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Characterizations of $(\in, \in \vee q_k)$-fuzzy fantastic ideals in BCI-algebras

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ABSTRACT

In this paper, the concepts of $(\in, \in \vee q_k)$-fuzzy fantastic ideals and $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals in BCI-algebras are introduced and investigate some of their properties. The concept of a fuzzy fantastic ideal with thresholds, properties of a fuzzy fantastic ideal, an $(\in, \in \vee q_k)$-fuzzy fantastic ideal and an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal are discussed.

Keywords: BCI-algebra; fuzzy fantastic ideal; $(\in, \in \vee q_k)$-fuzzy fantastic ideals; $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals; fuzzy fantastic ideal with thresholds.

INTRODUCTION

The concept of fantastic ideal in BCI-algebra was initiated by (Saeid, 2010). The concept of a fuzzy set was initiated by (Zadeh, 1965), was applied by many researchers to generalize some of the basic concepts of algebra. The fuzzy algebraic structures play a vital role in Mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, real analysis, measure theory etc. In (Xi, 1991) applied fuzzy subsets in BCK-algebras and studied fuzzy BCK-algebras.

In (Rosenfeld, 1971) laid the foundation of fuzzy groups. (Murali, 2004) defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. In (Pu & Liu, 1980), the idea of fuzzy point and its belongingness to and quasi-coincidence with a fuzzy subset were used to define (α, β)-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. This concept was further discussed by (Bhakat, 1999; Bhakat, 2000; Das, 1981; Bhakat & Das 1996; Yuan *et al.*, 2003). In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfelds fuzzy subgroups. (Jun, 2009) introduced the concept of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. (Zhan *et al.,* 2009) studied $(\in, \in \vee q)$-fuzzy ideals of BCI-algebras. In (Jun, 2004; Jun, 2005; Jun, 2007) studied (α, β)-fuzzy subalgebras (ideals) of BCK/BCI-algebras. A generalization of a fuzzy ideal in a BCK/BCI-algebra was discussed by (Guangji *et al.,* 2006; Jun *et al.,* 2010). In (Ma *et al.,* 2008) discussed $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras. (Zhan & Jun, 2009) studied generalized fuzzy ideals of BCI-algebras. In (Zhan & Jun, 2011) give the idea on $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals of BCI-algebras.

In this paper, the concepts of $(\in, \in \vee q_k)$-fuzzy fantastic ideals and $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals in BCI-algebras are introduced and investigate some of their properties. The concept of a fuzzy fantastic ideal with thresholds, properties of a fuzzy fantastic ideal, an $(\in, \in \vee q_k)$-fuzzy fantastic ideal and an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal are discussed.

2. Preliminaries

In what follow let X denote a BCI-algebra unless otherwise specified.

Definition 2.1. (Huang, 2006) By a BCI-algebra, we mean an algebra $(X, *, 0)$ of type (2, 0) satisfying the axioms:

(BCI-I) $((x * y) * (x * z)) * (z * y) = 0$

(BCI-II) $(x * (x * y)) * y = 0$

(BCI-III) $x * x = 0$

(BCI-IV) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$.

We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. (Meng & Jun, 1994; Meng & Guo, 2005; Huang, 2006) In any BCI-algebra X, the following are true:

(1) $(x * y) * z = (x * z) * y$

(2) $(x * z) * (y * z) \leq x * y$

(3) $(x * y) * (x * z) \leq z * y$

(4) $x * 0 = x$

(5) $x * (x * (x * y)) = x * y$

for all $x, y, z \in X$.

Definition 2.3. (Meng 1997; Meng & Guo, 2005) A nonempty subset I of a BCI-algebra X is called an ideal of X if it satisfies (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all $x, y \in X$.

Definition 2.4. (Saeid, 2010) A nonempty subset I of a BCI-algebra X is called a fantastic ideal of X, denoted by $I \triangleright X$, if it satisfies (I1) and (I3), where

(I1) $0 \in I$,

(I3) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$,

for all $x, y, z \in X$.

3. Fuzzy fantastic ideals in BCI-algebras

We now review some fuzzy logic concepts.

Definition 3.1. (Zadeh, 1965) A fuzzy subset ζ of a universe X is a function from X into the unit closed interval [0, 1], that is $\zeta : X \to [0, 1]$.

Definition 3.2. (Jun, 2004) For a fuzzy set ζ of a BCI-algebra X and $t \in (0, 1]$, the crisp set

$$\zeta_t = \{x \in X \mid \zeta(x) \geq t\}$$

is called the level subset of ζ.

Definition 3.3. (Meng & Guo 2005; Meng *et al.,* 1997) A fuzzy set ζ of a BCI-algebra X is called a fuzzy ideal of X if it satisfies (F1) and (F2), where

(F1) $\zeta(0) \geq \zeta(x)$,

(F2) $\zeta(x) \geq \zeta(x * y) \wedge \zeta(y)$,

for all $x, y \in X$.

Definition 3.4. A fuzzy set ζ of a BCI-algebra X is called a fuzzy fantastic ideal of X if it satisfies (F1) and (F3), where

(F1) $\zeta(0) \geq \zeta(x)$,

(F3) $\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z)$,

for all $x, y, z \in X$.

Example 3.5. Let $X = \{0, a, b, c\}$ in which $*$ is given by the Table 1

Table 1.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

From Table 1, X is a BCI-algebra. Let $u_0, u_1, u_2 \in [0, 1]$ be such that $u_0 > u_1 > u_2$. We define a map $\zeta : X \to [0, 1]$ by $\zeta(0) = u_0$, $\zeta(a) = u_1$ and $\zeta(b) = \zeta(c) = u_2$. By simple calculations give that ζ is a fuzzy fantastic ideal of X.

Theorem 3.6. A fuzzy set ζ in BCI-algebras X is a fuzzy fantastic ideal of X if and only if for every $t \in (0, 1]$, $\zeta_t = \{x \in X \mid \zeta(x) \geq t\}$ is a fantastic ideal of X, where $t \leq \zeta(0)$.

Proof. Straightforward.

Definition 3.7. (Zhan & Jun, 2009) A fuzzy set ζ of a BCI-algebra X having the form

$$\zeta(y) = \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set ζ in a set X, (Pu & Liu, 1980) gave meaning to the symbol $x_t\alpha\zeta$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set ζ, written as $x_t \in \zeta$ (resp., $x_t q\zeta$) if $\zeta(x) \geq t$ (resp., $\zeta(x) + t > 1$).

To say that $x_t \in \vee q\zeta$ means that $x_t \in \zeta$ or $x_t q\zeta$. $x_t\bar{\alpha}\zeta$ if $x_t\alpha\zeta$ does not hold for $\alpha \in \{\in, q, \in \vee q\}$.

4. $(\in, \in \vee q_k)$**-fuzzy fantastic ideals in BCI-algebras**

Let k denote an arbitrary element of [0, 1) unless otherwise specified. For a fuzzy point x_t and a fuzzy subset ζ of a BCI-algebra X, (Jun *et al.,* 2010) defined the following:

(A) $x_t q_k \zeta$ if $\zeta(x) + t + k > 1$.

(B) $x_t \in \vee q_k \zeta$ if $x_t \in \zeta$ or $x_t q_k \zeta$.

(C) $x_t\bar{\alpha}\zeta$ if $x_t\alpha\zeta$ does not hold for $\alpha \in \{q_k, \in \vee q_k\}$.

Theorem 4.1. Let ζ be a fuzzy subset of a BCI-algebra X. Then the following are equivalent:

(D) $\zeta_t \neq \phi \Rightarrow \zeta_t \triangleright X$, for all $t \in (\frac{1-k}{2}, 1]$

(E) ζ satisfies the following assertions:

(i) $\zeta(x) \leq \zeta(0) \vee \frac{1-k}{2}$,

(ii) $\zeta((x * y) * z) \wedge \zeta(z) \leq \zeta(x * (y * (y * x))) \vee \frac{1-k}{2}$,

for all x, y, z $\in$ X.

Proof. Suppose that (D) is hold. If there is $a \in X$ such that the condition (i) is not true, that is, there exists $a \in X$ such that

$$\zeta(a) > \zeta(0) \vee \frac{1-k}{2}$$

then $\zeta(a) \in (\frac{1-k}{2}, 1]$ and $a \in \zeta_{\zeta(a)}$. But $\zeta(0) < \zeta(a)$ implies that $0 \notin \zeta_{\zeta(a)}$, a contradiction. Hence (i) is hold. Suppose that (ii) is false, i.e.,

$$w = \zeta((a * b) * c) \wedge \zeta(c) > \zeta(a * (b * (b * a))) \vee \frac{1-k}{2}$$

for some a, b, c $\in$ X. Then

$$w \in (\frac{1-k}{2}, 1] \text{ and } (a * b) * c, c \in \zeta_w.$$

But $a * (b * (b * a)) \notin \zeta_w$ since $\zeta(a * (b * (b * a))) < w$. This is a contradiction, and so (ii) holds.

Conversely, suppose that ζ satisfies conditions (i) and (ii). Let $t \in (\frac{1-k}{2}, 1]$ be such that $\zeta_t \neq \phi$. For any $x \in \zeta_t$, we have

$$\zeta(0) \vee \frac{1-k}{2} \geq \zeta(x) \geq t > \frac{1-k}{2}$$

and so

$$\zeta(0) = \zeta(0) \vee \frac{1-k}{2} \geq t.$$

Hence $0 \in \zeta_t$. Let x, y, z $\in$ X be such that

$$(x * y) * z \in \zeta_t \text{ and } z \in \zeta_t.$$

Then

$$\zeta(x * (y * (y * x))) \vee \frac{1-k}{2} \geq \zeta((x * y) * z) \wedge \zeta(z)$$

$$\geq t \wedge t$$

$$\geq t$$

$$> \frac{1- \ k}{2}$$

and thus

$$\zeta(x*(y*(y*x))) = \zeta(x*(y*(y*x))) \vee \frac{1- \ k}{2} \geq t,$$

i.e., $x*(y*(y*x)) \in \zeta_t$. Therefore ζ_t is a fantastic ideal of X.

Putting k = 0 in Theorem 4.1, then we have the following corollary.

Corollary 4.2. Let ζ be a fuzzy subset of a BCI-algebra X. Then the following are equivalent:

(F) $\zeta_t \neq \phi \Rightarrow \zeta_t \triangleright X$, for all $t \in (0.5, 1]$.

(G) ζ satisfies the following assertions:

(iii) $\zeta(x) \leq \zeta(0) \vee 0.5$,

(iv) $\zeta((x*y)*z) \wedge \zeta(z) \leq \zeta(x*(y*(y*x))) \vee 0.5$,

for all $x, y, z \in X$.

Definition 4.3. A fuzzy subset ζ of a BCI-algebra X is called an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X if it satisfies (H) and (I), where

(H) $x_t \in \zeta \Rightarrow 0_t \in \vee\, q_k \zeta$,

(I) $((x*y)*z)_t \in \zeta, z_t \in \zeta \Rightarrow (x*(y*(y*x)))_t \in \zeta$

for all $x, y, z \in X$ and $t \in (0, 1]$.

Example 4.4. Let X = {0, 1, 2, 3, 4} in which is defined by Table 2

Table 2.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

From Table 2, X is a BCI-algebra. Define $\zeta : X \to [0, 1]$ by $\zeta(0) = 0.6$, $\zeta(1) = \zeta(3) = 0.7$ and $\zeta(2) = \zeta(4) = 0.2$. By simple calculations show that ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X for k = 0.3.

An $(\in, \in \vee q_k)$-fuzzy fantastic ideal of a BCI-algebra X with k = 0 is called an $(\in, \in \vee q)$-fuzzy fantastic ideal of X.

Theorem 4.5. Let X be a BCI-algebra. A fuzzy subset ζ of X is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X if and only if it satisfies (J) and (K), where

(J) $\zeta(0) \geq \zeta(x) \wedge \frac{1-k}{2}$,

(K) $\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1-k}{2}$,

for all x, y, z $\in$ X.

Proof. Assume that ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X. Let x $\in$ X and suppose that

$$\zeta(x) < \frac{1-k}{2}.$$

If $\zeta(0) < \zeta(x)$, then

$$\zeta(0) < t \leq \zeta(x)$$

for some $t \in (0, \frac{1-k}{2})$. It follows that $x_t \in \zeta$ but $0_t \bar{\in} \zeta$. Since

$$\zeta(0) + t < 2t < 1 - k,$$

we get $0_t \bar{q}_k \zeta$. Therefore

$$0t\overline{\in \vee q_k}\zeta,$$

which is a contradiction. Hence $\zeta(0) \geq \zeta(x)$.

Now if $\zeta(x) \geq \frac{1-k}{2}$, then

$$x_{\frac{1-k}{2}} \in \zeta \text{ and so } 0_{\frac{1-k}{2}} \in \vee\, q_k\zeta.$$

This implies that

$$\zeta(0) \geq \frac{1-k}{2}$$

or

$$\zeta(0) + \frac{1-k}{2} > 1 - k.$$

Hence

$$\zeta(0) \geq \frac{1-\ k}{2}.$$

Otherwise,

$$\zeta(0) + \frac{1-\ k}{2} < \frac{1-\ k}{2} + \frac{1-\ k}{2}$$
$$= 1-k,$$

a contradiction. Consequently,

$$\zeta(0) \geq \zeta(x) \wedge \frac{1-\ k}{2}$$

for all x $\in$ X. Let x, y, z $\in$ X and suppose that

$$\zeta((x*y)*z) \wedge \zeta(z) < \frac{1-\ k}{2}.$$

Now we have to prove that

$$\zeta(x*(y*(y*x))) \geq \zeta((x*y)*z) \wedge \zeta(z).$$

If not, then

$$\zeta(x*(y*(y*x))) < t \leq \zeta((x*y)*z) \wedge \zeta(z)$$

for some t $\in$ (0, $\frac{1-\ k}{2}$). It follows that

$$((x*y)*z)_t \in \zeta \textit{ and } z_t \in \zeta, \textit{ but } (x*(y*(y*x)_t \bar{\in} \zeta$$

and

$$\zeta(x*(y*(y*x))) + t < 2t < 1-k,$$

i.e., $(x*(y*(y*x)))_t \bar{q}_k \zeta$. This is a contradiction. Thus

$$\zeta(x*(y*(y*x))) \geq \zeta((x*y)*z) \wedge \zeta(z)$$

whenever

$$\zeta((x*y)*z) \wedge \zeta(z) < \frac{1-\ k}{2}.$$

If $\zeta((x * y) * z) \wedge \zeta(z) \geq \frac{1-k}{2}$, then

$$((x * y) * z)_{\frac{1-k}{2}} \in \zeta \text{ and } z_{\frac{1-k}{2}} \in \zeta.$$

Since ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal, it follows that

$$(x * (y * (y * x)))_{\frac{1-k}{2}} \in \vee q_k \zeta.$$

So that

$$\zeta(x * (y * (y * x))) \geq \frac{1-k}{2}$$

or $$\zeta(x * (y * (y * x))) + \frac{1-k}{2} > 1 - k.$$

If $\zeta(x * (y * (y * x))) < \frac{1-k}{2}$, then

$$\begin{aligned}\zeta(x * (y * (y * x))) + \frac{1-k}{2} &< \frac{1-k}{2} + \frac{1-k}{2} \\ &= 1-k\end{aligned}$$

a contradiction. Therefore

$$\zeta(x * (y * (y * x))) \geq \frac{1-k}{2}.$$

Consequently

$$\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1-k}{2}.$$

Conversely, suppose that (J) and (K) are hold. Let $x \in X$ and $t \in (0, 1]$ be such that $x_t \in \zeta$. Then $\zeta(x) \geq t$. Suppose $\zeta(0) < t$. If $\zeta(x) < \frac{1-k}{2}$, then

$$\zeta(0) \geq \zeta(x) \wedge \frac{1-k}{2} = \zeta(x) \geq t$$

a contradiction. Hence

$$\zeta(x) \geq \frac{1-k}{2}.$$

This implies that

$$\begin{aligned}\zeta(0) + t &> 2\zeta(0)\\ &\geq 2(\zeta(x) \wedge \frac{1- k}{2})\\ &= 1 - k.\end{aligned}$$

Thus $0_t \in \vee\, q_k\zeta$. Let x, y, z ∈ **X** and t_1, $t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \in \zeta \text{ and } z_{t_2} \in \zeta.$$

Then

$$\zeta((x * y) * z) \geq t_1 \text{ and } \zeta(z) \geq t_2.$$

Suppose that

$$\zeta(x * (y * (y * x))) < t_1 \wedge t_2.$$

If $\zeta((x * y) * z) \wedge \zeta(z) < \frac{1- k}{2}$, then

$$\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1- k}{2}$$

$$= \zeta((x * y) * z) \wedge \zeta(z)$$

$$\geq t_1 \wedge t_2.$$

This is not possible, and so

$$\zeta((x * y) * z) \wedge \zeta(z) \geq \frac{1- k}{2}.$$

It follows that

$$\begin{aligned}\zeta(x * (y * (y * x))) + t_1 \wedge t_2 &> 2\zeta(x * (y * (y * x)))\\ &\geq 2(\zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1- k}{2})\\ &= 1 - k.\end{aligned}$$

So that

$$(x * (y * (y * x)))_{t_1 \wedge t_2} \in \vee q_k \zeta.$$

Hence ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

Letting k = 0 in Theorem 4.5, then we have the following corollary.

Corollary 4.6. Let X be a BCI-algebra. A fuzzy subset ζ of X is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X if and only if it satisfies (L) and (M), where

(L) $\zeta(0) \geq \zeta(x) \wedge 0.5$,

(M) $\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge 0.5$,

for all x, y, z $\in$ X.

Clearly, every fuzzy fantastic ideal is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal.

Here we give a condition for an $(\in, \in \vee q_k)$-fuzzy fantastic ideal to be a fuzzy fantastic ideal.

Theorem 4.7. Let ζ be an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of a BCI-algebra X. If

$$\zeta(0) < \frac{1 - k}{2},$$

then ζ is a fuzzy fantastic ideal of X.

Proof. Straightforward.

Example 4.8. Let X = {0, 1, 2, 3} in which is defined by Table 3

Table 3.

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

From Table 3, X is a BCI-algebra. Define $\zeta : X \rightarrow [0, 1]$ by $\zeta(0) = 0.3$, $\zeta(1) = \zeta(2) = \zeta(3) = 0.6$. By simple calculations show that ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X for k = 0.4. But ζ is not fuzzy fantastic ideal of X.

Corollary 4.9. Let ζ be an $(\in, \in \vee q)$-fuzzy fantastic ideal of a BCI-algebra X. If $\zeta(0) < 0.5$, then ζ is a fuzzy fantastic ideal of X.

Proof. It follows from Theorem 4.7 by setting k = 0.

Theorem 4.10. Let X be a BCI-algebra. If $0 \leq k < r < 1$, then every $(\in, \in \vee q_k)$-fuzzy fantastic ideal is an $(\in, \in \vee q_r)$-fuzzy fantastic ideal.

Proof. Straightforward.

Theorem 4.11. For a fuzzy subset ζ of a BCI-algebra X, the following are equivalent:

(N) ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

(O) $\zeta_t \neq \phi \Rightarrow \zeta_t \rhd X$, for all $t \in (0, \frac{1-k}{2}]$.

We say that ζ_t is an $(\in, \in \vee q_k)$-level fantastic ideal of ζ in X.

Proof. The proof of the Theorem is obvious.

Setting $k = 0$ in Theorem 4.11 induces the following corollary.

Corollary 4.12. For a fuzzy subset ζ of a BCI-algebra X, the following are equivalent:

(P) ζ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X.

(Q) $\zeta_t \neq \phi \Rightarrow \zeta_t \rhd X$, for all $t \in (0, 0.5]$.

For a fuzzy point x_t and a fuzzy subset ζ of a BCI-algebra X, (Jun *et al.*, 2010) defined the following:

(R) $x_t \underline{q} \zeta$ if $\zeta(x) + t > 1$,

(S) $x_t \underline{q}_k \zeta$ if $\zeta(x) + t + k > 1$.

Denote by $Q(\zeta; t)$ (resp. $\underline{Q}(\zeta; t)$) the set $\{x \in X \mid x_t q \zeta\}$ (resp. $\{x \in X \mid x_t \underline{q} \zeta\}$), and

$$Q^k(\zeta; t) = \{x \in X \mid x_t q_k \zeta\}$$

$$\underline{[\zeta]_t^k} = \{x \in X \mid x_t \in \vee \underline{q}_k \zeta\}$$

$$\underline{Q}^k(\zeta; t) = \{x \in X \mid x_t \underline{q}_k \zeta\}$$

$$\underline{[\zeta]_t^k} = \{x \in X \mid x_t \in \vee \underline{q}_k \zeta\}$$

Obviously

$$[\zeta]_t^k = \zeta_t \cup Q^k(\zeta; t)$$

and

$$\underline{[\zeta]_t^k} = \zeta_t \cup \underline{Q}^k(\zeta; t).$$

Theorem 4.13. If ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of a BCI-algebra X, then

$$\underline{Q}^k(\zeta; t) \neq \phi \Rightarrow \underline{Q}^k(\zeta; t) \triangleright X,$$

for all t $\in (\frac{1-\ k}{2}, 1]$.

Proof. Straightforward.

Corollary 4.14. If ζ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of a BCI-algebra X, then

$$\underline{Q}(\zeta; t) \neq \phi \quad \Rightarrow \quad \underline{Q}(\zeta; t) \triangleright X,$$

for all t $\in$ (0.5, 1].

Corollary 4.15. Let ζ be an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of a BCI-algebra X. If k < r < 1, then

$$\underline{Q}^r(\zeta; t) \neq \phi \Rightarrow \underline{Q}^r(\zeta; t) \triangleright X,$$

for all t $\in (\frac{1-\ r}{2}, 1]$.

Proof. It is straightforward by Theorems 4.10 and 4.13.

Theorem 4.16. For a fuzzy subset ζ of a BCI-algebra X, the following are equivalent:

(T) ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

(U) $\underline{[\zeta]_t^k} \neq \phi \Rightarrow \underline{[\zeta]_t^k} \triangleright$ X, for all t $\in$ (0, 1].

We call $\underline{[\zeta]_t^k}$ an $(\in \vee \underline{q}_k)$-level fantastic ideal of ζ.

Proof. Suppose that ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X and let t $\in$ (0, 1] such that $\underline{[\zeta]_t^k} \neq \phi$. Then there exists $a \in \underline{[\zeta]_t^k} \neq \phi$, *and so* $a \in \zeta_t$ *or* $a \in \underline{Q}^k(\zeta;\ t)$, *i.e.,*

$$\zeta(x) \geq t \quad \text{or} \quad \zeta(x) + t \geq 1 - k.$$

By Theorem 4.5(J), we get

$$\zeta(0) \geq \zeta(a) \wedge \frac{1-\ k}{2} \tag{1}$$

Here we consider two cases:

Case 1: $\zeta(a) \leq \dfrac{1-\ k}{2}$ and

Case 2: $\zeta(a) > \dfrac{1-\ k}{2}$.

Case 1: We have $\zeta(0) \geq \zeta(a)$ by (1). Thus if $\zeta(a) \geq$ t, then $\zeta(0) \geq$ t and so

$$0 \in \zeta_t \subseteq \underline{[\zeta]_t^k}.$$

If $\zeta(a) + t \geq 1 - k$, then

$$\zeta(0) + t \geq \zeta(a) + t$$
$$\geq 1 - k.$$

This implies that $0_t \ \underline{q}_k \zeta$, i.e.,

$$0 \in \underline{Q}^k(\zeta; t) \subseteq \underline{[\zeta]_t^k}.$$

Combining the Case 2 and (1) induces

$$\zeta(0) \geq \frac{1-\ k}{2}.$$

If t $\leq \dfrac{1-\ k}{2}$, then $\zeta(0) \geq$ t and hence $0 \in \zeta_t \subseteq \underline{[\zeta]_t^k}$.

Case 2: If t $> \dfrac{1-\ k}{2}$, then

$$\zeta(0) + t > \frac{1-\ k}{2} + \frac{1-\ k}{2} = 1 - k.$$

This implies that

$$0 \in Q^k(\zeta; t) \subseteq \underline{Q}^k(\zeta; t) \subseteq \underline{[\zeta]_t^k}.$$

Therefore $\underline{[\zeta]_t^k}$ satisfies the condition (I1). Let x, y, z $\in$ X be such that

$$(x * y) * z \in \underline{[\zeta]_t^k} \ \text{ and } z \in \underline{[\zeta]_t^k}.$$

Then

$$(x * y) * z \in \zeta_t \text{ or } ((x * y) * z)_t \ \underline{q}_k \zeta \ \text{ and } z \in \zeta_t \text{ or } \ zt\underline{q}_k \zeta$$

that is

$\zeta((x * y) * z) \geq t$ or $\zeta((x * y) * z) + t \geq 1 - k$ and $\zeta(z) \geq t$ or $\zeta(z) + t \geq 1 - k$.

Since ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X, we have

$$\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1 - k}{2} \tag{2}$$

by Theorem 4.5(K). If $\zeta((x * y) * z) \wedge \zeta(z) \leq \frac{1 - k}{2}$, then

$$\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z)$$

$$\zeta(x * (y * (y * x))) \geq t \wedge t$$

$$\zeta(x * (y * (y * x))) \geq t$$

i.e.,

$$x * (y * (y * x)) \in \zeta_t \subseteq \underline{[\zeta]_t^k}.$$

If $\zeta((x * y) * z) \wedge \zeta(z) > \frac{1 - k}{2}$, then

$$\zeta(x * (y * (y * x))) \geq \frac{1 - k}{2}$$

and so

$$\zeta(x * (y * (y * x))) + t > \frac{1 - k}{2} + \frac{1 - k}{2} = 1 - k.$$

Hence

$$x * (y * (y * x)) \in Q^k(\zeta; t) \subseteq \underline{Q}^k(\zeta; t) \subseteq \underline{[\zeta]_t^k}.$$

Consequently, $\underline{[\zeta]_t^k}$ is a fantastic ideal of X.

Conversely, assume that (U) is hold. If there exists $a \in X$ such that

$$\zeta(0) < \zeta(a) \wedge \frac{1 - k}{2},$$

then

$$\zeta(0) < t_0 \leq \zeta(a) \wedge \frac{1 - k}{2}$$

for some $t_0 \in (0, \frac{1 - k}{2}]$. It follows that $a \in \zeta_{t_0} \subseteq \underline{[\zeta]_{t_0}^k}$ but $0 \notin \zeta_{t_0}$. Also, we have

$$\zeta(0) + t_0 < 2t_0 \leq 1 - k,$$

and so $0_{t_0} \overline{q_k}\, \zeta$, i.e., $0 \notin \underline{Q}^k(\zeta; t)$. Therefore $0 \notin \underline{[\zeta]_{t_0}^k}$, a contradiction. Hence

$$\zeta(0) \geq \zeta(x) \wedge \frac{1 - k}{2}$$

for all $x \in X$. Suppose that there exist a, b, c $\in X$ such that

$$\zeta(a * (\mathrm{b} * (\mathrm{b} * a))) < \zeta((a * \mathrm{b}) * \mathrm{c}) \wedge \zeta(\mathrm{c}) \wedge \frac{1-k}{2}.$$

Then

$$\zeta(a * (\mathrm{b} * (\mathrm{b} * a))) < \mathrm{t}_w \leq \zeta((a * \mathrm{b}) * \mathrm{c}) \wedge \zeta(\mathrm{c}) \wedge \frac{1-k}{2}$$

for some $\mathrm{t}_w \in (0, \frac{1-k}{2}]$. It follows that $(a * \mathrm{b}) * \mathrm{c}, \mathrm{c} \in \zeta_{t_w} \subseteq \underline{[\zeta]}_{t_w}^{k}$ from (I3) that $a * (\mathrm{b} * (\mathrm{b} * a)) \in \underline{[\zeta]}_{t_w}^{k}$. Thus

$$\zeta(a * (\mathrm{b} * (\mathrm{b} * a))) \geq \mathrm{t}_w \text{ or } \zeta(a * (\mathrm{b} * (\mathrm{b} * a))) + \mathrm{t}_w \geq 1 - \mathrm{k},$$

a contradiction. Therefore

$$\zeta(\mathrm{x} * (\mathrm{y} * (\mathrm{y} * \mathrm{x}))) \geq \zeta((\mathrm{x} * \mathrm{y}) * \mathrm{z}) \wedge \zeta(\mathrm{z}) \wedge \frac{1-k}{2}$$

for all x, y, z $\in$ X. From Theorem 4.5, we conclude that ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

Corollary 4.17. For a fuzzy subset ζ of a BCI-algebra X, the following are equivalent:

(V) ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X,

(W) $\underline{[\zeta]}_t \neq \phi \Rightarrow \underline{[\zeta]}_t \triangleright \mathrm{X}$, for all $\mathrm{t} \in (0, 1]$,

where $\underline{[\zeta]}_t = \{\mathrm{x} \in \mathrm{X} \mid \mathrm{x}_t \in \vee \mathrm{q}\zeta\} = \zeta_t \cup \underline{Q}(\zeta; \mathrm{t})$.

A fuzzy subset ζ of a BCI-algebra X is said to be proper if Im(ζ) has at least two elements. Two fuzzy subsets are said to be equivalent if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

Theorem 4.18. Let ζ be an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of a BCI-algebra X such that

$$\# \{\zeta(\mathrm{x}) \mid \zeta(\mathrm{x}) < \frac{1-k}{2}\} \geq 2.$$

Then there exist two proper non-equivalent $(\in, \in \vee q_k)$-fuzzy fantastic ideals of X such that ζ can be expressed as the union of them.

Proof. The proof of the Theorem is obvious.

Theorem 4.19. Let $\{\zeta_i \mid \mathrm{i} \in \Lambda\}$ be a family of $(\in, \in \vee q_k)$-fuzzy fantastic ideals of a BCI-algebra X. Then $\zeta = \bigcap_{i \in \Lambda} \zeta_i$ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

Proof. Let $\mathrm{x} \in \mathrm{X}$ and $\mathrm{t} \in (0, 1]$ be such that $\mathrm{x}_t \in \zeta$. Suppose that $0_t \overline{\in \vee q_k} \zeta$. Then

$$\zeta(0) < \mathrm{t} \text{ and } \zeta(\mathrm{x}) + \mathrm{t} \leq 1 - \mathrm{k}.$$

This imply that

$$\zeta(0) < \frac{1-k}{2}. \tag{3}$$

Let

$$\Omega_1 = \{i \in \Lambda \mid \zeta_i(0) \geq t\} \text{ and } \Omega_2 = \{i \in \Lambda \mid 0_t q_k \zeta_i \text{ and } \zeta_i(0) < t\}.$$

Then

$$\Lambda = \Omega_1 \cup \Omega_2 \text{ and } \Omega_1 \cap \Omega_2 = \phi.$$

If $\Omega_2 = \phi$, then $\zeta_i(0) \geq t$ for all $i \in \Lambda$, and so $\zeta(0) \geq t$ is a contradiction. Hence $\Omega_2 \neq \phi$, and so

$$\zeta_i(0) + t > 1 - k \text{ and } \zeta_i(0) < t$$

for every $i \in \Omega_2$. It follows that $t > \frac{1-k}{2}$ so that

$$\zeta_i(x) \geq \zeta(x) \geq t > \frac{1-k}{2}$$

for all $i \in \Lambda$. Now, assume that

$$t_0 = \zeta_i(x) < \frac{1-k}{2}$$

for some $i \in \Lambda$. Let $t_0' \in (0, \frac{1-k}{2})$ be such that $t_0 < t_0'$. Then

$$\zeta_i(x) > \frac{1-k}{2} > t_0',$$

i.e., $x_{t_0'} \in \zeta_i$. But

$$\zeta_i(0) = t_0 < t_0' \text{ and } \zeta_i(0) + t_0' < 1 - k,$$

that is

$$x_{t_0'} \overline{\in \vee q_k} \zeta_i.$$

Which is a contradiction, and so

$$\zeta_i(x) \geq \frac{1-k}{2}$$

for all $i \in \Lambda$. Thus

$$\zeta(x) \geq \frac{1-k}{2}.$$

This is not possible. Therefore $0_t \in \vee q_k \zeta$. Let $x, y, z \in X$ and $t_1, t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \in \zeta \text{ and } z_{t_2} \in \zeta.$$

Suppose that

$$(x * (y * (y * x)))_{t_1 \wedge t_2} \overline{\in \vee q_k} \zeta.$$

Then

$$\zeta(x * (y * (y * x))) < t_1 \wedge t_2 \text{ and } \zeta(x * (y * (y * x))) + t_1 \wedge t_2 \leq 1 - k.$$

It follows that

$$\zeta(x * (y * (y * x))) < \frac{1-k}{2}.$$

Let

$$\Omega_3 = \{i \in \Lambda \mid \zeta_i(x * (y * (y * x))) \geq t_1 \wedge t_2\}$$

and

$$\Omega_4 = \{i \in \Lambda \mid (x * (y * (y * x)))_{t_1 \wedge t_2} q_k \zeta_i \text{ and } \zeta_i(x * (y * (y * x))) < t_1 \wedge t_2\}.$$

Then

$$\Lambda = \Omega_3 \cup \Omega_4 \text{ and } \Omega_3 \cap \Omega_4 = \phi.$$

If $\Omega_4 = \phi$, then

$$\zeta_i(x * (y * (y * x))) \geq t_1 \wedge t_2$$

for all $i \in \Lambda$ and so

$$\zeta(x * (y * (y * x))) \geq t_1 \wedge t_2.$$

This is a contradiction. Hence $\Omega_4 \neq \phi$ and thus

$$(x * (y * (y * x)))_{t_1 \wedge t_2} q_k \zeta_i$$

i.e.,

$$\zeta_i(x * (y * (y * x))) + t_1 \wedge t_2 > 1 - k, \text{ and } \zeta_i(x * (y * (y * x))) < t_1 \wedge t_2.$$

It follows that

$$t_1 \wedge t_2 > \frac{1-k}{2}.$$

So that

$$\begin{aligned} \zeta_i((x * y) * z) &\geq \zeta((x * y) * z) \\ &\geq t_1 \\ &\geq t_1 \wedge t_2 \\ &> \frac{1-k}{2} \end{aligned}$$

for all $i \in \Lambda$. Similarly, we have

$$\zeta_i(z) > \frac{1-k}{2}$$

for all $i \in \Lambda$. Now, suppose that

$$t = \zeta_i(x * (y * (y * x))) < \frac{1-k}{2}$$

for some $i \in \Lambda$. Let $t' \in (0, \frac{1-k}{2})$ be such that $t < t'$. Then

$$\zeta_i((x * y) * z) > \frac{1-k}{2} > t'$$

and
$$\zeta_i(z) > \frac{1-k}{2} > t'$$

that is,
$$((x * y) * z)_{t'} \in \zeta_i \text{ and } z_{t'} \in \zeta_i.$$

But
$$\zeta_i(x * (y * (y * x))) = t < t' \text{ and } \zeta_i(x * (y * (y * x))) + t' < 1 - k$$

that is
$$(x * (y * (y * x)))_{t'} \overline{\in \vee q_k} \zeta_i.$$

Which is a contradiction, and hence
$$\zeta_i(x * (y * (y * x))) \geq \frac{1-k}{2}$$

for all $i \in \Lambda$. Therefore
$$\zeta(x * (y * (y * x))) \geq \frac{1-k}{2}.$$

This is not hold. Consequently
$$(x * (y * (y * x)))_{t_1 \wedge t_2} \in \vee q_k \zeta.$$

Hence ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

Letting k = 0 in Theorem 4.19, we have the following corollary.

Corollary 4.20. Let $\{\zeta_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee q)$-fuzzy fantastic ideals of a BCI-algebra X. Then $\zeta = \bigcap_{i \in \wedge} \zeta_i$ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X.

We note that the union of $(\in, \in \vee q_k)$-fuzzy fantastic ideals of a BCI-algebra X may not be an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

Consider the number $t \in (\frac{1-k}{2}, 1]$ for which ζ_t is a fantastic ideal of X, we consider a new kind of fuzzy fantastic ideals as follows.

Definition 4.21. A fuzzy subset ζ of a BCI-algebra X is called an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X if it satisfies (X) and (Y), where

(X) $0_t \bar{\in} \zeta \Rightarrow x_t \bar{\in} \vee \bar{q}_k \zeta$,

(Y) $(x * (y * (y * x)))_{t_1 \wedge t_2} \bar{\in} \zeta \Rightarrow ((x * y) * z)_{t_1} \bar{\in} \vee \bar{q}_k \zeta$ or $z_{t_2} \bar{\in} \vee \bar{q}_k \zeta$,

for all $x, y, z \in X$ and $t, t_1, t_2 \in (0, 1]$.

Example 4.22. Let $X = \{0, 1, 2, a, b\}$, in which is defined by Table 4

Table 4

*	0	1	2	*a*	b
0	0	0	0	*a*	*a*
1	1	0	1	b	a
2	2	2	0	*a*	*a*
a	*a*	*a*	*a*	0	0
b	b	*a*	b	1	0

From Table 4, X is a BCI-algebra (Jun *et al.,* 2010). Define $\zeta : X \to [0, 1]$ by

$$\zeta = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.8 & 0.3 & 0.6 & 0.5 & 0.2 \end{pmatrix}.$$

By simple calculations show that ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X for k = 0.08.

Example 4.23. Let X = $\{0, a, \mathrm{b}, \mathrm{c}\}$ in which $*$ is defined by Table 5

Table 5.

*	0	*a*	b	c
0	0	c	0	*a*
a	*a*	0	*a*	c
b	b	c	0	*a*
c	c	*a*	c	0

From Table 5, X is a BCI-algebra. Define $\zeta : X \to [0, 1]$ by $\zeta(0) = 0.3$, $\zeta(a) = \zeta(\mathrm{b}) = \zeta(\mathrm{c}) = 0.4$. By simple calculations show that ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X for k = 0, but it is not fuzzy fantastic ideal of X.

An $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of a BCI-algebra X with k = 0 is called an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy fantastic ideal of X.

Clearly, every fuzzy fantastic ideal is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal.

Let ζ be an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of a BCI-algebra X. Assume that there exists $a \in X$ such that

$$\zeta(a) > \zeta(0) \vee \frac{1-\ k}{2}.$$

Then

$$\zeta(a) \geq \mathrm{t} > \zeta(0) \vee \frac{1-\ k}{2}$$

for some $t \in (\frac{1-k}{2}, 1]$. It follows that

$$0_t \bar{\in} \zeta, a_t \in \zeta \text{ and } \zeta(0) + t \geq 2t > 1 - k$$

i.e., $a_t q_k \zeta$. Which is a contradiction, and so the following inequality is hold.

(i) $\zeta(x) \leq \zeta(0) \vee \frac{1-k}{2}$ for all $x \in X$.

Assume that

$$\zeta(a * (b * (b * a))) \vee \frac{1-k}{2} < \zeta((a * b) * c) \wedge \zeta(c)$$

for some a, b, c $\in$ X. Then there exists $t \in (\frac{1-k}{2}, 1]$ such that

$$\zeta(a * (b * (b * a))) \vee \frac{1-k}{2} < t \leq \zeta((a * b) * c) \wedge \zeta(c).$$

Thus $(a * (b * (b * a)))_t \bar{\in} \zeta$. From $t \leq \zeta((a * b) * c) \wedge \zeta(c)$, we have

$$((a * b) * c)_t \in \zeta, c_t \in \zeta, \zeta((a * b) * c) + t \geq 2t > 1 - k$$

i.e.,

$$((a * b) * c)_t\, q_k \zeta, \text{ and } \zeta(c) + t \geq 2t > 1 - k$$

i.e., $c_t q_k \zeta$. Which is not possible and hence we know that ζ satisfies the following assertion:

(ii) $\zeta(x * (y * (y * x))) \vee \frac{1-k}{2} \geq \zeta((x * y) * z) \wedge \zeta(z)$, for all x, y, z $\in$ X.

Suppose ζ is a fuzzy subset of a BCI-algebra X satisfying (i) and (ii). Let $t \in (\frac{1-k}{2}, 1]$ be such that $\zeta_t \neq \phi$. Then there exists $a \in \zeta_t$, and so

$$\frac{1-k}{2} < t \leq \zeta(a) \leq \zeta(0) \vee \frac{1-k}{2} = \zeta(0)$$

by (i). Hence $0 \in \zeta_t$. Let x, y, z $\in$ X be such that

$$(x * y) * z \in \zeta_t \text{ and } z \in \zeta_t.$$

Then

$$\zeta((x * y) * z) \geq t > \frac{1-k}{2} \text{ and } \zeta(z) \geq t > \frac{1-k}{2}.$$

By using (ii), we get

$$\begin{aligned} \zeta(x * (y * (y * x))) \vee \frac{1-k}{2} &\geq \zeta((x * y) * z) \wedge \zeta(z) \\ &\geq t \wedge t \\ &\geq t \\ &> \frac{1-k}{2} \end{aligned}$$

This implies that

$$\zeta(x * (y * (y * x))) = \zeta(x * (y * (y * x))) \vee \frac{1- k}{2} \geq t.$$

Thus $x * (y * (y * x)) \in \zeta_t$. Consequently, $\zeta_t \triangleright X$. Thus, we conclude that if a fuzzy subset ζ of X satisfies conditions (i) and (ii), then the following assertion is hold.

(iii) $\zeta_t \neq \phi \Rightarrow \zeta_t \triangleright X$, for all $t \in (\frac{1- k}{2}, 1]$.

Let ζ be a fuzzy subset of a BCI-algebra X satisfying (iii). Let $x \in X$ and $t \in (0, 1]$ be such that

$$x_t \overline{\bar{\in} \vee \bar{q}_k} \zeta.$$

Then $x_t \in \zeta$ and $x_t q_k \zeta$. Hence $x \in \zeta_t$, i.e., $\zeta_t \neq \phi$ and so $\zeta_t \triangleright X$ by (iii). Thus $0 \in \zeta_t$ and thus $\zeta(0) \geq t$, i.e., $0_t \in \zeta$. This shows that condition (X) is hold. Let x, y, $z \in X$ and $t_1, t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \overline{\bar{\in} \vee \bar{q}_k} \zeta \text{ and } z_{t_2} \overline{\bar{\in} \vee \bar{q}_k} \zeta.$$

Then

$$((x * y) * z)_{t_1} \in \zeta, z_{t_2} \in \zeta, ((x * y) * z)_{t_1} q_k \zeta \text{ and } z_{t_2} q_k \zeta.$$

This imply that

$$(x * y) * z \in \zeta_{t_1} \subseteq \zeta_{t_1 \wedge t_2} \text{ and } z \in \zeta_{t_2} \subseteq \zeta_{t_1 \wedge t_2}.$$

Since $\zeta_{t_1 \wedge t_2} \triangleright X$ by (iii), it follows from (I3) that $x * (y * (y * x)) \in \zeta_{t_1 \wedge t_2}$ i.e.,

$$\zeta(x * (y * (y * x))) \geq t_1 \wedge t_2 .$$

So that

$$(x * (y * (y * x)))_{t_1 \wedge t_2} \in \zeta.$$

Hence (Y) holds and so ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

So we have the following theorem.

Theorem 4.24. For a fuzzy subset ζ of a BCI-algebra X, the following are equivalent:

(Z) ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

(A1) ζ satisfies the condition (iii).

(B1) ζ satisfies conditions (i) and (ii).

Corollary 4.25. For a fuzzy subset ζ of a BCI-algebra X, the following are equivalent:

(C1) ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy fantastic ideal of X.

(D1) $\zeta_t \neq \phi \Rightarrow \zeta_t \triangleright X$, for all $t \in (0.5, 1]$.

(E1) ζ satisfies the following conditions:

(i) $\zeta(x) \leq \zeta(0) \vee 0.5$,

(ii) $\zeta(x * (y * (y * x))) \vee 0.5 \geq \zeta((x * y) * z) \wedge \zeta(z)$,

for all $x, y, z \in X$.

For a fuzzy subset ζ of a BCI-algebra X, we consider the following set

$$\gamma = \{t \in (0, 1] \mid \zeta_t \neq \phi \Rightarrow \zeta_t \triangleright X\}.$$

Then

(*a*) If $\gamma = [0, 1]$, then ζ is a fuzzy fantastic ideal of X.

(b) If $\gamma = (0, \frac{1-k}{2}]$, then ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

(c) If $\gamma = (\frac{1-k}{2}, 1]$, then ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

Definition 4.26. A fuzzy subset ζ of a BCI-algebra X is called a fuzzy fantastic ideal with thresholds ϵ and δ of X, $\epsilon, \delta \in (0, 1]$ with $\epsilon < \delta$, if it satisfies (G1) and (H1), where

(G1) $\zeta(0) \vee \epsilon \geq \zeta(x) \wedge \delta$,

(H1) $\zeta(x * (y * (y * x))) \vee \epsilon \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \delta$,

for all $x, y, z \in X$.

Example 4.27. Let $X = \{0, 1, 2, a, b\}$ in which $*$ is defined by Table 6

Table 6.

*	0	1	2	*a*	b
0	0	0	0	*a*	*a*
1	1	0	1	b	*a*
2	2	2	0	*a*	*a*
a	*a*	*a*	*a*	0	0
b	b	*a*	b	1	0

From Table 6, X is a BCI-algebra (Jun *et al.,* 2010). Define $\zeta : X \to [0, 1]$ by $\zeta(0) = 0.4$, $\zeta(1) = \zeta(b) = 0.1$, $\zeta(2) = 0.5$ and $\zeta(a) = 0.3$. By simple calculations show that ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X with thresholds $\epsilon = 0.3$ and $\delta = 0.4$.

Theorem 4.28. Let ζ be a fuzzy subset of a BCI-algebra X, $\epsilon, \delta \in [0, 1]$ with $\epsilon < \delta$.

Then ζ is a fuzzy fantastic ideal with thresholds ϵ and δ of X if and only if it satisfies

$$\zeta_t \neq \phi \Rightarrow \zeta_t \triangleright X, \tag{4}$$

for all t $\in (\epsilon, \delta]$.

Proof. The proof is similar to the proof of Theorems 4.9 and 4.20.

Theorem 4.29. Let ζ be a fuzzy subset of a BCI-algebra X, $\epsilon, \delta \in [0, 1]$ with $\epsilon < \delta$. Then

(J1) ζ is a fuzzy fantastic ideal of X if and only if ζ is a fuzzy fantastic ideal of X with thresholds $\epsilon = 0$ and $\delta = 1$.

(K1) ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X if and only if ζ is a fuzzy fantastic ideal of X with thresholds $\epsilon = 0$ and $\delta = \frac{1-k}{2}$.

(L1) ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X if and only if ζ is a fuzzy fantastic ideal of X with thresholds $\epsilon = \frac{1-k}{2}$ and $\delta = 1$.

Proof. Straightforward.

5. Implication-based fuzzy fantastic ideals in BCI-algebras

Fuzzy propositional calculus is an extension of the Aristotelean propositional calculus. In fuzzy propositional calculus the truth set is taken [0, 1] instead of {0, 1}, which is the truth set in Aristotelean propositional calculus. In fuzzy logic some of the operators, like $\wedge$, $\vee$, $\neg$, $\rightarrow$ can be defined by using truth tables. One can also use the extension principle to obtain the definitions of these operators.

In fuzzy logic the truth value of a fuzzy proposition ζ is denoted by $[\zeta]$: In the following we give fuzzy logic and its corresponding set theoretical notations, which we will use in the paper hereafter.

$$[x \in \zeta] = \zeta(x) \tag{5}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\} \tag{6}$$

$$[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\} \tag{7}$$

$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)] \tag{8}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{9}$$

The truth valuation rules given in (7) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators can be similarly defined. We consider in the following some important implication operators:

(*a*) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise}, \end{cases}$$

(b) Gödel implication operator (I_G):

$$I_G(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise}, \end{cases}$$

(c) The contraposition of Gödel implication operator ($\bar{I}_G$):

$$\bar{I}_G(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 1-x & \text{otherwise}, \end{cases}$$

Where x is the degree of truth (or degree of membership) of the premise and y is the respective values for the consequence and I is the resulting degree of truth for the implication. The quality of these implication operators could be evaluated either by empirically or by axiomatically methods.

(Ying, 1991) introduced the concept of fuzzifying topology. Here we can expand his idea to BCI-algebras, and we define a fuzzifying fantastic ideal as follows.

Definition 5.1. A fuzzy subset ζ of a BCI-algebra X is called a fuzzifying fantastic ideal of X if it satisfies (M1) and (N1), where

(M1) $| = [x \in \zeta] \rightarrow [0 \in \zeta]$,

(N1) $| = [(x * y) * z \in \zeta] \wedge [z \in \zeta] \rightarrow [x * (y * (y * x)) \in \zeta]$,

for all x, y, z $\in$ X.

Obviously, conditions (M1) and (N1) are equivalent to (F1) and (F3), respectively. Therefore a fuzzifying fantastic ideal is an ordinary fuzzy fantastic ideal. In (Ying, 1988), the concept of t-tautology is introduced, i.e.,

$$| =_t \Phi \text{ if and only if } [\Phi] \geq t \text{ for all valuations.} \tag{10}$$

Definition 5.2. Let ζ be a fuzzy subset of a BCI-algebra X and t $\in$ (0, 1], ζ is called a t-implication-based fuzzy fantastic ideal of X if it satisfies (O1) and (P1), where

(O1) $\mid =_t [x \in \zeta] \to [0 \in \zeta]$,

(P1) $\mid =_t [(x * y) * z \in \zeta] \wedge [z \in \zeta] \to [x * (y * (y * x)) \in \zeta]$,

for all x, y, z $\in$ X.

Let I be an implication operator. Clearly, ζ is a t-implication-based fuzzy fantastic ideal of a BCI-algebra X if and only if it satisfies (Q1) and (R1), where

(Q1) $I(\zeta(x), \zeta(0)) \geq t$,

(R1) $I(\zeta((x * y) * z) \wedge \zeta(z), \zeta(x * (y * (y * x)))) \geq t$,

for all x, y, z $\in$ X.

Theorem 5.3. For any fuzzy subset ζ of a BCI-algebra X, we have

(S1) If $I = I_{GR}$, then ζ is a 0.5-implication-based fuzzy fantastic ideal of X if and only if ζ is a fuzzy fantastic ideal of X.

(T1) If $I = I_G$, then ζ is a $\frac{1-k}{2}$-implication-based fuzzy fantastic ideal of X if and only if ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X.

(U1) If $I = \bar{I}_G$, then ζ is a $\frac{1-k}{2}$-implication-based fuzzy fantastic ideal of X if and only if ζ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal X.

Proof. (S1) Straightforward.

(T1) Suppose that ζ is a $\frac{1-k}{2}$-implication-based fuzzy fantastic ideal of X. Then $I_G(\zeta(x), \zeta(0)) \geq \frac{1-k}{2}$

and

$$I_G(\zeta((x * y) * z) \wedge \zeta(z), \zeta(x * (y * (y * x)))) \geq \frac{1-k}{2}.$$

It follows that

$$\zeta(0) \geq \zeta(x))$$

or

$$\zeta(x) \geq \zeta(0) \geq \frac{1-k}{2},$$

and

$$\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z)$$

or

$$\zeta((x * y) * z) \wedge \zeta(z) \geq \zeta(x * (y * (y * x))) \geq \frac{1-k}{2}.$$

Hence

$$\zeta(0) \vee 0 = \zeta(0) \geq \zeta(x) \wedge \frac{1-k}{2}$$

and

$$\zeta(x * (y * (y * x))) \vee 0 = \zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1-k}{2}$$

Therefore ζ is a fuzzy fantastic ideal of X with thresholds $\epsilon = 0$ and $\delta = \frac{1-k}{2}$, and hence ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X by Theorem 4.29.

Conversely, assume that ζ is an $(\in, \in \vee q_k)$-fuzzy fantastic ideal of X. Then

$$\zeta(0) = \zeta(0) \vee 0 \geq \zeta(x) \wedge \frac{1-k}{2}$$

and

$$\zeta(x * (y * (y * x))) = \zeta(x * (y * (y * x))) \vee 0 \geq \zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1-k}{2}.$$

Case 1: If $\zeta(x) \wedge \frac{1-k}{2} = \zeta(x)$, then

$$I_G(\zeta(x), \zeta(0)) = 1 \geq \frac{1-k}{2}.$$

If $\zeta(x) \wedge \frac{1-k}{2} = \frac{1-k}{2}$, then

$$\zeta(0) \geq \frac{1-k}{2}$$

and so

$$I_G(\zeta(x), \zeta(0)) \geq \frac{1-k}{2}.$$

Case 2: If $\zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1-k}{2} = \zeta((x * y) * z) \wedge \zeta(z)$, then

$$\zeta(x * (y * (y * x))) \geq \zeta((x * y) * z) \wedge \zeta(z)$$

and thus

$$I_G(\zeta((x * y) * z) \wedge \zeta(z), \zeta(x * (y * (y * x)))) = 1 \geq \frac{1-k}{2}.$$

Let

$$\zeta((x * y) * z) \wedge \zeta(z) \wedge \frac{1-k}{2} = \frac{1-k}{2}.$$

Then $\zeta(x * (y * (y * x))) \geq \frac{1-k}{2}$, and hence

$$I_G(\zeta((x * y) * z) \wedge \zeta(z), \zeta(x * (y * (y * x)))) \geq \frac{1-k}{2}.$$

Therefore ζ is a $\dfrac{1-k}{2}$-implication-based fuzzy fantastic ideal of X.

(U1) Straightforward.

REFERENCES

Bhakat, S. K. 1999. $(\in \vee q)$-level subsets. Fuzzy Sets and Systems **103**: 529-533.

Bhakat, S. K. 2000. $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups. Fuzzy Sets and Systems **112**: 299-312.

Das, P. S. 1981. Fuzzy groups and level subgroups. Journal of Mathematical Analysis and Applications **84**: 264-269.

Bhakat, S. K. & Das, P. 1996. $(\in, \in \vee q)$-fuzzy subgroups. Fuzzy Sets and Systems **80**: 359-368.

Guangji, Z., Cheng, Z., Zixin, L. & Jiatai, G. 2006. New kinds of fuzzy ideals in BCI-algebras. Fuzzy Optimization and Decision Making **5**: 177-186.

Huang, Y. S. 2006. BCI-algebras. Science Press, China.

Jun, Y. B. 2004. On (α, β)-fuzzy ideals of BCK/BCI-algebras. Scientiae Mathematicae Japonicae **60**: 613-617.

Jun, Y. B. 2005. On (α, β)-fuzzy subalgebras of BCK/BCI-algebras. Bulletin of the Korean Mathematical Society **42**: 703-711.

Jun, Y. B. 2007. Fuzzy subalgebras of type (α, β) in BCK/BCI-algebras. Kyungpook Mathematical Journal **47**: 403-410.

Jun, Y. B. 2009. Generalizations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. Computers and Mathematics with Applications **58:** 1383-1390.

Jun, Y. B., Lee, K. J. & Park, C. H. 2010. New types of fuzzy ideals in BCK/ BCI- algebras. Computers and Mathematics with Applications **60**: 771-785.

Ma, X., Zhan, J., Davvaz, B. & Jun, Y. B. 2008. Some kinds of $(\in, \in \vee q)$-interval- valued fuzzy ideals of BCI-algebras. Information Sciences **178**: 3738-3754.

Meng, J. 1997. On ideals in BCK-algebras. Mathematica Japonica **40**: 143-154.

Meng, J. & Guo, X. 2005. On fuzzy ideals in BCK/ BCI-algebras. Fuzzy sets and Systems **149:** 509-525.

Meng, J. & Jun, Y. B. 1994. BCK-algebras. Kyung Moon Sa Co., Seoul, Korea.

Meng, J., Jun, Y. B. & Kim, H. S. 1997. Fuzzy implicative ideals of BCK-algebras. Fuzzy sets and Systems **89**: 243-248.

Murali, V. 2004. Fuzzy points of equivalent fuzzy subsets. Information Sciences **158**: 277-288.

Pu, P. M. & Liu, Y. M. 1980. Fuzzy topology I: neighourhood structure of a fuzzy point and Moore-Smith convergence. Journal of Mathematical Analysis and Applications **76**: 571-599.

Rosenfeld, A. 1971. Fuzzy groups. Journal of Mathematical Analysis and Applications **35:** 512-517.

Saeid, A. B. 2010. Fantastic ideals in BCI-algebras. World Applied Sciences Journal **8** (5): 550-554.

Xi, O. G. 1991. Fuzzy BCK-algebras. Mathematica Japonica 36: 935-942.

Ying, M. S. 1991. A1991. A new approach for fuzzy topology (I). Fuzzy Sets and Systems **39**: 303-321.

Ying, M. S. 1988. On standard models of fuzzy modal logics. Fuzzy Sets and Systems **26**: 357-363.

Yuan, X., Zhang, C. & Ren, Y. 2003. Generalized fuzzy groups and many-valued implications. Fuzzy Sets and Systems **138**: 205-211.

Zadeh, L. A. 1965. Fuzzy sets. Information and Control **8**: 338-353.

Zhan, J. & Jun, Y. B. 2009. Generalized fuzzy ideals of BCI-algebras. Bulletin of the Malaysian Mathematical Sciences Society **32**: 119-130.

Zhan, J. & Jun, Y. B. 2011. On $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals of BCI-algebras. Neural Computing and Applications **20:** 319-328.

Zhan, J., Jun, Y. B. & Davvaz, B. 2009. On $(\in, \in \vee q)$-fuzzy ideals of BCI-algebras. Iranian Journal of Fuzzy Systems **6** (1): 81-94.

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خصائص المثاليات الخيالية المشوشة في الجبريات من نوع بي سي آي

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خلاصة

نقدم في هذا البحث مفاهيم المثاليات الخيالية المشوشة والمثاليات الخيالية المشوشة في الجبريات من نوع بي سي آي. كما نقوم بدراسة خصائص هذه المفاهيم ونناقش بعض الحالات المعينة مثل: المثاليات الخيالية المشوشة ذات الحد، والمثاليات الخيالية المشوشة من النوعين $(\in, \in \vee q_k)$ و $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$.