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A note on the simplicity of C^*-algebras of edge-colored graphs

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ABSTRACT

We modify the definitions of hereditary and saturated subsets and Condition (L) for edge-colored graphs. We then give three necessary conditions for the simplicity of C^*-algebras of edge-colored graphs.

Keywords: Edge-colored graph C^*-algebras; amalgamated free product; simplicity.

INTRODUCTION

In (Duncan, 2010), to compute free products of graph C^*-algebras, the C^*-algebra of an edge-colored graph was introduced. An edge-colored graph $G = (V, E, r, s, d)$ is a directed graph $(V, E, r, s,)$consisting of a countable vertex set V, a countable edge set E, and range and source maps with an edge-coloring map $d: E \to \mathbb{N}$. We will denote the edge-colored graph by (G, d) when the coloring map is obscure. Whereas C^*-algebras of edge-colored graphs and usual directed graphs have similar properties, the structure of C^*-algebras of edge-colored graphs are more complicated. One basic difference is this: the generating partial isometries of a C^*-algebras of edge-colored graphs may have non-orthogonal ranges. However, for an arbitrary edge-colored graph G,$C^*(G)$ may be considered as a certain free product of $C^*(G_i)$ where G_i are 1-colored subgraphs of G. In (Duncan, 2010), it is claimed that if $C^*(G)$ is a simple C^*-algebra, then all $C^*(G_i)$ are also simple. By an example, we show that this claim is not correct. The aim of this short paper is to modify this proposition. In Theorem 2.5, we give three necessary conditions for G such that $C^*(G)$ is simple.

PRELIMINARIES

In this section, we recall some definitions and properties of C^*-algebras associated to edge-colored graphs.

If $e_1, \ldots, e_n$ are edges in G such that $s(e_i) = r(e_{i+1})$ for $1 \leq i < n$, then $\alpha = e_1 \ldots e_n$ is called a *path* of length $|\alpha| = n$ with the range $r(\alpha) = r(e_1)$ and the source $s(\alpha) = s(e_n)$. For $n \geq 0$, we define G^n to be the set of paths of length n, and $G^* := \bigcup_{i=0}^{\infty} G^n$ the set of all finite paths. Note that we consider the vertices in V to be paths of length zero. A *closed path based at* v is a path $\alpha \in G^* \setminus V$ such that$v = r(\alpha) = s(\alpha)$. If $r(\alpha) = s(\alpha)$ and $r(e_i) \neq r(e_1)$ for every $i > 1$, then α is called a *simple closed path*.

A Cuntz-Krieger G-family is a set $\{P_v : v \in V\}$ of mutually orthogonal projections and a set $\{S_e : e \in E\}$ of partial isometries which satisfy the Cuntz-Krieger relations:

(CK1) $S_e^* S_f = \delta_{e,f} P_{s(e)}$ if $e, f \in E$ with $d(e) = d(f)$,

(CK2) $S_e S_e^* \leq P_{r(e)}$ if $e \in E$, and

(CK3) $P_v = \sum_{r(e)=v, d(e)=i} S_e S_e^*$ if $i \in$ rang (d) and $0 < |r^{-1}(v) \cap d^{-1}(i)| < \infty$.

Definition 1.1. Let Gbe an edge-colored graph. The C*-algebra of G, denoted by $C^*(G)$, is the universal C*-algebra generated by a Cuntz-Krieger$G$-family $\{s_e, p_v : v \in V, e \in E$.

The universality of $C^*(G)$ means that if $\{t_e, q_v : v \in V, e \in E\}$ is a Cuntz-Krieger G-family in a C*-algebra A, then there exists a $*$-homomorphism $\varphi : C^*(G) \to A$ such that $\varphi(p_v) = q_v$ and $\varphi(s_e) = t_e$ for $v \in V,\ e \in E$. Note that the universal property yields that $C^*(G)$ is unique (up to isomorphism). For this, let Abe a universal C*-algebra generated by a Cuntz-Krieger G-family $\{t_e, q_v : v \in V, e \in E$. Then there is a $*$-homomorphism $\varphi : C^*(G) \to A$ such that $\varphi(p_v) = q_v$ and $\varphi(s_e) = t_e$ for all$v \in V,\ e \in E$. On the other hand, the universality of A implies that there is a $*$-homomorphism $\psi : A \to C^*(G)$ with $\psi(q_v) = p_v$ and $\psi(t_e) = s_e$ for $v \in V, e \in E$. Then, by checking on the generators, we see that ψ is the inverse of φ and so $A \cong C^*(G)$.

Remark 1.2. There are two basic differences between C*-algebras of edge-colored graphs and that of usual graphs. First, if $e, f \in r^{-1}(v)$ with $d(e) \neq d(f)$, then $s_e^* s_f \neq 0$. (This follows from the free product construction in Proposition 1.5.) Second, if a vertex $v \in V$ receives edges with different colors, then there may be edges $e_1 \ldots e_n \in r^{-1}(v)$ such that $p_v \quad \sum_{i-1}^{n} s_{ei} s_{ei}^*$.

The construction of C*-algebras of edge-colored graphs in the following proposition is similar to that of C*-algebras of 1-colored graphs (AnHuef & Raeburn, 1997). For a path $\alpha = e_1 \ldots e_n$, we define $s_\alpha := s_{e_1} \ldots s_{e_n}$. Also, we use the notation $\hat{\alpha} := e_1$ for the first edge of path $\alpha = e_1 \ldots e_n$.

Proposition 1.3. *If G is an edge-colored graph, then*

$$C^*(G) = \overline{\text{span}}\Big\{\prod_{t=1}^{n} s_{\alpha_i}s^*_{\beta_i} : \alpha_i, \beta_i \in G^*, s(\alpha_i) = s(\beta_i) \text{ for } 1 \leq i \leq n, \\ r(\alpha_{i+1}) = r(\beta_i) \text{ and } d(\widehat{\alpha_{i+1}}) \neq d(\widehat{\beta_i}) \text{ for } 1 \leq i \leq n\Big\}$$

Proof. Since $s^*_e s_f = \delta_{e,f} p_{s(e)}$ when $d(e) = d(f)$, and $s^*_e s_f = 0$ when $r(e) \neq r(f)$, we see that every multiplication $(s_\alpha s^*_\beta)(s_\gamma s^*_\delta)$ may be written as $(s_{\alpha'} s^*_{\beta'})(s_{\gamma'} s^*_{\delta'})$ such that $s(\alpha') = s(\beta'), s(\gamma') = s(\delta')$, $r(\beta') = r(\gamma')$, and $d(\hat{\beta'}) \neq d(\hat{\gamma'})$. AnHuef & Raeburn (1997) for some similar formulas.) This allows us to write any words in the generators $\{s_e, p_v\}$ as a multiplication $(s_{\alpha_1} s^*_{\beta_1}) \ldots (s_{\alpha_n} s^*_{\beta_n})$ satisfying $s(\alpha_i) = s(\beta_i)$ for $1 \leq i \leq n$, and $r(\alpha_{i+1}) = r(\beta_i)$, $d(\widehat{a_{i+1}}) \neq d(\widehat{\beta_i})$ for $1 \leq i < n$. Since the set of these multiplications are closed under involution, we conclude the statement.

Definition 1.4. Let G be an edge-colored graph. An element $x = \alpha_1\beta_1^* \ldots \alpha_n\beta_n^*$ is called a *chain of admissible paths* (or more concisely, a *chain*) *in* G whenever $\alpha_i, \beta_i \in G^*$, $s(\alpha_i) = s(\beta_i)$ for $1 \leq i \leq n, r(\alpha_{i+1}) = r(\beta_i)$ and $d(\widehat{a_{i+1}}) \neq d(\widehat{\beta_i})$ for $1 \leq i < n$. The vertices$r(\alpha_{i+1}) = r(\beta_i), 1 \leq i \leq n$, are called *nodes* of the chain.

Now we establish some notations. For each i in $rang(d)$, we denote $G_i := (V, d^{-1}(i), r, s)$ the 1-colored subgraph associated to i. Also, for any vertex $v \in V$, we denote G_v the edge-colored subgraph of G with the vertices set V and the edges set $r^{-1}(v) := \{e \in E : r(e) = v\}$.

The following propositions show that $C^*(G)$ may be considered as a certain amalgamated free product of the C*-algebra of subgraphs. Recall that if $\{A_i\}$ is a collection of $C*$-algebras, the *free product*$*A_i$ is the universal $C*-$algebra generated by copies of the A_i with no additional relations. More generally, if P is a $C*$-subalgebra of each A_i, the *amalgamated free product*$*_P A_i$ is the universal $C*$-algebra generated by copies of the A_i with the copies of P identified.

Proposition 1.5 (Duncan, 2010)**.** *Let* $\boldsymbol{G}$ *be an edge-colored graph. Then* $\boldsymbol{C^*(G) = *_P C^*(G_i)}$*, where* P *is the* C*-*subalgebra of each* $\boldsymbol{C^*(G_i)}$ *generated by* $\{\boldsymbol{p_v : v \in V}\}$.

If $e,f \in r^{-1}(v)$ with $d(e) \neq d(f)$, then the free product construction of $C^*(G)$ in Proposition 1.5 implies that $s^*_e s_f \neq 0$. This concludes that $s_{\alpha_1} s^*_{\beta_1} \ldots s_{\alpha_n} s^*_{\beta_n}$ is nonzero, where $x = \alpha_1\beta_1^* \ldots \alpha_n\beta_n^*$ is a chain in G.

Definition 1.6. Let G be a graph and let d_1, d_2 be two colorings on G. We say that d_1 and d_2 are *equivalent* on G if for each $v \in V$ there exists a bijective map $\psi : d_1(r^{-1}(v)) \to d_2(r^{-1}(v))$ such that $\{e \in r^{-1}(v) : d_1(e) = i\} = \{e \in r^{-1}(v) : d_2(e) = \psi(i)\}$ for all $i \in \mathbb{N}$.

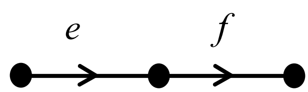
Suppose that d_1 and d_2 are two equivalent colorings on G. By Definition 1.6, we observe that every Cuntz-Krieger(G, d_1)-family is also a Cuntz-Krieger (G, d_2)-family and vice-versa (because the relations are identical). So, the universal property implies that the $C*$-algebras of (G, d_1) and (G, d_2) are isomorphic. Therefore, we have the following corollary.

Corollary 1.7. *Let G be a graph. If d_1 and d_2 are two equivalent colorings on G, then $C^*(G, d_1) \cong C^*(G, d_2)$.*

Remark 1.8. Let G be an edge-colored graph and ψ: $\mathbb{N} \to \mathbb{N}$ is a bijective map. Corollary 1.7 implies that for each vertex $v \in V$, $C^*(G)$ will not be changed (up to isomorphism) if we change the colors of $r^{-1}(v)$ by setting the coloring $\psi \circ d$ on $r^{-1}(v)$.

SIMPLICITYOF$C^*(G)$

In Duncan (2010), it is claimed that if $C^*(G)$ is simple, then all $C^*(G_i)$ are also simple. It seems that this claim is incorrect. For example, suppose that G is the edge-colored graph with the coloring $d(e) = 1$ and $d(f) = 2$. Since G is 1-colorable by Remark 1.8 (for example, one may change the coloring by redefining $d(f) = 1$), we have which is a simple C*-algebra. While, both $C^*(G_1)$ and $C^*(G_2)$ are not simple.



Let G be a 1-colored graph. It is well-known that $C^*(G)$ is simple if and only if V has no nontrivial saturated hereditary subsets and G satisfies Condition (L) (Szymanski, 2001; Bates *et al,*. 2000; Tomforde, 2006). When G is not 1-colorable, the structure of $C^*(G)$ is more complicated than that in the 1-colored setting. However, we can modify the definitions of saturated and hereditary subset and Condition (L) for edge-colored graphs. Then, in Theorem 2.4, we give necessary conditions for the simplicity of $C^*(G)$.

Definition 2.1. Let $G = (V, E, r, s, d)$ be an edge-colored graph. A subset $H \subseteq V$ is called *hereditary* if for any edge $e \in E$ with $r(e) \in H$ we have $s(e) \in H$. Also, we say that $H \subseteq V$ is *saturated* if for each vertex$v$ with $0 < \#\{e \in E : r(e) = v, d(e) = i\} < \infty$ and $\{s(e) : r(e) = v, d(e) = i\} \subseteq H$ for some $i \in$ rang (d), we have $v \in H$. It is not difficult to verify by these definitions that H is a saturated hereditary subset of V if and only if H is hereditary and saturated in all 1-colored subgraphs G_i.

Definition 2.2. Suppose that $G = (V, E, r, s, d)$ is an edge-colored graph and H is a saturated hereditary subset of V. For each $i \in$ rang (d),we denote by B_H^i the set of all vertices in V which receive infinitely many edges with color i such thatonly finitely many of the memitted from$V \backslash H$; that is

$$B_H^i := \{v \in V : |r^{-1}(v) \cap d^{-1}(i)| = \infty \text{ and } 0 < |r^{-1}(v) \cap d^{-1}(i) \cap s^{-1}(V \backslash H)| < \infty\}.$$

Also, for any $v \in B_H^i$, we denote

$$p_{v,H}^i := p_v - \sum_{r(e)=v, d(e)=i, s(e)\notin H} s_e s_e^*.$$

Recall from (Bate *et al.,* 2002) that if G is a 1-colored graph, I_{H,B_H} is the (closed two sided) ideal of $C^*(G)$ generated by $\{p_v, p_{w,H} : v \in H,\ w \in B_H\}$ and $\pi : C^*(G) \to C^*(G)/I_{H,B_H}$ is the quotient map, then $\pi(p_v) \neq 0$ for all $v \in V \backslash H$. Also, we have $C^*(G)/I_{H,B_H} \cong C^*(G \backslash H)$, where $G \backslash H := (V \backslash H, s^{-1}(V \backslash H), r, s)$.

Definition 2.3. Let $\alpha = e_1 \ldots e_n$ be a closed path in G. An edge f is called *an entrance for α with same color* if there exists $1 \leq i \leq n$ such that $f \neq e_i$, $r(f) = r(e_i)$, and $d(f) = d(e_i)$. An edge-colored graph G is said *to satisfy Condition (L)* if every closed path in G has entrances with same color.

Note that if G satisfies Condition (L), then each 1-colored subgraph G_i also satisfies Condition (L). But the converse is not true.

The following lemma is well-known for C^*-algebras of 1-colored graphs. However, for sake of completeness we give a proof.

Lemma 2.4. *Let G be a 1-colored graph. If G has a closed path without any entrance, then $C^*(G)$ contains an ideal J such that $p_v \notin J$ for all $v \in V$.*

Proof. Let $\alpha = e_1 \ldots e_n$ be a closed path in G that has no entrances. Then $X := \{r(e_i)\}_{i=1}^n$ is a hereditary subset in G. If we denote $G_X := (X, r^{-1}(X), r, s)$ and $p_X := \sum_{v \in X} p_v$, then I_X the ideal of $C^*(G)$ generated by p_X is Morita-equivalent to $C^*(G_X)$ and the map $I \mapsto p_X I p_X$ is a one-to-one correspondence between ideals in I_X and ideals in $C^*(G_X)$ (Bate *et al.,* 2002). Recall that $C^*(G_X) \cong C(\mathrm{T}) \otimes M_n(\mathbb{C})$, where T is the unit circle in $\mathbb{C}$ (Raeburn, 2005), and so $C^*(G_X)$ contains infinitely many non-gauge-invariant ideals (corresponding to nontrivial closed subsets of T). Thus we may choose an ideal $\tilde{J}$ in $C^*(G_X)$ which contains no vertex projections and let J be the ideal in I_X corresponding to $\tilde{J}$. Note that J is also an ideal of $C^*(G_X)$. If there exists a vertex $v \in X$ such that $p_v \in J$, then $\tilde{J}$ contains a vertex projection which is a contradiction. If there is $v \in V \backslash X$ such that $p_v \in J \subseteq I_X$, then v must belong to $\overline{X}$ the hereditary and saturated closure of X (this follows from the facts $I_X = I_{\overline{X}}$ and $p_v \notin I_{\overline{X}}$ for all $v \in V \backslash \overline{X}$). But the construction $\overline{X}$ of X in (Bate *et al.,* 2002) implies that there is a path from α to v, and so $p_{r(\alpha)} \in J$ which is impossible by the above argument. Therefore, J is the desired ideal.

Theorem 2.5. *Let G be an edge-colored graph. If $C^*(G)$ is simple, then*

(1) *the only saturated hereditary subsets of V are $\emptyset$ and V,*

(2) *G satisfies Condition (L), and*

(3) *for any (not necessary distinct) pair of vertices v and wthat* $E_{v,w} := \{e \in E : r(e) = v, s(e) = w\}$ *is finite, the subgraph* $G_{v,w} := (\{v, w\}, E_{v,w}, r, s, d)$ *is 1-colored.*

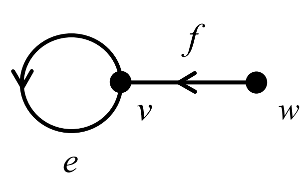
Proof.

(1) Suppose that H is a nontrivial saturated hereditary subset of V. So, for each $i \in \text{rang}\,(d)$, H is hereditary and saturated in G_i. If I^i_{H,B_H} is the ideal of $C^*(G_i)$ generated by $\{p_v, p^i_{w,H} : v \in H, w \in B^i_H\}$, then each I^i_{H,B_H} is a nontrivial ideal in $C^*(G_i)$. Hence, the quotient maps $\pi_i : C^*(G_i) \to C^*(G_i)/I^i_{H,B_H} \cong C^*(G_i \backslash H)$ induce a nontrivial representation$*\pi_i : *_P C^*(G_i) \to *_{P_{V\backslash H}}(C^*(G_i \backslash H))$ and so $C^*(G)$ is not simple.

(2) Suppose on the contrary that G does not satisfy Condition (L) and $\alpha = e_1 \ldots e_n$ is a closed path based at v which has no entrances with same color. By replacing α with a closed subpath of it with minimal length, we may assume that $r(e_i) \neq r(e_j)$for $i \neq j$. Since the vertices $X := \{r(e_i) : 1 \leq i \leq n\}$ are distinct and α is a closed path which has no entrances with same color, by Remark 1.8, we may define an equivalent coloring (denote it again by d) such that $d(e_1) = \cdots = d(e_n)$. Without loss of generality, we suppose that $d(e_i) = 1$. Since α has no entrances with same color, α is a simple closed path with no entrances in G_1. By Lemma 2.4, there exists an ideal J in $C^*(G_1)$ such that $p_v \notin J$ for all $v \in V$. Now if $\pi_1 : C^*(G) \to C^*(G_1)/J$ is the quotient map, and for $i \geq 2$, $\pi_i : C^*(G_i) \to C^*(G_i)$ are the identity maps, π_i's induce a representation $*\pi_i : C^*(G) \to (C^*(G_1)/J) *_P (*_P\{C^*(G_i) : i \geq 2\})$. Note that the orthogonality of p_v's implies that $J \cap P = \emptyset$, and so the elements $p + J$, for $p \in P$, are all nonzero in $C^*(G_1)/J$. This ensures that the free product $(C^*(G_1)/J) *_P (*_P\{C^*(G_i) : i \geq 2\})$ is not the zero algebra (because it contains a copy of P). Since $J \subseteq \ker *\pi_i$, we see that $*\pi_i$ is nontrivial and hence $C^*(G)$ is not simple.

(3) The proof of this part is similar to that of (Duncan, 2010).

Example 2.6. Let Gbe the edge-colored graph



with the coloring $d(e) = 1$ and $d(f) = 2$. Since G does not satisfy Condition (L), Theorem 2.5 implies that $C^*(G)$ is not simple. In fact $C^*(G) \cong C(\mathbb{T}) \otimes M_2(\mathbb{C})$. For this, consider the matrix algebra $C(\mathbb{T}) \otimes M_2(\mathbb{C})$ with the involution

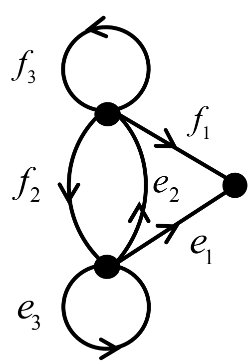
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

If we set

$$p_v := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ p_w := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ s_e := \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \ and\, s_f := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

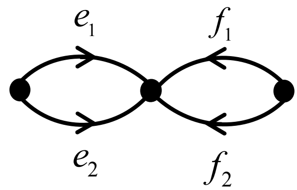
Where u is the unitary generating $C(\mathbb{T})$, then the set $X = \{s_e, s_f, p_v, p_w\}$ is a Cuntz-Krieger G-family in $C(\mathbb{T}) \otimes M_2(\mathbb{C})$. Let A be the C*-subalgebra of $C(\mathbb{T}) \otimes M_2(\mathbb{C})$ generated by X. It is not difficult to see that A is isomorphic to $C^*(G)$. Indeed, if $\{t_e, t_f, q_v, q_w\}$ is a Cuntz-Krieger$G$- family in a C*-algebra B, then the fact $s_e^* s_f \neq 0$ induces a well-defined $*$-homomorphism $\varphi : A \to B$ such that $\varphi(p_v) = q_v$, $\varphi(p_w) = q_w$, $\varphi(s_e) = t_e$, and $\varphi(s_f) = t_f$. Therefore, A is a universal C*-algebra generated by a Cuntz-Krieger G-family and so it is isomorphic to $C^*(G)$. On the other hand, A contains $C(\mathbb{T}) \otimes \theta_{1,1}$ and all the matrix units, and so it is equal to $C(\mathbb{T}) \otimes M_2(\mathbb{C})$.

Example 2.7. The edge-colored graph



with the coloring $d(e_i) = 1$ and $d(f_i) = 2$ was considered in (Duncan 2010). Since G has closed paths without any entrance with same color, Theorem 2.4 implies that $C^*(G)$ is not simple. In (Duncan, 2010), the author showed this by constructing a specific isomorphism between $C^*(G)$ and the Toeplitz algebra. However, our criteria for non-simplicity are simpler to verify.

However, it seems that our conditions in Theorem 2.5 are not sufficient for simplicity. For example, let G be the edge-colored graph



with the coloring $d(e_i) = 1$ and $d(f_i) = 2$. If we set $I := \langle s_{e_1}^* s_{f_1} \rangle$, then $I$is a nontrivial ideal of $C^*(G)$ (because it contains no vertex projections). Therefore, $C^*(G)$ is not simple while G satisfies the conditions of Theorem 2.5.

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مذكرة حول بساطة جبريات C^* للبيانات الملونة الحروف

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خلاصة

نقوم في هذا البحث بتحوير التعاريف للمجموعات الجزئية الوراثية المشبعة، وللشرط (L) للبيانات الملونة الحروف. ثم نعطي بعد ذلك ثلاثة شروط ضرورية لبساطة جبريات C^* للبيانات الملونة الحروف.