \\ 0 & \text{if } i < j. \end{cases}$$

Theorem 6. The Cholesky factorization of the generalized Lehmer matrix is given by

$$\mathcal{F}_n = C_n C_n^T$$

where C_n is the lower triangular matrix defined as above.

Proof. The proof can be done similarly as in Theorem 1.

Next, we give the inverse of the Lehmer matrix with recursive coefficients $\mathcal{F}_n^{-1}$ by considering its *LU* factorization. Before this, we give the inverses of the matrices L_n and U_n in the following lemma, whose proof is as routine as before.

Lemma 7.

(i) Let $U_n^{-1} = (a_{ij})$ denote the inverse of U_n. Then

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \\ -\dfrac{X_{i+1}^2}{X_i^2 - X_{i+1}^2} & \text{if } i = j > 1, \\ \dfrac{X_{i+1}X_{i+2}}{X_{i+1}^2 - X_{i+2}^2} & \text{if } i = j - 1, \\ 0 & \textit{otherwise}. \end{cases}$$

(ii) Let $L_n^{-1} = (b_{ij})$ denote the inverse of L_n. Then

$$b_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{X_i}{X_{i+1}} & \text{if } i = j + 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Thus the inverse of the matrix $\mathcal{F}_n$ is given by in the following theorem.

Theorem 8. Let $\mathcal{F}_n^{-1} = (h_{ij})$, then

$$h_{ij} = \begin{cases} \dfrac{X_3^2}{X_3^2 - X_2^2} & \text{if } i = j = 1, \\ \dfrac{X_{i+1}X_{i+2}}{X_{i+1}^2 - X_{i+2}^2} & \text{if } 1 \le i \le n-1 \text{ and } j = i \pm 1, \\ \dfrac{X_{n+1}^2}{X_{n+1}^2 - X_n^2} & \text{if } i = j = n \\ \dfrac{X_{i+1}^2\left(X_{i+2}^2 - X_i^2\right)}{\left(X_{i+1}^2 - X_i^2\right)\left(X_{i+2}^2 - X_{i+1}^2\right)} & \text{if } 2 \le i \le n-1 \text{ and } j = i \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Considering $\mathcal{F}_n^{-1} = U_n^{-1}L_n^{-1}$, the proof is obtained by the previous lemma and matrix multiplication.

REFERENCES

Kılıç, E. 2010. The Generalized Fibonomial Matrix, European Journal of Combinatorics **31**: 193-209.

Kılıç, E. & Prodinger, H. 2010a. A Generalized Filbert Matrix, The Fibonacci Quarterly **48(1)**: 29-33.

Kılıç, E. & Prodinger, H. 2013. Variants of the Filbert Matrix, The Fibonacci Quarterly **51(2)**: 153–162.

Kılıç, E. & Stanica, P. 2010b. The Lehmer matrix and its recursive analogue, Journal of Combinatrial Mathematics and Combinatorial Computing **74**: 193-207.

Marcus, M. 1960. Basic theorems in matrix theory, National Bureau of Standards Applied Mathematics Series **57**: 21–24.

Newman, M. & Todd, J. 1958. The evaluation of matrix inversion programs, Journal of the Society Industrial and Applied Mathematics **6**: 466 - 476.

Stanica, P. 2005. Cholesky factorizations of matrices associated with *r*-order recurrent sequences, Integers: Electronic Journal of Combinatiorial Number Theory **5(2)**: #A16.

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مصفوفة لهمر ذات المدخلات العاملية المرتدة

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