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On the developable ruled surfaces Kinematically generated in Minkowski 3-Space

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ABSTRACT

In this paper, we present a method to be developable of a ruled surface, generated in Minkowski 3-space $\mathbb{R}_1^3$, corresponding to the dual Lorentzian curves according to E. Study's transference principle and some theorems and examples.

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INTRODUCTION

Rather unexpectedly, dual numbers have been applied to study the motion of a line in space; in Euclidean space $\mathbb{R}^3$, they even seem to be the most appropriate apparatus for this purpose.

It was first done by Study (1891), and since his time, dual numbers had an established place in kinematics as a tool to solve problems dealing with lines in space. The application of dual numbers to the lines of the Euclidean 3-space is carried out by the principle of transference which has been formulated by E. Study. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry, by means of dual number extension, i.e., replacing all ordinary quantities by the corresponding dual number quantities (Veldkamp, 1976).

Expressing the differential geometry of ruled surfaces in terms of dual vector calculus has rederived the curvature theory of a line trajectory and exposed the fundamental curvature functions that characterize the shape of ruled surface. The curvature theory of line trajectories has also been studied by (Hacısalihoğlu, 1972; Müller, 1980; Chen & Pottmann, 1999; Köse, 1999; Karadağ & Keleş, 2005). Using the geometry of curves and developable ruled surfaces, some spatial design problems were investigated by (Schaaf & Ravani, 1998). On the other hand, the ruled surfaces in a three-dimensional Minkowski space were studied by (Turgut & Hacısalihoğlu, 1998; Kim & Yoon, 2004).

In this paper, we study the ruled surfaces generated by the dual Lorentzian vectors in the Lorentzian dual unit sphere S_1^2 and the hyperbolic dual unit sphere H_0^2. This paper is organized as follows: In section 2, we give some basic concepts related to the dual Lorentzian space $\mathbb{D}_1^3$. In section 3 (and respectively in section 4), we obtain a method to be developable of a ruled surface generated by the dual Lorentzian vectors in S_1^2 (and respectively in H_0^2) and we give some examples.

PRELIMINARIES

The set $\mathbb{D}=\{\hat{A}=a+\varepsilon a^*: a, a^* \in \mathbb{R}, \varepsilon^2=0\}$ of dual numbers is a commutative ring with respect to the operations

$$i)\ \hat{A} \bigoplus \hat{B}=(a+b)+\varepsilon(a^*+b^*),$$

$$ii)\ \hat{A} \odot \hat{B}=ab+\varepsilon(ab^*+a^*b).$$

$(\mathbb{D}^3, <\ ,\ >)$ is called a dual Lorentzian space (or $\mathbb{D}-$modul) and denoted by $\mathbb{D}_1^3$, where

$$\mathbb{D}^3=\{\hat{X}:\hat{X}=x+\varepsilon x^*;\quad x, x^* \in \mathbb{R}^3\},$$

$$<\hat{X}, \hat{Y}> = <x, y> + \varepsilon(<x^*, y> + <x, y^*>),$$

$$<x, y> = -x_1y_1+x_2y_2+x_3y_3, \tag{1}$$

for any $\hat{X}, \hat{Y} \in \mathbb{D}_1^3$ and $x, y \in \mathbb{R}_1^3$.

Also, the Lorentzian vector product of dual vectors $\hat{X}, \hat{Y} \in \mathbb{D}_1^3$ is defined by

$$\hat{X} \wedge \hat{Y}=x \wedge y+\varepsilon(x \wedge y^*+x^* \wedge y),$$

where

$$x \wedge y=(-x_2y_3+x_3y_2, x_3y_1-x_1y_3, x_1y_2-x_2y_1). \tag{2}$$

On the other hand, Taylor series expression of an analytic dual variable function is given by

$$f(\hat{X})=f(x+\varepsilon x^*)=f(x)+\varepsilon x^* f'(x), \tag{3}$$

where f' is derivative of the analitic dual variable function f.

It is clear that any dual vector $\hat{X}$ in $\mathbb{D}_1^3$ consists of any two real vectors x and x^* in $\mathbb{R}_1^3$, which are expressed in the natural orthonormal frame in the 3-dimensional Lorentzian Euclidean space $\mathbb{R}_1^3$. A dual vector $\hat{X} \in \mathbb{D}_1^3$ is said to be spacelike, timelike or lightlike (null) if the vector $x \in \mathbb{R}_1^3$ is spacelike, timelike or lightlike (null), respectively. If $x \neq 0$, then the norm of the dual vector $\hat{X} \in \mathbb{D}_1^3$ is defined by $\| \hat{X} \| = \sqrt{| < \hat{X}, \hat{X} > |}$.

On the other hand,

$$S_1^2 = \{\hat{X} = x + \varepsilon x^* \colon \| \hat{X} \| = (1,0); \quad x, x^* \in \mathbb{R}^3, x\text{−spacelike}\}$$

is called the Lorentzian dual unit sphere in $\mathbb{D}_1^3$ and

$$H_0^2 = \{\hat{X} = x + \varepsilon x^* \colon \| \hat{X} \| = (1,0); \quad x, x^* \in \mathbb{R}^3, x\text{−timelike}\}$$

is called the hyperbolic dual unit sphere in $\mathbb{D}_1^3$, (O'Neill, 1983). Oriented spacelike and timelike lines in $\mathbb{R}_1^3$ may be represented by spacelike and timelike unit vectors with three-components in the dual Lorentzian space $\mathbb{D}_1^3$, respectively. While a differentiable curve on the Lorentzian dual unit sphere S_1^2 corresponds to any ruled surface in $\mathbb{R}_1^3$, a differentiable curve on the hyperbolic dual unit sphere H_0^2 corresponds to a timelike ruled surface $\mathbb{R}_1^3$.

A differentiable Lorentzian curve $\hat{X}$ on a dual unit sphere, depending on real parameter t, represents a differentiable family of straight line in R_1^3 which we call a ruled surface. This ruled surface $\Phi(t,u) = \alpha(t) + u x(t)$ is written as the Lorentzian dual vector function $\hat{X}(t)$ given by

$$\hat{X}(t) = x(t) + \varepsilon \alpha(t) \wedge x(t) = x(t) + \varepsilon x^*(t). \tag{4}$$

Since the spherical indicatrix of $\hat{X}(t)$ is a dual unit Lorentzian vector $\hat{X}(t)$, it has unit magnitude, i.e., $<\hat{X}, \hat{X}> = < x, x> + 2\varepsilon < x, \alpha \wedge x> + \varepsilon^2 < \alpha \wedge x, \alpha \wedge x> = < x, x> = \varepsilon, \quad \varepsilon = \mp 1$.

Thus, the ruled surface can be represented by the Lorentzian dual curve on the surface of a Lorentzian dual unit sphere S_1^2 (or a hyperbolic dual unit sphere H_0^2).

The dual arc-length of the Lorentzian dual curve $\hat{X}(t)$ is defined as

$$\hat{s} = \int_0^t \| \frac{d\hat{X}}{dt} \| \, dt. \tag{5}$$

From the integrant of equation (5), the dual speed of $\hat{X}(t)$ is

$$\widehat{\delta} = \delta(1 + \varepsilon \Delta), \tag{6}$$

where $\delta = \| \frac{dx}{dt} \|$ and $\Delta = \frac{< \frac{dx}{dt}, \frac{dx^*}{dt} >}{\| dxdt \|^2}$. The curvature function Δ is the well-known distribution parameter (or drall) of the ruled surface. The relationship between the Gaussian curvature K and the distribution parameter (or drall) Δ of the ruled surface is given by the following formula:

$$K = \varepsilon \frac{\Delta^2}{(\Delta^2 + u^2)^2}, \quad \varepsilon = \mp 1. \tag{7}$$

If the ruled surface is a timelike surface, then the Gauss curvature is positive (Köse, 1999). If K is zero everywhere, that is, Δ is zero everywhere, then the ruled surface is said to be developable.

ON THE DEVELOPABLE RULED SURFACE, IN $\mathbb{R}_1^3$, WHICH IS CORRESPONDING TO A SPACELIKE DUAL UNIT VECTOR ON S_1^2

In this section, we give a method of determining to be developable of the ruled surface generated by a spacelike curve in S_1^2 .

The dual coordinates $\hat{X}_i = x_i + \varepsilon x_i^*$, $(i = 1, 2, 3)$, of an arbitrary point $\hat{X}$ of a Lorentzian dual unit sphere S_1^2 may be expressed as

$$(\hat{X}_1, \hat{X}_2, \hat{X}_3) = (\sinh\widehat{\Theta}\sin\widehat{\Phi}, \cosh\widehat{\Theta}\sin\widehat{\Phi}, \cos\widehat{\Phi}) \tag{8}$$

where $\widehat{\Theta} = \widehat{\Theta}(t) = \theta(t) + \varepsilon\theta^*(t)$, $\widehat{\Phi} = \widehat{\Phi}(t) = \varphi(t) + \varepsilon\varphi^*(t)$ are dual angles with $-\pi < \varphi < \pi$ and $0 \neq \theta \in \mathbb{R}$. Since $\varepsilon^2 = \varepsilon^3 = ... = 0$, according to the Taylor series expansion from equation (8), we obtain

$$x_1 = \sinh\theta\sin\varphi, \qquad x_1^* = \varphi^*\sinh\theta\cos\varphi + \theta^*\cosh\theta\sin\varphi,$$

$$x_2 = \cosh\theta\sin\varphi, \qquad x_2^* = \varphi^*\cosh\theta\cos\varphi + \theta^*\sinh\theta\sin\varphi,$$

$$x_3 = \cos\varphi, \qquad x_3^* = -\varphi^*\sin\varphi.$$

Thus, the Lorentzian dual vector $\hat{X}(t)$ can be written as

$$\hat{X}(t) = x(t) + \varepsilon x^*(t) = (\sinh\theta\sin\varphi, \cosh\theta\sin\varphi, \cos\varphi) \tag{9}$$

$$+\varepsilon(\varphi^*\sinh\theta\cos\varphi + \theta^*\cosh\theta\sin\varphi, \varphi^*\cosh\theta\cos\varphi + \theta^*\sinh\theta\sin\varphi, -\varphi^*\sin\varphi).$$

Now, let us consider the dual curve $\hat{X}(t)$ given by equation (9) on the dual unit sphere corresponding to the ruled surface

$$\Psi(t,u) = \alpha(t) + ux(t), \tag{10}$$

where $\alpha(t), x(t) \in \mathbb{R}_1^3$ and $u \in \mathbb{R}$. On the other hand, It is well known that (Hacısalihoğlu, 1972),

$$x^*(t) = \alpha(t) \wedge x(t). \tag{11}$$

From equations (2) and (11), we have

$$x^* = (\alpha_3 \cosh\theta \sin\varphi - \alpha_2 \cos\varphi, \alpha_3 \sinh\theta \sin\varphi - \alpha_1 \cos\varphi, \alpha_1 \cosh\theta \sin\varphi - \alpha_2 \sinh\theta \sin\varphi). \tag{12}$$

Thus, from equations (9) and (12), we get

$$\begin{aligned} &\alpha_3 \cosh\theta \sin\varphi - \alpha_2 \cos\varphi = \varphi^* \sinh\theta \cos\varphi + \theta^* \cosh\theta \sin\varphi, \\ &\alpha_3 \sinh\theta \sin\varphi - \alpha_1 \cos\varphi = \varphi^* \cosh\theta \cos\varphi + \theta^* \sinh\theta \sin\varphi, \\ &\alpha_1 \cosh\theta \sin\varphi - \alpha_2 \sinh\theta \sin\varphi = -\varphi^* \sin\varphi. \end{aligned} \tag{13}$$

The matrix of coefficients of unknowns α_1, α_2 and α_3 is

$$A = \begin{bmatrix} 0 & -\cos\varphi & \cosh\theta \sin\varphi \\ -\cos\varphi & 0 & \sinh\theta \sin\varphi \\ \cosh\theta \sin\varphi & -\sinh\theta \sin\varphi & 0 \end{bmatrix}$$

and since $\det A = 0$, $rank A < 3$. For $\varphi \neq \frac{(2k+1)\pi}{2,} k \in \mathbb{Z}$, $det\ A_1 \neq 0$ and its *rank* is 2, where

$$A_1 = \begin{bmatrix} 0 & -\cos\varphi \\ -\cos\varphi & 0 \end{bmatrix}.$$

Let

$$A_2 = \begin{bmatrix} 0 & -\cos\varphi & \varphi^* \sinh\theta \cos\varphi + \theta^* \cosh\theta \sin\varphi \\ -\cos\varphi & 0 & \varphi^* \cosh\theta \cos\varphi + \theta^* \sinh\theta \sin\varphi \\ \cosh\theta \sin\varphi & -\sinh\theta \sin\varphi & -\varphi^* \sin\varphi \end{bmatrix}.$$

The *rank* of the augmented matrix A_2 is 2. Hence, this system of linear equations has infinite solutions given by

$$\begin{aligned} &\alpha_1(t) = (\alpha_3(t) - \theta^*(t)) \sinh\theta(t) \tan\varphi(t) - \varphi^*(t) \cosh\theta(t), \\ &\alpha_2(t) = (\alpha_3(t) - \theta^*(t)) \cosh\theta(t) \tan\varphi(t) - \varphi^*(t) \sinh\theta(t), \\ &\alpha_3(t) = \alpha_3(t). \end{aligned} \tag{14}$$

Since $\alpha_3(t)$ can be chosen arbitrarily, then we may take $\alpha_3(t) = \theta^*(t)$. In this case, equation (14) reduces to

$$(\alpha_1(t),\alpha_2(t),\alpha_3(t)) = (-\varphi^*(t)\cosh\theta(t), -\varphi^*(t)\sinh\theta(t),\alpha_3(t)). \tag{15}$$

From equation (15), we have

$$\coth\theta(t) = \frac{\alpha_1(t)}{\alpha_2(t)}, \quad \varphi^*(t) = \mp\sqrt{|\alpha_1^2(t)-\alpha_2^2(t)|}, \quad \theta^*(t) = \alpha_3(t). \tag{16}$$

Using equations (6) and (9), the distribution parameter (drall) of the ruled surface given by equation (10) is obtained as follows

$$\Delta = \frac{\frac{d\theta}{dt}\frac{d\theta^*}{dt}\sin^2\varphi + \varphi^*(\frac{d\theta}{dt})^2\sin\varphi\cos\varphi - \frac{d\varphi}{dt}\frac{d\varphi^*}{dt}}{(\frac{d\theta}{dt})^2\sin^2\varphi - (\frac{d\varphi}{dt})^2}. \tag{17}$$

If this ruled surface is developable , then $\Delta = 0$. Thus, equation (17) is reduced to

$$\frac{d\theta}{dt}\frac{d\theta^*}{dt}\sin^2\varphi + \varphi^*(\frac{d\theta}{dt})^2\sin\varphi\cos\varphi - \frac{d\varphi}{dt}\frac{d\varphi^*}{dt} = 0. \tag{18}$$

From equation (18), we get

$$\frac{d}{dt}(-\cot\varphi) + \frac{\varphi^*(\frac{d\theta}{dt})^2}{\frac{d\varphi^*}{dt}}(-\cot\varphi) - \frac{\frac{d\theta}{dt}\frac{d\theta^*}{dt}}{\frac{d\varphi^*}{dt}} = 0. \tag{19}$$

Setting

$$y(t) = -\cot\varphi(t), \quad P(t) = \frac{\varphi^*(\frac{d\theta}{dt})^2}{\frac{d\varphi^*}{dt}}, \quad Q(t) = \frac{\frac{d\theta}{dt}\frac{d\theta^*}{dt}}{\frac{d\varphi^*}{dt}}, \tag{20}$$

we are lead to a linear differential equation of the first degree

$$\frac{dy}{dt} + P(t)y(t) = Q(t). \tag{21}$$

In the case that $\varphi(t)$ and $\theta(t)$ are both constant, equation (21) is identically zero. In other words, the ruled surface $\hat{X}(t)$ is a Lorentzian cylinder. Now, we try to answer an important question, that is, when we are given a Lorentzian curve $\alpha{=}\alpha(t)$, can we find a developable ruled surface such that its base curve is the curve $\alpha{=}\alpha(t)$? The answer is positive; in fact, from equation (15) we have

$$\theta(t) = \coth^{-1}\left(\frac{\alpha_1(t)}{\alpha_2(t)}\right), \quad \varphi^*(t) = \mp\sqrt{|\alpha_1^2(t) - \alpha_2^2(t)|}, \quad \theta^*(t) = \alpha_3(t).$$

Now only $\varphi(t)$ remains to be determined. From equation (21), if $Q(t) = 0$, then $\frac{d\theta}{dt} = 0$ or $\frac{d\theta^*}{dt} = 0$, that is, θ=constant or θ^*=constant. Thus, equation (21) becomes

$$\frac{dy}{dt} + P(t)y(t) = 0.$$

The solution of this equation is

$$y = k exp\left[\int P(t)\right]dt.$$

Let $Q(t) \neq 0$, then the generalized solution of equation (21), when $k{=}v(t)$, is given by

$$y = exp\left[-\int P(t)dt\right]\left[\int exp\left[\int P(t)dt\right]Q(t)dt + c\right], \tag{22}$$

where c is an arbitrary constant. Furthermore, it can be shown that, this one-parameter family of solutions of the linear differential equation (21) includes all solutions of equation (22).

Thus, from equation (20), the solutions of the linear differential equation (21) give $-\cot\varphi(t)$. This solution includes an integral; therefore we have infinitely many developable ruled surfaces such that each having a base curve, $\alpha(t)$.

On the other hand, it is to be noted that $\varphi^*(t)$, given by equation (16), has two values; when we use the minus sign, we obtain the reciprocal of the ruled surface $\hat{X}(t)$ obtained by using the plus sign for a given integral constant. Thus, we have the following theorem:

Theorem 1 *Let α be a differentiable curve in S_1^2. Then, there exists the family of developable ruled surface in $\mathbb{R}_1^3$ which is corresponding to α in S_1^2 such that*

$$\Psi(t,u)=(\alpha_1(t),\alpha_2(t),\alpha_3(t))+u\left(-\frac{\alpha_2(t)}{\varphi^*(t)}\sin\varphi(t),-\frac{\alpha_1(t)}{\varphi^*(t)}\sin\varphi(t),\cos\varphi(t)\right). \quad (23)$$

ON THE DEVELOPABLE RULED SURFACE, IN $\mathbb{R}_1^3$, WHICH IS CORRESPONDING TO TIMELIKE DUAL UNIT VECTOR ON H_0^2

In this section, we give a method of determining to be developable of the ruled surface generated by a timelike curve in H_0^2 .

Let $\hat{X}(t)=x(t)+\varepsilon x^*(t)$ be a unit dual timelike vector. Then, we may chose

$$\hat{X}(t)=(\cosh\widehat{\Theta}(t),\sinh\widehat{\Theta}(t)\sin\widehat{\Phi}(t),\sinh\widehat{\Theta}(t)\cos\widehat{\Phi}(t)), \quad (24)$$

where $\widehat{\Theta}=\widehat{\Theta}(t)=\theta(t)+\varepsilon\theta^*(t)$, $\widehat{\Phi}=\widehat{\Phi}(t)=\varphi(t)+\varepsilon\varphi^*(t)$. According to the Taylor series expansion, from equations (3) and (24), we obtain

$$x(t)=(\cosh\theta,\sinh\theta\sin\varphi,\sinh\theta\cos\varphi), \quad (25)$$

$$x^*(t)=(\theta^*\sinh\theta,\varphi^*\sinh\theta\cos\varphi+\theta^*\cosh\theta\sin\varphi,-\varphi^*\sinh\theta\sin\varphi+\theta^*\cosh\theta\cos\varphi).$$

Let $\alpha=\alpha(t)=(\alpha_1(t),\alpha_2(t),\alpha_3(t))$ be a regular Lorentzian curve. From equations (2), (11) and(25), we have the following linear system of equations:

$$\begin{aligned}
&-\alpha_1\sinh\theta\cos\varphi+\alpha_3\cosh\theta = \varphi^*\sinh\theta\cos\varphi+\theta^*\cosh\theta\sin\varphi,\\
&-\alpha_2\sinh\theta\cos\varphi+\alpha_3\sinh\theta\sin\varphi=\theta^*\sinh\theta, \\
&-\alpha_1\sinh\theta\sin\varphi-\alpha_2\cosh\theta= -\varphi^*\sinh\theta\sin\varphi+\theta^*\cosh\theta\cos\varphi.
\end{aligned} \quad (26)$$

The matrix of coefficients of unknowns $\alpha_1(t)$,$\alpha_2(t)$ and $\alpha_3(t)$ is

$$B=\begin{bmatrix} 0 & -\sinh\theta\cos\varphi & \sinh\theta\sin\varphi \\ -\sinh\theta\cos\varphi & 0 & \cosh\theta \\ \sinh\theta\sin\varphi & -\cosh\theta & 0 \end{bmatrix}$$

and since $detB=0$, $rankB<3$. For $\theta\neq 0$ and $\varphi\neq\frac{(2k+1)\pi}{2}$, $k\in Z$, $detB_1\neq 0$ and its *rank* is 2, where

$$B_1=\begin{bmatrix} 0 & -\sinh\theta\cos\varphi \\ -\sinh\theta\cos\varphi & 0 \end{bmatrix}.$$

Also the *rank* of the augmented matrix

$$B_2=\begin{bmatrix} 0 & -\sinh\theta\cos\varphi & \theta^*\sinh\theta \\ -\sinh\theta\cos\varphi & 0 & \varphi^*\sinh\theta\cos\varphi+\theta^*\cosh\theta\sin\varphi \\ \sinh\theta\sin\varphi & -\cosh\theta & -\varphi^*\sinh\theta\sin\varphi+\theta^*\cosh\theta\cos\varphi \end{bmatrix}$$

is 2. Hence, this system of linear equation has infinite solutions given by

$$\begin{aligned} \alpha_1(t) &= \alpha_1(t), \\ \alpha_2(t) &= (\alpha_1(t)+\varphi^*(t))\tanh\theta(t)\sin\varphi(t)+\theta^*(t)\cos\varphi(t), \\ \alpha_3(t) &= (\alpha_1(t)+\varphi^*(t))\tanh\theta(t)\cos\varphi(t)+\theta^*(t)\sin\varphi(t). \end{aligned} \tag{27}$$

Since $\alpha_1(t)$ can be chosen arbitrarily, we may take $\alpha_1(t) = -\varphi^*(t)$. In this case, equation (27) reduces to

$$(\alpha_1(t),\alpha_2(t),\alpha_3(t)) = (-\varphi^*(t),\theta^*(t)\cos\varphi(t),\theta^*(t)\sin\varphi(t)). \tag{28}$$

From equation (28), we have

$$\cot\varphi(t) = \frac{\alpha_2(t)}{\alpha_3(t)}, \quad \theta^*(t) = \mp\sqrt{\alpha_1^2(t)+\alpha_3^2(t)}, \quad \varphi^*(t) = -\alpha_1(t). \tag{29}$$

The distribution parameter (drall) of the ruled surface given by equation (24), from equation (6), is

$$\Delta=\frac{\frac{d\theta}{dt}\frac{d\theta^*}{dt}+\theta^*(\frac{d\varphi}{dt})^2\sinh\theta\cosh\theta+\frac{d\varphi}{dt}\frac{d\varphi^*}{dt}\sinh^2\theta}{(\frac{d\varphi}{dt})^2\sinh^2\theta+(\frac{d\theta}{dt})^2}. \tag{30}$$

If this ruled surface is a developable one, then equation (30) becomes

$$\frac{d\theta}{dt}\frac{d\theta^*}{dt}+\theta^*(\frac{d\varphi}{dt})^2\sinh\theta\cosh\theta+\frac{d\varphi}{dt}\frac{d\varphi^*}{dt}\sinh^2\theta= 0. \tag{31}$$

Then, from equation (31), we have

$$\frac{d}{dt}(\coth\theta)-\frac{\theta^*(\frac{d\varphi}{dt})^2}{\frac{d\theta^*}{dt}}(\coth\theta)-\frac{\frac{d\varphi}{dt}\frac{d\varphi^*}{dt}}{\frac{d\theta^*}{dt}}= 0. \tag{32}$$

Setting

$$y(t) = \coth\theta(t), \quad P(t) = -\frac{\theta^*(\frac{d\varphi}{dt})^2}{\frac{d\theta^*}{dt}}, \quad Q(t) = \frac{\frac{d\varphi}{dt}\frac{d\varphi^*}{dt}}{\frac{d\theta^*}{dt}}, \tag{33}$$

we are lead to a linear differential equation of first degree

$$\frac{dy}{dt} + P(t)y(t) = Q(t). \tag{34}$$

The solution of this linear differential equation gives $\coth\theta$. This solution includes an integral, therefore we have infinitely many developable ruled surfaces such that its base curve is $\alpha(t)$.

On the other hand, it is to be noted that $\theta^*(t)$, given by equation (29), has two values; when we use the minus sign, we obtain the reciprocal of the ruled surface $\hat{X}(t)$ obtained by using the plus sign for a given integral constant.

Theorem 2 *Let α be a differentiable curve in H_0^2. Then there exists the family of developable ruled surface in $\mathbb{R}_1^3$ which is corresponding to α in H_0^2 such that*

$$\Psi(t,u) = (\alpha_1, \alpha_2, \alpha_3) + u(\cosh\theta, \frac{\alpha_3}{\theta^*}\sinh\theta, \frac{\alpha_2}{\theta^*}\sinh\theta). \tag{35}$$

Now, we can give the following examples for each type.

Example 3 *We consider a differentiable Lorentzian curve $\alpha = \alpha(t) = (t, t, 1)$. We note that this curve is a lightlike curve. Thus, weget*

$$\frac{d\varphi}{dt} = -\frac{1}{t^2+1}, \quad \frac{d\varphi^*}{dt} = -1, \quad \frac{d\theta^*}{dt} = \frac{t}{\sqrt{t^2+1}}, P(t) = -\frac{1}{t(t^2+1)}, \quad Q(t) = \frac{1}{t(\sqrt{t^2+1})} \quad .$$

Hence, the linear differential equation is written as

$$\frac{dy}{dt} - \frac{1}{t(t^2+1)}y(t) = \frac{1}{t(\sqrt{t^2+1})} \quad .$$

The general solution of this linear differential equation is $y = \frac{ct-1}{\sqrt{t^2+1}}$. Since $y = -\coth\theta(t)$, we obtain

$$\sinh\theta(t) = \mp\frac{\sqrt{t^2+1}}{\sqrt{(c^2-1)t^2-2ct}}, \quad \cosh\theta(t) = \mp\frac{ct-1}{\sqrt{(c^2-1)t^2-2ct}} \quad ,$$

$$\sin\varphi(t) = \frac{1}{\sqrt{t^2+1}}, \quad \cos\varphi(t) = \frac{t}{\sqrt{t^2+1}} \quad .$$

If we choose plus sign, then the family of developable ruled surface is given by

$$\Psi(t,u) = (t,t,1) + \frac{u}{\sqrt{(c^2-1)t^2-2ct}}(ct-1,1,t) \quad .$$

The graph of the developable ruled surface given by this equation for $c=2$ in domain $D=\{2 \leq t \leq 4 \text{ and } 5 \leq u \leq 6\}$ is given in Figure 1.

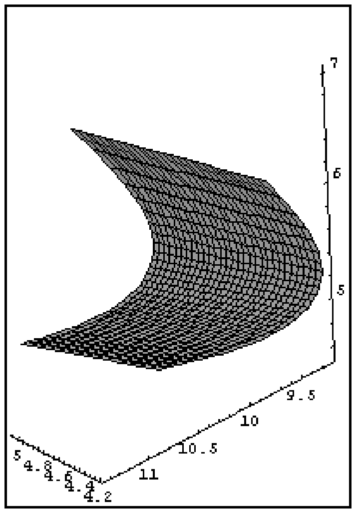

Fig.1. Developable ruled surface corresponding to a unit Lorentzian vector on S_1^2, in $\mathbb{R}_1^3$.

Example 4. *Consider a regular Lorentzian curve* $\alpha = \alpha(t) = (\frac{t^2+2}{2}, \frac{t^2}{2}, t)$. *We note that* α *is a timelike curve. Then, we obtain*

$$\frac{d\theta}{dt} = \frac{t}{t^2+1}, \qquad \frac{d\varphi^*}{dt} = \frac{t}{\sqrt{t^2+1}}, \qquad \frac{d\theta^*}{dt} = 1, P(t) = \frac{t}{t^2+1}, \qquad Q(t) = \frac{1}{\sqrt{t^2+1}} \quad .$$

Hence, the linear differential equation is written as

$$\frac{dy}{dt} + \frac{t}{t^2+1}y(t) = \frac{1}{\sqrt{t^2+1}} \quad .$$

The general solution of this linear differential equation is $y = \frac{t+c}{\sqrt{t^2+1}}$.

Since $y = -\cot\varphi(t)$, we obtain

$$\sin\varphi(t) = \mp \frac{\sqrt{t^2+c}}{\sqrt{2t^2+2ct+c^2+c}}, \qquad \cos\varphi(t) = \pm \frac{t^2+c}{\sqrt{2t^2+2ct+c^2+c}},$$

$$\sinh\theta(t) = \mp \frac{t^2}{2\sqrt{t^2+1}}, \qquad \cosh\theta(t) = \mp \frac{t^2+2}{2\sqrt{t^2+1}} \quad .$$

If we choose plus sign, then the family of developable ruled surface is given by

$$\Psi(t,u) = (\frac{t^2+2}{2},\frac{t^2}{2},t)+\frac{u}{\sqrt{2t^2+2ct+c^2+c}}(\frac{t^2\sqrt{t^2+c}}{2\sqrt{t^2+1}},\frac{(t^2+2)\sqrt{t^2+c}}{2\sqrt{t^2+1}},\quad -(t+c)) \quad .$$

The graph of the developable ruled surface given by this equation for c= 2 in domain D=$\{-2 \leq t \leq 2$ and $-3 \leq u \leq 3\}$ is given in Figure 2.

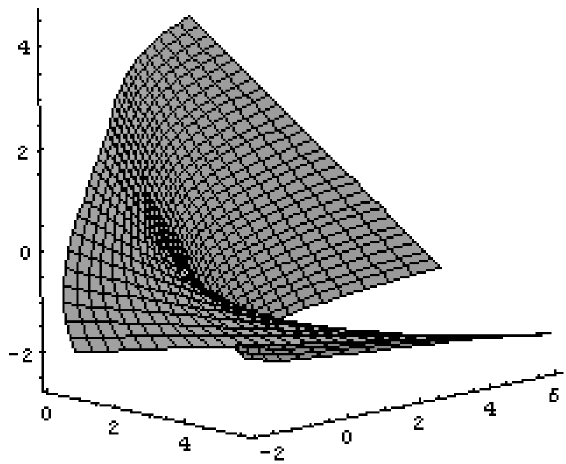

Fig.2. Developable ruled surface corresponding to a unit Lorentzian vector on H_0^2, in $\mathbb{R}_1^3$.

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حول السطوح المسطّرة القابلة للانبساط والمؤكدة حركياً في فضاء مينكوفسكي الثلاثي البعدية

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خلاصة

نقدم في هذا البحث طريقة لدراسة السطوح المسطرة القابلة للانبساط والمولدة في فضاء مينكوفسكي الثلاثي البعدة. وتقابل طريقتنا طريقة منحنيات لورنتز وفق دراسة ستادي حول مبدأ النقل. كما نعطي أيضاً بعض المبرهنات والأمثلة.