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Omega and related polynomials of polyomino chains of 4 *k*-cycles

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ABSTRACT

Omega polynomial of a graph G is defined, on the ground of "opposite edge strips" *ops*: $\Omega(G;x) = \sum_c m(G,c)x^c$, where $m(G,c)$is the number of *ops* strips of length c. The Sadhana polynomial $Sd(G;x)$can also be calculated by *ops* counting. In this paper we compute these polynomials for polyomino chains of 4k-cycles. Also by using Omega polynomial we can compute the (edge) PI_e polynomial for this graph.

Keywords: Omega polynomial; sadhana polynomial; PI_e polynomial; strips; polyomino chain.

INTRODUCTION

Let $G = (E, V)$be a connected graph, with the vertex set $V(G)$ and edge set $E(G)$. Two edges $e = uv$ and $f = xy$ of G are called *codistant e co f*, if they obey the following relation (John *et al.,* 2007):

$$d(v,x) = d(v.y) + 1 = d(u,x) + 1 = d(u,y)$$

Relation *co* is reflexive, that is, e *co* e holds for any edge $e$of G; it is also symmetric, if e *co* f then f *co* e. In general, relation *co* is not transitive; an example showing this fact is the complete bipartite graph $K_{2,n}$. If “*co*” is also transitive, thus an equivalence relation, then G is called a *co-graph* and the set of edges $C(e) = \{f \in E(G) | f\, co\, e\}$ is called an *orthogonal cut co* of G, $E(G)$ being the union of disjoint orthogonal cuts:

$$E(G) = C_1 \cup C_2 \cup \cdots \cup C_k, \ C_i \cap C_j = \phi, \ i \neq j.$$

Let $e = uv$ and $f = xy$ be two edges of G,which are *opposite* or topologically parallel and denote this relation by *e op f*. A set of opposite edges, within the same face/ring, eventually forming a strip of adjacent faces/rings, is called an *opposite* edge *strip ops*, which is a quasi-ortogonal cut *qoc* (*i.e.*, the transitivity relation is not necessarily obeyed). Note that *co* relation is defined in the whole graph, while *op* is defined only in a face/ring. The length of *ops* is maximal, irrespective of the starting edge. Let $m(G,s)$ be the number of *ops* strips of length s. The Omega polynomial is defined as (Diudea, 2006)

$$\Omega(G; x) = \sum_c m(G, c) \cdot x^c.$$

The first derivative (in $x = 1$) equals the number of edges in the graph

$$\Omega'(G; 1) = \sum_c m(G, c) \cdot c = |E(G)|.$$

The Sadhana index $Sd(G)$ was defined by Khadikar *et al.* (2002) as

$$Sd(G) = \sum_c m(G, c)(|E(G)| - c),$$

where $m(G,c)$ is the number of strips of length c.

The Sadhana polynomial $Sd(G; x)$ was defined by Ashrafi *et al.* (2008b) as:

$$Sd(G; x) = \sum_c m(G, c) \cdot x^{|E)G)|-c}.$$

Clearly, the Sadhana polynomial can be derived from the definition of Omega polynomial by replacing the exponent $c$by $|E(G) - c|$. Then the Sadhana index will be the first derivative of $Sd(G; x)$ evaluated at $x = 1$.

THE MAIN RESULTS

In this section we compute the Omega polynomial of k-polyomino chain, which is the main results in this paper. A *k-polyomino system* is a finite 2-connected plane graph, such that each interior face (also called *cell*) is surrounded by a regular $4k$-cycle of length one. In other words, it is an edge-connected union of cells. For the origin of polyominoes see, for example, Klarner (1997) and Golomb (1954; 1965).

For calculating the omega polynomial of a k-polyomino chain, we introduce some concepts for a k-polyomino chain. A *kink* of a k-polyomino chain is any branched or angularly connected $4k$-cycle. A *segment* of a k-polyomino chain is a maximal linear chain in the polyomino chain, including the kinks and/or terminal $4k$-cycles at its end. The number of $4k$-cycles in a segment S is called its *length* and is denoted by $\ell(S)$. For any segment $S$of a polyomino chain with $n \geq 2$ $4k$-cycles one has $2 \leq \ell(S) \leq n$.

In particular, a k-polyomino chain is a *linear chain,* if and only if, it contains exactly one segment, see Figure 1. A k-polyomino chain is a *zig-zag chain,* if and only if, the length of each segment is 2, see Figure 2.

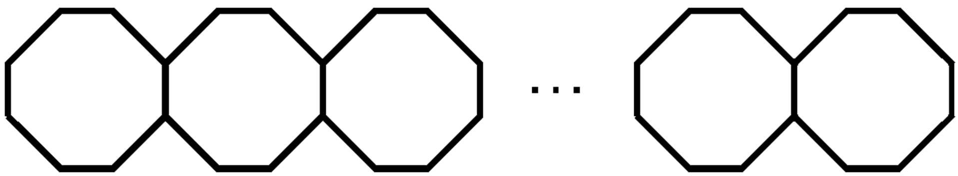

Fig. 1. The linear chain of 8-cycles

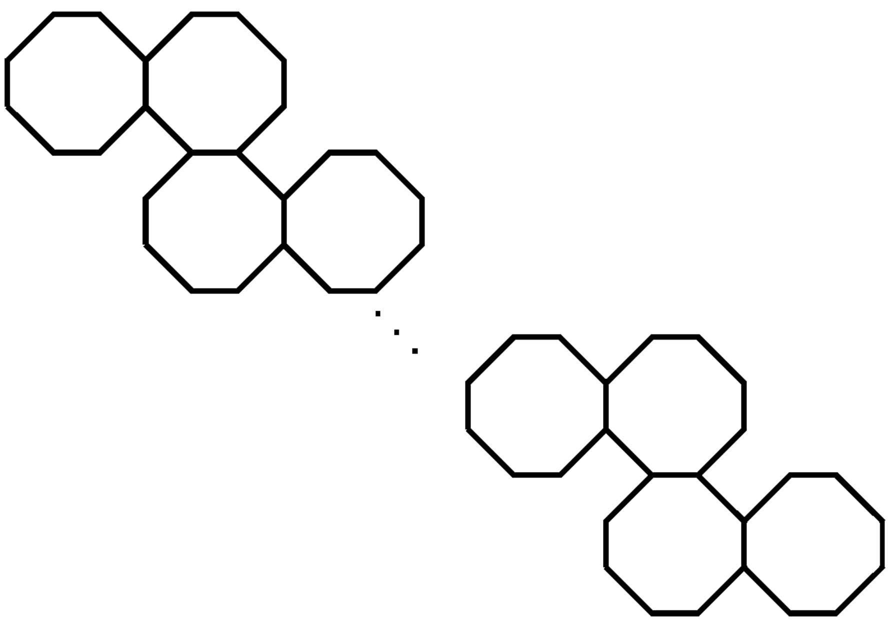

Fig. 2. The zig-zag chain of 8-cycles

A k-polyomino chain consists of a sequence of segments $S_1, S_2, \ldots, S_s$, $s \geq 1$, with Lengths $\ell(S_i) \equiv \ell_i, \quad i = 1, 2, \ldots, s$, where

$$\sum_{i=1}^{s} \ell_i = n + s - 1$$

(n denotes the number of $4k$-cycles of the polyomino chain), since two neighboring segments have always one $4k$-cycle in common. In the following we will abbreviate the vector of lengths by ℓ, i.e., $\ell = (\ell_1, \ldots, \ell_s)$.

Ghorbani & Ghazi (2010) computed these polynomials only for the zig-zag chain of 8-cycles. In this paper, we compute the mention polynomails of polyomino chains of $4k$-cycles with arbitrary k. Here our notations are standard and mainly taken from Ashrafi *et al.,* (2008a) and (Klarner, (1997).

Theorem 1. Let $B_{n,k|\ell}$ be a k-polyomino chain with n $4k$-cycles consisting of $s \geq 1$ segments $S_1, S_2, \ldots, S_s$ with lengths $\ell_1, \ell_2, \ldots, \ell_s$. Then

$$\Omega(B_{n,k|\ell}; x) = [(2k-1)n - s + 1]x^2 + \sum_{i=1}^{s} x^{(\ell_i+1)} \text{and}$$

$$Sd(B_{n,k|\ell}; x) = [(2k-1)n - s + 1]x^{(4k-1)n-1} + \sum_{i=1}^{s} x^{(4k-1)n-\ell_i}.$$

Proof. Every segment S_i has two kinds of strips: one of lengths 2 and one of lengths $\ell_i + 1$. So $B_{n,k|\ell}$ has $s+1$strips $C_0, C_1, C_2, \ldots, C_s$ of length $2, \ell_1 + 1, \ell_2 + 1, \ldots, \ell_s + 1$, respectively, see Figure 3. There is one strip equivalent with $|C_i|$, $i = 1, \ldots, s$, i.e., $m(B_{n,k|\ell}, \ell_i + 1) = 1,\ 1 \leq i \leq s$. Thus the number of strips equivalent with $|C_0| = 2$ is $\frac{1}{2}[|E(B_{n,k|\ell})| - \sum_{i=1}^{s}(\ell_i + 1)]$. We have $|E(B_{n,k|\ell})| = (4k-1)n + 1$ and $\sum_{i=1}^{s} \ell_i = n + s - 1$.

Hence $m(B_{n,k|\ell}, 2) = (2k-1)n - s + 1$. Now, by the definitions of Omega and Sadhana polynomials, the proof is completed.

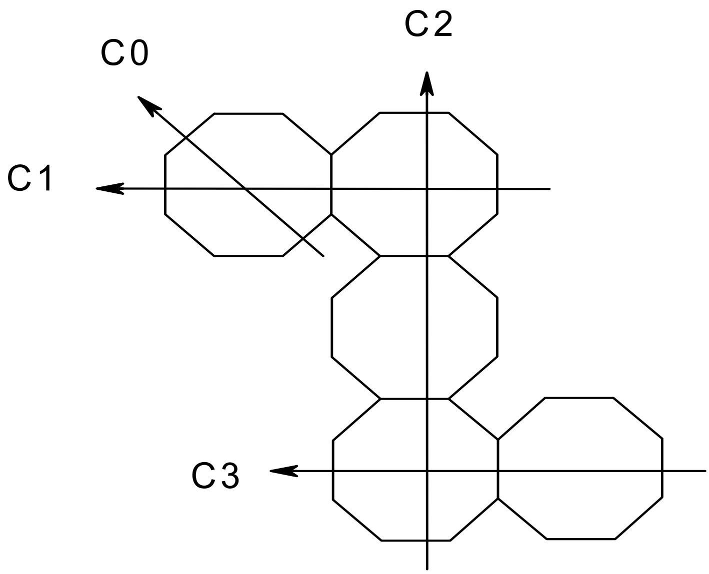

Fig. 3. The graph $B_{5,2|3}$

The following corollaries are resulted from Theorems 1.

Corollary 2. (*Linear chain*) In the case s = 1 and $\ell_1 = n$ the graph $B_{n,k|\ell}$ is the linear chain $L_{n,k}$ of n 4k-cycles. . Then

$$\Omega(L_{n,k}; x) = (2k-1)nx^2 + x^{n+1} \quad \text{and}$$

$$Sd(L_{n,k}; x) = (2k-1)nx^{(4k-1)n-1} + x^{(4k-2)n}.$$

Corollary 3. (*Zig-zag chain*) In the case $s = n - 1$ and $\ell_i = 2$ for all i the graph $B_{n,k|\ell}$ is the zig-zag chain $Z_{n,k}$ of n 4k-cycles. Then

$$\Omega(Z_{n,k}; x) = ((2k-2)n + 2)x^2 + (n-1)x^3 \text{ and}$$

$$Sd(Z_{n,k}; x) = 2((k-1)n + 1)nx^{(4k-1)n-1} + (n-1)x^{(4k-1)n-2}.$$

Let G be a connected graph, u and $v$be vertices of G and $e = uv$. The number of edges of G lying closer to u than to v is denoted by $n_{eu}(e|G)$ and the number of edges of G lying closer to v than to u is denoted by $n_{ev}(e|G)$. The PI_e polynomial of G is defined as (Ashrafi *et al.,* 2006)

$$PI(G; x) = \sum\nolimits_{\{u,v\} \subseteq V} x^{N(u,v)},$$

where $N(u,v) = n_{eu}(e|G) + n_{ev}(e|G)$, if $e = uv$; and $= 0$, otherwise.

Now, If G be a bipartite graph, PI_e can be computed by (Diudea *et al.*, 2008):

$$PI_e(G;x) = \sum\nolimits_c m(G,c) \cdot c \cdot x^{|E(G)|-c} \quad .$$

Then the PI index will be the first derivative of $PI(G;x)$ evaluated at $x = 1$.In other words, by using Omega polynomial in bipartite graph we can compute the PI_e polynomial and then PI index.

Hence, the following Theorems are resulted from Theorem 1 and Corollaries 2 and 3, respectively.

Theorem 4. PI_e polynomial of a k-polyomino $B_{n,k|\ell}$ is

$$PI_e(B_{n,k|\ell};x) = 2[(2k-1)n - s + 1]x^{(4k-1)n-1} + \sum_{i=1}^{s} (\ell_i + 1)x^{(4k-1)n-\ell_i}.$$

Theorem 5. Let $L_{n,k}$ and $Z_{n.k}$ be the *linear chain* and *zig-zag chain* respectively. Then

$PI_e(L_{n,k};x) = 2((2k-1)nx^{(4k-1)n-1} + (n+1)x^{(4k-2)n}$, PI_e

$(Z_{n,k};x) = 4((k-1)n+1)x^{(4k-1)n-1} + 3(n-1)x^{(4k-1)n-2}$.

The mentioned polynomials for linear chain and zig-zag chain of k-polyomino in case $k = 2$ are shown in Table 1.

Table 1. Formulas for the polynomials in linear and zig-zag chains of k-polyomino chains, case k = 2.

$\Omega(L_{n,2};x) = 3nx^2 + x^{n+1}$	$\Omega(Z_{n,2};x) = (2n+2)x^2 + (n-1)x^3$
$Sd(L_{n,2};x) = 3nx^{7n-1} + x^{6n}$	$Sd(Z_{n,2};x) = (2n+2)x^{7n-1} + (n-1)x^{7n-2}$
$PI_e(L_{n,2};x) = 6nx^{7n-1} + (n+1)x^{6n}$	$PI_e(Z_{n,2};x) = 4(n+1)x^{7n-1} + 3(n-1)x^{7n-2}$

CONCLUSION

In this paper, we have calculated the Omega, Sadhana and PI_e polynomials of a polyomino chain of $4k$-cycles with arbitrary k. The following generalizations of the situation considered here seem to deserve further study. At first, one should allow regular $2k$-cycles instead of $4k$-cycles (thus including, e.g., configurations of hexagons). Then one should allow ‘‘heterogeneous’’ segments where $2k$-cycles with different k are allowed. Finally, one should allow more general configurations, where the segments do not necessarily meet at the endpoints (and with an angle of 90 degree).

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حدوديات اوميغا والحدوديات المرتبطة بها من سلاسل بولي اومينو ذات الدورات 4K

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خلاصة

نعرّف حدوديات اوميغا للبيان على أساس أشرطة الحروف المتقابلة. ويمكن بهذه الطريقة أن نحسب أيضاً حدودية سادانا. نقوم في هذا البحث بحساب هذه الحدوديات من سلاسل بولي أومينو ذات الدورات 4K. ويمكن أن نحسب أيضاً باستخدام حدوديات اوميغا، حدوديات الحروف لهذا البيان.