Omega and related polynomials of polyomino chains of 4 k-cycles

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ABSTRACT

Omega polynomial of a graph G is defined, on the ground of "opposite edge strips" ops:
Ω(G; x) = \sum c m(G, c)x^c, where m(G, c) is the number of ops strips of length c. The
Sadhana polynomial Sd(G; x) can also be calculated by ops counting. In this paper we
compute these polynomials for polyomino chains of 4k-cycles. Also by using Omega
polynomial we can compute the (edge) PIe polynomial for this graph.

Keywords: Omega polynomial; sadhana polynomial; PIe polynomial; strips;
polyomino chain.

INTRODUCTION

Let G = (E, V) be a connected graph, with the vertex set V(G) and edge set E(G).
Two edges e = uv and f = xy of G are called codistant e co f, if they obey the
following relation (John et al., 2007):

\[ d(v, x) = d(v, y) + 1 = d(u, x) + 1 = d(u, y) \]

Relation co is reflexive, that is, e co e holds for any edge eof G; it is also
symmetric, if e co f then f co e. In general, relation co is not transitive; an
example showing this fact is the complete bipartite graph \( K_{2,n} \). If “co” is also
transitive, thus an equivalence relation, then G is called a co-graph and the set of
edges C(e) = \{f ∈ E(G)|f co e\} is called an orthogonal cut co of G, E(G) being
the union of disjoint orthogonal cuts:
\[ E(G) = C_1 \cup C_2 \cup \cdots \cup C_k, \quad C_i \cap C_j = \emptyset, \quad i \neq j. \]

Let \( e = uv \) and \( f = xy \) be two edges of \( G \), which are opposite or topologically parallel and denote this relation by \( e \text{ op } f \). A set of opposite edges, within the same face/ring, eventually forming a strip of adjacent faces/rings, is called an opposite edge strip ops, which is a quasi-orthogonal cut qoc (i.e., the transitivity relation is not necessarily obeyed). Note that co relation is defined in the whole graph, while op is defined only in a face/ring. The length of ops is maximal, irrespective of the starting edge. Let \( m(G,s) \) be the number of ops strips of length \( s \). The Omega polynomial is defined as (Diudea, 2006)

\[
\Omega(G; x) = \sum_{c} m(G, c) \cdot x^c.
\]

The first derivative (in \( x = 1 \)) equals the number of edges in the graph

\[
\Omega'(G; 1) = \sum_{c} m(G, c) \cdot c = |E(G)|.
\]

The Sadhana index \( Sd(G) \) was defined by Khadikar et al. (2002) as

\[
Sd(G) = \sum_{c} m(G, c)(|E(G)| - c),
\]

where \( m(G,c) \) is the number of strips of length \( c \).

The Sadhana polynomial \( Sd(G; x) \) was defined by Ashrafi et al. (2008b) as:

\[
Sd(G; x) = \sum_{c} m(G, c) \cdot x^{|E(G)|-c}.
\]

Clearly, the Sadhana polynomial can be derived from the definition of Omega polynomial by replacing the exponent \( c \) by \( |E(G)| - c \). Then the Sadhana index will be the first derivative of \( Sd(G; x) \) evaluated at \( x = 1 \).

**THE MAIN RESULTS**

In this section we compute the Omega polynomial of \( k \)-polyomino chain, which is the main results in this paper. A \( k \)-polyomino system is a finite 2-connected plane graph, such that each interior face (also called cell) is surrounded by a regular \( 4k \)-cycle of length one. In other words, it is an edge-connected union of cells. For the origin of polyominoes see, for example, Klarner (1997) and Golomb (1954; 1965).
For calculating the omega polynomial of a $k$-polyomino chain, we introduce some concepts for a $k$-polyomino chain. A *kink* of a $k$-polyomino chain is any branched or angularly connected $4k$-cycle. A *segment* of a $k$-polyomino chain is a maximal linear chain in the polyomino chain, including the kinks and/or terminal $4k$-cycles at its end. The number of $4k$-cycles in a segment $S$ is called its *length* and is denoted by $\ell(S)$. For any segment $S$ of a polyomino chain with $n \geq 2$ $4k$-cycles one has $2 \leq \ell(S) \leq n$.

In particular, a $k$-polyomino chain is a *linear chain*, if and only if, it contains exactly one segment, see Figure 1. A $k$-polyomino chain is a *zig-zag chain*, if and only if, the length of each segment is 2, see Figure 2.

**Fig. 1.** The linear chain of 8-cycles

**Fig. 2.** The zig-zag chain of 8-cycles
A \( k \)-polyomino chain consists of a sequence of segments \( S_1, S_2, \ldots, S_s, s \geq 1 \), with lengths \( \ell(S_i) \equiv \ell_i, \quad i = 1, 2, \ldots, s \), where

\[
\sum_{i=1}^{s} \ell_i = n + s - 1
\]

\((n\) denotes the number of \( 4k \)-cycles of the polyomino chain), since two neighboring segments have always one \( 4k \)-cycle in common. In the following we will abbreviate the vector of lengths by \( \ell \), i.e., \( \ell = (\ell_1, \ldots, \ell_s) \).

Ghorbani & Ghazi (2010) computed these polynomials only for the zig-zag chain of \( 8 \)-cycles. In this paper, we compute the mention polynomials of polyomino chains of \( 4k \)-cycles with arbitrary \( k \). Here our notations are standard and mainly taken from Ashrafi et al., (2008a) and (Klarner, 1997).

**Theorem 1.** Let \( B_{n,k|\ell} \) be a \( k \)-polyomino chain with \( n \) \( 4k \)-cycles consisting of \( s \geq 1 \) segments \( S_1, S_2, \ldots, S_s \) with lengths \( \ell_1, \ell_2, \ldots, \ell_s \). Then

\[
\Omega(B_{n,k|\ell}; x) = [(2k - 1)n - s + 1]x^2 + \sum_{i=1}^{s} x^{(\ell_i + 1)} \text{and}
\]

\[
Sd(B_{n,k|\ell}; x) = [(2k - 1)n - s + 1]x^{(4k-1)n-1} + \sum_{i=1}^{s} x^{(4k-1)n-\ell_i}.
\]

**Proof.** Every segment \( S_i \) has two kinds of strips: one of lengths 2 and one of lengths \( \ell_i + 1 \). So \( B_{n,k|\ell} \) has \( s + 1 \) strips \( C_0, C_1, C_2, \ldots, C_s \) of length 2, \( \ell_i + 1, \ell_i + 2, \ldots, \ell_s + 1 \), respectively, see Figure 3. There is one strip equivalent with \( |C_i|, i = 1, \ldots, s \), i.e., \( m(B_{n,k|\ell}; \ell_i + 1) = 1, 1 \leq i \leq s \). Thus the number of strips equivalent with \( |C_0| = 2 \) is \( \frac{1}{2} |E(B_{n,k|\ell})| - \sum_{i=1}^{s} (\ell_i + 1) \). We have \( |E(B_{n,k|\ell})| = (4k - 1)n + 1 \) and \( \sum_{i=1}^{s} \ell_i = n + s - 1 \).

Hence \( m(B_{n,k|\ell}, 2) = (2k - 1)n - s + 1 \). Now, by the definitions of Omega and Sadhana polynomials, the proof is completed.
The following corollaries are resulted from Theorems 1.

**Corollary 2.** (Linear chain) In the case \( s = 1 \) and \( \ell_1 = n \) the graph \( B_{n,k|\ell} \) is the linear chain \( L_{n,k} \) of \( n \) 4k-cycles. Then

\[
\Omega(L_{n,k}; x) = (2k - 1)nx^2 + x^{n+1} \quad \text{and} \\
Sd(L_{n,k}; x) = (2k - 1)nx^{(4k-1)n-1} + x^{(4k-2)n}.
\]

**Corollary 3.** (Zig-zag chain) In the case \( s = n - 1 \) and \( \ell_i = 2 \) for all \( i \) the graph \( B_{n,k|\ell} \) is the zig-zag chain \( Z_{n,k} \) of \( n \) 4k-cycles. Then

\[
\Omega(Z_{n,k}; x) = ((2k - 2)n + 2)x^2 + (n - 1)x^3 \quad \text{and} \\
Sd(Z_{n,k}; x) = 2((k - 1)n + 1)nx^{(4k-1)n-1} + (n - 1)x^{(4k-1)n-2}.
\]

Let \( G \) be a connected graph, \( u \) and \( v \) be vertices of \( G \) and \( e = uv \). The number of edges of \( G \) lying closer to \( u \) than to \( v \) is denoted by \( n_{eu}(e|G) \) and the number of edges of \( G \) lying closer to \( v \) than to \( u \) is denoted by \( n_{ev}(e|G) \). The PL\(_{e}\) polynomial of \( G \) is defined as (Ashrafi et al., 2006)

\[
PI(G; x) = \sum_{\left\{u,v\right\} \subseteq V} x^{N(u,v)},
\]
where \( N(u, v) = n_{eu}(e|G) + n_{ev}(e|G) \), if \( e = uv \); and \( = 0 \), otherwise.

Now, If \( G \) be a bipartite graph, \( P_e \) can be computed by (Diudea et al., 2008):

\[
\text{PI}_e(G; x) = \sum_{c} m(G, c) \cdot c \cdot x^{|E(G)|-c} .
\]

Then the PI index will be the first derivative of \( \text{PI}(G; x) \) evaluated at \( x = 1 \). In other words, by using Omega polynomial in bipartite graph we can compute the \( \text{PI}_e \) polynomial and then PI index.

Hence, the following Theorems are resulted from Theorem 1 and Corollaries 2 and 3, respectively.

**Theorem 4.** \( \text{PI}_e \) polynomial of a \( k \)-polyomino \( B_{n,k|\ell} \) is

\[
\text{PI}_e(B_{n,k|\ell}; x) = 2[(2k - 1)n - s + 1]x^{(4k - 1)n-1} + \sum_{i=1}^{s}(\ell_i + 1)x^{(4k - 1)n-\ell_i}.
\]

**Theorem 5.** Let \( L_{n,k} \) and \( Z_{n,k} \) be the linear chain and zig-zag chain respectively. Then

\[
\text{PI}_e(L_{n,k}; x) = 2((2k - 1)n + 1)x^{(4k - 1)n-1} + (n + 1)x^{(4k - 2)n},
\]

\[
\text{PI}_e(Z_{n,k}; x) = 4((k - 1)n + 1)x^{(4k - 1)n-1} + 3(n - 1)x^{(4k - 1)n-2}.
\]

The mentioned polynomials for linear chain and zig-zag chain of \( k \)-polyomino in case \( k = 2 \) are shown in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Formulas for the polynomials in linear and zig-zag chains of ( k )-polyomino chains, case ( k = 2 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega(L_{n,2}; x) = 3nx^2 + x^{n-1} )</td>
</tr>
<tr>
<td>( \text{Sd}(L_{n,2}; x) = 3nx^{7n-1} + x^{6n} )</td>
</tr>
<tr>
<td>( \text{PI}<em>e(L</em>{n,2}; x) = 6nx^{7n-1} + (n + 1)x^{6n} )</td>
</tr>
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</table>

**CONCLUSION**

In this paper, we have calculated the Omega, Sadhana and \( \text{PI}_e \) polynomials of a polyomino chain of \( 4k \)-cycles with arbitrary \( k \). The following generalizations of the situation considered here seem to deserve further study. At first, one should allow regular \( 2k \)-cycles instead of \( 4k \)-cycles (thus including, e.g., configurations of hexagons). Then one should allow “heterogeneous” segments where \( 2k \)-cycles with different \( k \) are allowed. Finally, one should allow more general configurations, where the segments do not necessarily meet at the endpoints (and with an angle of 90 degree).
REFERENCES


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حدوديات اومیغا والحدوديات المرتبطة بها من سلاسل بولي اومینو

4K ذات الدورات

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خلاصه

نعرف حدوديات اومیغا للبيان على أساس أشرطة الحروف المتقابلة. ويمكن بهذه
الطريقة أن نحسب أيضاً حدودية سادانا. تقوم في هذا البحث بحساب هذه الحدوديات
من سلاسل بولي اومینو ذات الدورات 4K. ويمكن أن نحسب أيضاً باستخدام
حدوديات اومیغا، حدوديات الحروف لهذا البيان.