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Numerical solutions of the MRLW equation by cubic B-spline Galerkin finite element method

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ABSTRACT

In this paper, a numerical solution of the modified regularized long wave (MRLW) equation has been obtained by a numerical technique based on a lumped Galerkin method using cubic B-spline finite elements. Solitary wave motion, interaction of two and three solitary waves have been studied to validate the proposed method. The three invariants (I_1, I_2, I_3) of the motion have been calculated to determine the conservation properties of the scheme. Error norms L_2 and L_∞ have been used to measure the differences between the exact and numerical solutions. Also, a linear stability analysis of the scheme is proposed.

Keywords: Cubic B-splines; finite element method; Galerkin; MRLW equation; solitary waves.

AMS classification: 97N40, 65N30, 65D07, 76B25, 74S05,74J35.

INTRODUCTION

The modified regularized long wave (MRLW) equation which will be discussed in this article is related to the modified equal width wave (MEW) (Karakoc, 2011) equation and the modified Korteweg-de Vries (mKdV) (Gardner *et al*., 1994) equation and is based upon the regularized long wave (RLW) equation. All the modified equations are nonlinear wave equations with cubic nonlinearities and all have solitary pulse like wave solutions. Due to dynamical balance between the nonlinear and dispersive effects, these waves retain a stable waveform. The RLW equation in the following form:

$$U_t + U_x + \delta U U_x - \mu U_{xxt} = 0, \tag{1}$$

where δ and μ are positive parameters belonging to a class of nonlinear evolution equations, which can be used to model a large number of problems arising in various areas of applied sciences. The equation, at first was proposed by Peregrine (1966) to describe the development of an undular bore. This equation plays an important role in several branches of science, especially in physics, and it is widely used in many physical phenomena such as nonlinear transverse waves in shallow water, phonon packets used in nonlinear crystals, magneto hydrodynamic waves in plasma and unidirectional propagation of the water waves having small amplitude but long wavelength. Benjamin *et al*. (1972) also introduced a mathematical theory of the equation. Bona & Pryant (1973) have discussed the existence and uniqueness of the equation. Due to nonlinear nature of the RLW equation, few exact solutions exist in the literature. So the numerical solution of the equation has been subject of many papers. Various numerical studies including finite difference (Eilbeck & McGuire, 1977; Jain *et al*., 1993), finite element (Gardner & Gardner, 1990; Esen & Kutluay, 2005) and pseudo-spectral (Gou & Cao, 1988) method have been used for the solution of the equation. One of the special property of the equation is that the solutions may exhibit solitons whose magnitudes, shapes and velocities are not changed after the collision. MRLW equation is a special case of the generalized regularized long wave (GRLW) equation having the form

$$U_t + U_x + \delta U^p U_x - \mu U_{xxt} = 0, \tag{2}$$

where p is a positive integer. Zhang (2005) used a finite difference method to solve the GRLW equation for a Cauchy problem. Kaya & Elsayed (2003) also studied the GRLW equation with Adomian decomposition method. A quasilinearization method based on finite differences was used by Ramos (2007) for solving the GRLW equation. Roshan (2012) solved the GRLW equation numerically by the Petrov-Galerkin method using a linear hat function as the trial function and a quintic B-spline function as the test function. Gardner *et al*. (1997) developed a collocation solution to the MRLW equation using quintic B-splines finite elemets. Khalifa *et al*. (2007;2008) obtained the numerical solutions of the MRLW equation using finite difference method and cubic B-spline collocation finite element method. Solutions based on collocation method with quadratic B-spline finite elements and the central finite difference method for time are investigated by Raslan (2009). Raslan & Hassan (2009) solved the MRLW equation by a collocation finite element method using quadratic, cubic, quartic and quintic B-splines to obtain the numerical solutions of the single solitary wave. Haq *et al*. (2010) have designed a numerical scheme based on quartic B-spline collocation method for the numerical solution of MRLW equation. Ali (2009) has formulated a classical radial basis functions (RBFs) collocation method for solving the MRLW equation. Karakoc *et al*. (2013) obtained numerical solutions of the MRLW equation by the method based on collocation of quintic B-splines and Petrov- Galerkin finite element method in which the element shape functions are cubic and weight functions are quadratic B-splines (Karakoc & Geyikli, 2013).

In this paper, we applied a lumped Galerkin method based on cubic B-spline finite elements to solve the MRLW equation. The numerical solution is constructed to the continuous model of the problem. Therefore we assume that the problem has a unique and convergent solution, see (Benjamin *et al.*, 1971; Alzubaidi, 2006). The proposed method is shown to represent accurately the migration of single solitary wave. Then, the interaction of two and three solitary waves are studied. A linear stability analysis based on the Fourier method is also investigated.

THE GOVERNING EQUATION AND CUBIC B-SPLINES

In this study, we will consider the MRLW equation, a special form of (2) with the choice $p=2$ and $\delta = 6$,

$$U_t + U_x + 6U^2 U_x - \mu U_{xxt} = 0, \tag{3}$$

with the physical boundary conditions $U \to 0$ as $x \to \pm\infty$, where μ is a positive parameter and the subscripts x and t denote the differentiation. To implement the numerical method, solution domain is restricted over an interval $a \le x \le b$. Boundary conditions will be selected from the following homogeneous boundary conditions:

$$\begin{aligned} &U(a,t)=0, \qquad U(b,t)=0, \\ &U_x(a,t)=0, \qquad U_x(b,t)=0, \qquad t>0, \end{aligned} \tag{4}$$

and the initial condition

$$U(x,0) = f(x) \quad a \le x \le b.$$

The cubic B-splines $\phi_m(x)$, (m= $-1(1)N+1$), at the knots x_m are defined over the interval $[a,b]$ by

$$\varphi_m(x) = \frac{1}{h^3}\begin{cases} (x-x_{m-2})^3, & x \in x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x-x_{m-1}) + 3h(x-x_{m-1})^2 - 3(x-x_{m-1})^3 & x \in x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1}-x) + 3h(x_{m+1}-x)^2 - 3(x_{m+1}-x)^3, & x \in x_m, x_{m+1}], \\ (x_{m+2}-x)^3, & x \in x_{m+1}, x_{m+2}], \\ 0 & otherwise. \end{cases} \tag{5}$$

The set of functions $\{\varphi_{-1}(x), \varphi_0(x), \ldots, \varphi_{N+1}(x)\}$ forms a basis for the space of B-splines functions defined over $[a,b]$. The approximate solution $U_N(x,t)$ to the exact solution $U(x,t)$ is given by

$$U_N(x,t) = \sum_{j=-1}^{N+1} \phi_j(x)\delta_j(t). \tag{6}$$

where $\delta_j(t)$ are time dependent parameters to be determined from the boundary, initial and weighted residual conditions. Each cubic B-spline covers 4 elements so that each element $[x_m, x_{m+1}]$ is covered by 4 splines. In each element, using the following local coordinate transformation

$$h\xi = x - x_m, \quad 0 \le \xi \le 1, \tag{7}$$

cubic B-spline shape functions in terms of ξ over the domain $[0,1]$ can be defined as

$$\begin{matrix} \phi_{m-1} \\ \phi_m \\ \phi_{m+1} \\ \phi_{m+2} \end{matrix} = \begin{cases} (1-\xi)^3, \\ 1+3(1-\xi)+3(1-\xi)^2-3(1-\xi)^3, \\ 1+3\xi+3\xi^2-3\xi^3, \\ \xi^3. \end{cases} \tag{8}$$

All splines apart from $\varphi_{m-1}(x), \varphi_m(x), \varphi_{m+1}(x)$ and $\phi_{m+2}(x)$ are zero over the element $[x_m, x_{m+1}]$. Variation of the function $U(x,t)$ over element $[x_m, x_{m+1}]$ is approximated by

$$U_N(\xi,t) = \sum_{j=m-1}^{m+2} \delta_j \phi_j, \tag{9}$$

where $\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}$ act as element parameters and B-splines $\phi_{m-1}, \phi_m, \phi_{m+1}, \phi_{m+2}$ as element shape functions. Using trial function (6) and cubic splines (5), the values of U, U', U'' at the knots are determined in terms of the element parameters δ_m by

$$\begin{aligned} U_m &= U(x_m,t) = \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ hU'_m &= U'(x_m,t) = 3(-\delta_{m-1} + \delta_{m+1}), \\ h^2U''_m &= U''(x_m,t) = 6(\delta_{m-1} - 2\delta_m + \delta_{m+1}) \end{aligned} \tag{10}$$

where the symbols $'$ and $''$ denotes first and second differentiation with respect to x, respectively. The splines $\phi_m(x)$ and its two principle derivatives vanish outside the interval $[x_{m-2}, x_{m+2}]$.

THE FINITE ELEMENT SOLUTION

By applying the Galerkin method to the (3) with weight function $W(x)$, we obtain the weak form of (3)

$$\int_a^b W\,(U_t + U_x + 6U^2U_x - \mu U_{xxt})dx = 0. \tag{11}$$

Since we are using Galerkin method and in the method the weight function $W(x)$ is taken as exactly the same as approximate functions, and also the approximate functions are taken as B-splines, the smoothness of the weight function is guaranteed. For a single element $[x_m, x_{m+1}]$. using transformation (7) into the equation (1) we obtain

$$\int_0^1 W\left(U_t + \left(\frac{1+6U^2}{h}\right)U_\xi - \frac{\mu}{h^2}U_{\xi t}\right)d\xi = 0. \tag{12}$$

Integrating (12) by parts and using (3) lead to

$$\int_0^1 [W(U_t + \lambda U_\xi) + \beta W_\xi U_{\xi t})]d\xi = \beta W U_{\xi t}\,|_0^1, \tag{13}$$

where $\lambda = \dfrac{1+6U^2}{h}$ and $\beta = \dfrac{\mu}{h^2}$. Taking the weight function as cubic B-spline shape functions given by equation (8) and substituting approximation (9) in integral equation (13) with some manipulation, we obtain the element contributions in the form

$$\sum_{j=m-1}^{m+2}[(\int_0^1 \varphi_i\varphi_j + \beta\varphi_i'\varphi_j')d\xi - \beta\varphi_i\varphi_j'\,|_0^1\;]\dot{\delta}_j^e + \sum_{j=m-1}^{m+2}(\lambda\int_0^1 \varphi_i\varphi_j' d\xi)\delta_j^e. \tag{14}$$

In matrix notation, this equation becomes

$$A^e + \beta(B^e - C^e)]\dot{\delta}^e + \lambda D^e\delta^e, \tag{15}$$

where $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$ are the element parameters and the dot denotes differentiation with respect to t. The element matrices A^e, B^e, C^e and D^e are given by the following integrals:

$$A_{ij}^e = \int_0^1 \varphi_i\varphi_j d\xi = \frac{1}{140}\begin{bmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 60 & 933 & 1188 & 129 \\ 1 & 60 & 129 & 20 \end{bmatrix}$$

$$B_{ij}^e = \int_0^1 \varphi_i'\varphi_j' d\xi = \frac{1}{10}\begin{bmatrix} 18 & 21 & -36 & -3 \\ 21 & 102 & -87 & -36 \\ -36 & -87 & 102 & 12 \\ -3 & -36 & 21 & 18 \end{bmatrix}$$

$$C_{ij}^e = \varphi_i\varphi_j'\,|_0^1 = 3\begin{bmatrix} 1 & 0 & -1 & 0 \\ 4 & -1 & -4 & 1 \\ 1 & -4 & -1 & 4 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$D_{ij}^e = \int_0^1 \varphi_i\varphi_j' d\xi = \frac{1}{20}\begin{bmatrix} -10 & -9 & 18 & 1 \\ -71 & -150 & 183 & 38 \\ -38 & -183 & 150 & 71 \\ -1 & -18 & 9 & 10 \end{bmatrix}$$

where the suffices i, j take only the values $m-1, m, m+1, m+2$ for the typical element $[x_m, x_{m+1}]$. A lumped value for λ is found from $(U_m + U_{m+1})^2/4$ as

$$\lambda = \frac{3}{4h}(\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2.$$

By assembling all contributions from all elements, (15) leads to the following matrix equation;

$$A^e + \beta(B^e - C^e)]\dot{\delta}^e + \lambda D^e \delta^e = 0, \qquad (16)$$

where $\delta = (\delta_{-1}, \delta_0 ... \delta_N, \delta_{N+1})^T$ are global element parameters. The matrices A, B and λD are septadiagonal and row of each has the following form:

$$A = \frac{1}{140}(1,120,1191,2416,1191,120,1),$$

$$B = \frac{1}{10}(-3,-72,-45,240,-45,-72,-3),$$

$$\lambda D = \frac{1}{20}(-\lambda_1, -18\lambda_1 - 38\lambda_2, 9\lambda_1 - 183\lambda_2 - 71\lambda_3, 10\lambda_1 + 150\lambda_2 - 150\lambda_3 - 10\lambda_4,$$
$$71\lambda_2 + 183\lambda_3 - 9\lambda_4, 38\lambda_3 + 18\lambda_4, \lambda_4),$$

where

$$\lambda_1 = \frac{3}{4h}(\delta_{m-2} + 5\delta_{m-1} + 5\delta_m + \delta_{m+1})^2,$$
$$\lambda_2 = \frac{3}{4h}(\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2,$$
$$\lambda_3 = \frac{3}{4h}(\delta_m + 5\delta_{m+1} + 5\delta_{m+2} + \delta_{m+3})^2,$$
$$\lambda_4 = \frac{3}{4h}(\delta_{m+1} + 5\delta_{m+2} + 5\delta_{m+3} + \delta_{m+4})^2.$$

Replacing the time derivative of the parameter $\dot{\delta}$ by usual forward finite difference approximation and parameter δ by the Crank-Nicolson formulation

$$\dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t}, \quad \delta = \frac{1}{2}(\delta^n + \delta^{n+1})$$

into (16), gives the $(N+3) \times (N+3)$ septadiagonal matrix system

$$A + \beta(B-C) + \frac{\lambda \Delta t}{2} D]\delta^{n+1} = [A + \beta(B-C) - \frac{\lambda \Delta t}{2} D]\delta^n \qquad (17)$$

where Δt is time step. Applying the boundary conditions (4) to the system (17), we obtain a $(N+1)\times(N+1)$ septadiagonal matrix system. This system is efficiently solved with a variant of the Thomas algorithm, but an inner iteration is also needed at each time step to cope with the non-linear term. A typical member of the matrix system (17) may be written in terms of the nodal parameters δ^n and δ^{n+1} as

$$\begin{aligned}&\gamma_1\delta_{m-2}^{n+1}+\gamma_2\delta_{m-1}^{n+1}+\gamma_3\delta_{m}^{n+1}+\gamma_4\delta_{m+1}^{n+1}+\gamma_5\delta_{m+2}^{n+1}+\gamma_6\delta_{m+3}^{n+1}+\gamma_7\delta_{m+4}^{n+1}=\gamma_7\delta_{m-2}^{n}+\gamma_6\delta_{m-1}^{n}+\\&\gamma_5\delta_{m}^{n}+\gamma_4\delta_{m+1}^{n}+\gamma_3\delta_{m+2}^{n}+\gamma_2\delta_{m+3}^{n}+\gamma_1\delta_{m+4}^{n}\end{aligned}\tag{18}$$

where

$$\begin{aligned}\gamma_1&=\frac{1}{140}-\frac{3\beta}{10}-\frac{\lambda\Delta t}{40},\\ \gamma_2&=\frac{120}{140}-\frac{72\beta}{10}-\frac{56\lambda\Delta t}{40},\\ \gamma_3&=\frac{1191}{140}-\frac{45\beta}{10}-\frac{245\lambda\Delta t}{40},\\ \gamma_4&=\frac{2416}{140}+\frac{240\beta}{10},\\ \gamma_5&=\frac{1191}{140}-\frac{45\beta}{10}+\frac{245\lambda\Delta t}{40},\\ \gamma_6&=\frac{120}{140}-\frac{72\beta}{10}+\frac{56\lambda\Delta t}{40},\\ \gamma_7&=\frac{1}{140}-\frac{3\beta}{10}+\frac{\lambda\Delta t}{40}\end{aligned}$$

which all depend on δ^n. The initial vector of parameters $\delta^0=(\delta_{-1}^0,\ldots,\delta_{N+1}^0)$ must be determined to iterate the system (17). To do this, the approximation is rewritten over the interval $[a,b]$ at time $t=0$ as follows:

$$U_N(x,0)=\sum_{m=-1}^{N+1}\phi_m(x)\delta_m^0,$$

where the parameters δ_m^0 will be determined. $U_N(x,0)$ are required to satisfy the following relations at the mesh points x_m:

$$\begin{aligned}&U_N(x_m,0)=U(x_m,0),\quad m=0,1,\ldots,N.\\&U_N'(x_0,0)=U'(x_N,0)=0\end{aligned}$$

The above conditions lead to a tridiagonal matrix system of the form

$$\begin{bmatrix} -3 & 0 & 3 & & & \\ 1 & 4 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & 4 & 1 \\ & & & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} \delta_{-1}^{0} \\ \delta_0^0 \\ \vdots \\ \delta_N^0 \\ \delta_{N+1}^0 \end{bmatrix} = \begin{bmatrix} 0 \\ U(x_0) \\ \vdots \\ U(x_N) \\ 0 \end{bmatrix}$$

which can be solved using a variant of the Thomas algorithm.

A LINEAR STABILITY ANALYSIS

To investiagte the stability analysis of the presented scheme, it is suitable to use Von Neumann theory. The growth factor g of the error in a typical mode of amplitude $\hat{\delta}^n$

$$\delta_j^n = \hat{\delta}^n e^{ijkh} \tag{19}$$

where k is the mode number and h the element size, is determined from a linearization of the numerical scheme. In order to apply the stability analysis, the MRLW equation can be linearized by assuming that the quantity U in the non-linear term U^2U_x is locally constant. Substituting the Fourier mode (19) into (18) gives the growth factor g of the form

$$g = \frac{a-ib}{a+ib}, \tag{20}$$

where

$$\begin{aligned} a &= 2416 + 3360\beta + (2382 - 1260\beta)\cos\theta h + (240 - 2016\beta)\cos 2\theta h + (2 - 84\beta)\cos 3\theta h, \\ b &= 5145\lambda\Delta t \sin\theta h + 1176\lambda\Delta t \sin 2\theta h + 21\lambda\Delta t \sin 3\theta h. \end{aligned} \tag{21}$$

According to the Fourier stability analysis, for the given scheme to be stable, the condition $|g| < 1$ must be satisfied. Using a symbolic programming software or using simple calculations, since $a^2 + b^2 = a^2 + (-b)^2$ it becomes evident that the modulus of $|g|$ is 1. Therefore the linearized scheme is unconditionally stable.

NUMERICAL EXAMPLES AND RESULTS

Numerical results of the MRLW equation are obtained for three problems: the motion of single solitary wave, interaction of two and three solitary waves. We use the error norm L_2

$$L_2 = \left\| U^{exact} - U_N \right\|_2 ; \sqrt{h \sum_{J=1}^{N} \left| U_j^{exact} - (U_N)_j \right|^2},$$

and the error norm L_∞

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty ; \max_j \left| U_j^{exact} - \left(U_N\right)_j \right|, j = 1,2,\ldots,N-1,$$

to calculate the difference between analytical and numerical solutions at some specified times. Olver (1979) proved that the MRLW equation (3) possesses only three conservation constants given by

$$I_1 = \int_a^b U dx \simeq h \sum_{J=1}^{N} U_j^n,$$

$$I_2 = \int_a^b [U^2 + \mu (U_x)^2] dx \simeq h \sum_{J=1}^{N} [(U_j^n)^2 + \mu \left(U_x\right)_j^n],$$

$$I_3 = \int_a^b (U^4 - \mu U_x^2) dx \simeq h \sum_{J=1}^{N} \left[(U_j^n)^4 - \mu (U_x)_j^n \right]$$

which correspond to conservation of mass, momentum and energy, respectively. In the simulation of solitary wave motion, the invariants I_1, I_2 and I_3 are monitored to check the conservation of the numerical algorithm.

THE MOTION OF SINGLE SOLITARY WAVE

As a first problem, (3) is considered with the boundary conditions $U \to 0$ as $x \to \pm\infty$ and the initial condition

$$U(x,0) = \sqrt{c} \sec h\left[p\left(x - x_0\right)\right].$$

The analytical solution of the MRLW can be written as

$$U(x,t) = \sqrt{c} \sec h\left[p\left(x - (c+1)t - x_0\right)\right],$$

where $p = \sqrt{\frac{c}{\mu(c+1)}}$, x_0 and c are arbitrary constants. The constants of motion, for a solitary wave of amplitude $\sqrt{c}$ and width depending on p may be evaluated analytically as in (Gardner *et al*., 1997)

$$I_1 = \frac{\pi\sqrt{c}}{p}, \; I_2 = \frac{2c}{p} + \frac{2\mu p}{3}, I_3 = \frac{4c^2}{3p} - \frac{2\mu p}{3}. \tag{22}$$

For the first experiment, parameters $c = 1$, $\mu = 1$, $h = 0.2$, $x_0 = 40$, $k = 0.025$ over the interval $[0,100]$ are chosen to coincide with those of earlier studies (Roshan, 2012; Khalifa *et al*., 2008; Raslan, 2009; Haq *et al*., 2010; Ali, 2009; Karakoc *et al*., 2013). For these parameters, the solitary wave has amplitude 1.0. Invariants and error norms L_2 and L_∞ are shown at selected times up to time $t=10$. The obtained results are tabulated in Table 1. It can be seen from the Table 1 that the error norms L_2 and L_∞ are found to be small enough and the computed values of invariants are in good agreement with their analytical values $I_1 = 4.4428829$, $I_2 = 3.2998316$, $I_3 = 1.4142135$. The percentage of the relative error of the conserved quantities I_1, I_2 and I_3 are calculated with respect to the conserved quantities at $t=0$. Percentage of relative changes of I_1, I_2 and I_3 are found to be 7×10^{-3} %, 14×10^{-3} %, 33×10^{-3} %, respectively. Thus, the quantities in the invariants remain almost constant during the computer run. Table 2 represents a comparison of the values of the invariants and error norms obtained by the present method with those obtained by other methods (Roshan, 2012; Gardner *et al*., 1997; Khalifa *et al*., 2008; Ali, 2009; Karakoc *et al*., 2013; Karakoc & Geyikli, 2013). It is clearly observed from the Table 2 that the error norms obtained by the present method are smaller than other methods (Roshan, 2012; Gardner *et al*., 1997; Khalifa *et al*., 2008; Ali, 2009; Karakoc *et al*., 2013). Figure (1) illustrates the motion of solitary wave with $c=1$, $h=0.2$, $k=0.025$ at different time levels.

Table 1. Invariants and error norms for single solitary wave with c =1 , h = 0.2, k= 0.025, $0 \le x \le 100$.

t	I_1	I_2	I_3	$L_2\times10^3$	$L_\infty\times10^3$
0	4.4428661	3.2998133	1.4142140	0.00000000	0.00000000
1	4.4429040	3.2998800	1.4142752	1.28062601	0.97327496
2	4.4429408	3.2999387	1.4143308	1.95082039	1.19160336
3	4.4429739	3.2999876	1.4143790	2.23507757	1.22256684
4	4.4430058	3.3000340	1.4144250	2.36484347	1.22370847
5	4.4430372	3.3000794	1.4144703	2.42609024	1.21382766
6	4.4430683	3.3001243	1.4145151	2.45181423	1.20000405
7	4.4430993	3.3001689	1.4145597	2.45719699	1.17913235
8	4.4431302	3.3002134	1.4146042	2.45030808	1.15204959
9	4.4431611	3.3002578	1.4146486	2.43599823	1.11925204
10	4.4431919	3.3003022	1.4146930	2.41750291	1.08099621

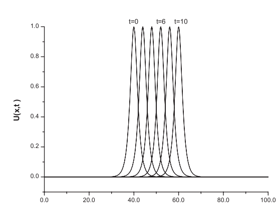

Fig. 1. Single solitary wave with c = 1 , h = 0.2, Δt = 0.025 , 0≤ x 100≤ t= 0, 2, 4, 6, 8 and 10

In addition, we have chosen the parameters $\mu = 1$, $c = 0.3$, $h = 0.1$, $k = 0.01$ and $x_0 = 40$ with range $[0,100]$ to enable comparison with the results of (Roshan, 2012; Khalifa *et al.*, 2008; Raslan, 2009; Haq *et al.*, 2010; Ali, 2009; Karakoc *et al.*, 2013). Error norms L_2 and L_∞ and conserved quantities are given in Table 3 up to time $t = 20$, together with the results obtained with those in (Roshan, 2012; Khalifa *et al.* 2008; Raslan, 2009; Haq *et al.*, 2010; Ali, 2009, Karakoc *et al.*, 2013). It is seen from the table that the error norm L_2 obtained by the present method is smaller than those given in Refs. (Khalifa *et al.* 2008; Raslan, 2009) and almost the same in Ref. (Roshan, 2012; Ali, 2009; Karakoc *et al.*, 2013), whereas error norm L_∞ is smaller than that given in Ref.(Khalifa *et al.* 2008; Raslan, 2009), but almost the same as those obtained with the other methods. Invariants are also reasonably in good agreement with their analytical values given by (22). Percentage of relative changes of I_1, I_2 and I_3 are found to be 0.001×10^{-3} %, 0.023×10^{-3} %, 0.052×10^{-3} %, respectively. Moreover, the invariants I_1 and I_2 change from their initial values by less than 3×10^{-7} and 1×10^{-7} respectively, during the time of running; whereas, the change of invariant I_3 approach to zero throughout the run. Figure (2) illustrates the motion of the solitary wave at different time leves. Error distributions at time $t = 10$ and $t = 20$ are depicted graphically for solitary waves amplitudes 1 and 0.3 in Figure (3). It is seen that the maximum errors are about the tip of the solitary waves and between -6×10^{-3} and 6×10^{-3}, -2×10^{-4} and 2×10^{-4}, respectively.

Table 2. Erros and invariants for single solitary wave with $c = 1$, $h = 0.2$, $k = 0.025$, $0 \le x \le 100$ at $t = 10$

Method	I_1	I_2	I_3	$L_2\times10^3$	$L_\infty\times10^3$
Analytical	4.4428829	3.2998316	1.4142135	0	0
Present	4.4431919	3.3003022	1.4146930	2.41750	1.08099
(Roshan, 2012)	4.44288	3.29981	1.41416	3.00533	1.68749
Cubic B-splines coll-CN(Gardner *et al.*, 1997)	4.442	3.299	1.413	16.39	9.24
Cubic B-splines coll+PA-CN(Gardner *et al.*, 1997)	4.440	3.296	1.411	20.3	11.2
Cubic B-splines coll (Khalifa *et al.*, 2008)	4.44288	3.29983	1.41420	9.30196	5.43718
MQ(Ali, 2009)	4.4428829	3.29978	1.414163	3.914	2.019
TPS(Ali, 2009)	4.4428821	3.29972	1.414104	4.428	2.306
(Karakoc *et al.*, 2013)	4.4428661	3.2997108	1.4143165	2.58891	1.35164
(Karakoc & Geyikli, 2013)	4.4431758	3.3003023	1.4146927	2.41552	1.07974

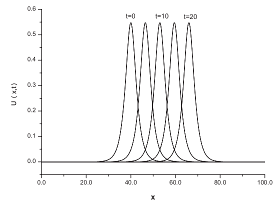

Fig. 2. Signle solitary wave with $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $0 \le x \le 100$, $t = 0$, 5, 10, 15 and 20

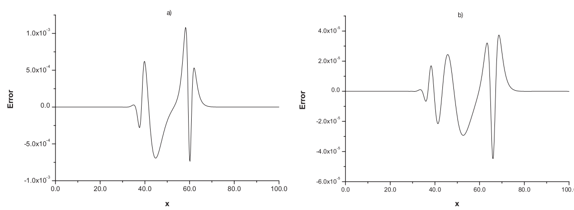

Fig. 3. Error with a) $c = 0.3$, $h = 0.2$, $\Delta t = 0.0.25$ $t = 10.0$, $0 \le x \le 100$, b) $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $t = 20.0$, $0 \le x \le 100$,

Table 3. Invariants and error norms for single solitary wave with c =0.3 , h = 0.1, k= 0.01, $0 \le x \le 100$.

t	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
	3.5820205	1.3450941	0.1537283	0.0000000	0.0000000
	3.5820206	1.3450941	0.1537284	0.5686216	0.3494378
	3.5820206	1.3450942	0.1537284	0.8766467	0.4283522
	3.5820206	1.3450942	0.1537284	1.0188630	0.4488258
	3.5820207	1.3450943	0.1537284	1.0933152	0.4225906
	3.5820207	1.3450943	0.1537284	1.1378242	0.4163976
	3.5820207	1.3450943	0.1537284	1.1671170	0.4254285
	3.5820207	1.3450943	0.1537284	1.1879104	0.4326199
	3.5820207	1.3450944	0.1537284	1.2036892	0.4388150
	3.5820207	1.3450944	0.1537284	1.2164373	0.4435480
	3.5820206	1.3450944	0.1537284	1.2273638	0.4472294
(Roshan, 2012)	3.58197	1.34508	0.153723	0.645295	0.301923
(Khalifa *et al*., 2008)	3.58197	1.34508	0.153723	6.06885	2.96650
(Raslan, 2009)	3.582265	1.345182	0.1538901	3.379583	7.672911
(Haq *et al*., 2010)	3.581967	1.345076	0.153723	0.5089274	0.2222848
(Ali, 2009)MQ	3.5819665	1.3450764	0.153723	0.51498	0.22551
(Ali, 2009)TPS	3.5819663	1.3450759	0.153723	0.51498	0.26605
(Karakoc *et al*., 2013)	3.5820204	1.3450974	0.1537250	0.8112594	0.3569076

INTERACTION OF TWO SOLITARY WAVES

In this problem, we consider the interaction of two solitary waves by using the initial condition given by the linear sum of two well seperated solitary waves having various amplitudes

$$U(x,0)=\sum_{j=1}^{2}A_j \sec h(p_j[x-x_j]), \tag{23}$$

where $A_j=\sqrt{c_j}$, $p_j=\sqrt{\dfrac{c_j}{\mu(c_j+1)}}$, $j=1,2,$ c_j and x_j are arbitrary constants.

The analytical values of the conservation laws can be found as (Gardner *et al*., 1997)

$$I_1=\sum_{j=1}^{2}\frac{\pi\sqrt{c_j}}{p_j}=11.467698,$$

$$I_2=\sum_{j=1}^{2}\left(\frac{2c_j}{p_j}+\frac{2\mu p_j c_j}{3}\right)=14.629243, \tag{24}$$

$$I_3=\sum_{j=1}^{2}\left(\frac{4c_j^2}{3p_j}-\frac{2\mu p_j c_j}{3}\right)=22.880466.$$

For the simulation, the parameters $\mu=1,$ $h=0.2,$ $k=0.025,$ $c_1=4, c_2=1,$ $x_1=25,$ $x_2=55$ are chosen over the range $0\le x\le 250$ to coincide with those used by Ref. (Roshan, 2012; Khalifa *et al*. 2008; Haq *et al*., 2010; Ali, 2009; Karakoc *et al*., 2013). The experiment are run from $t=0$ to $t=20$ and values of the invariant quantities I_1, I_2 and I_3 are tabulated in Table 4. Table 4 compares the calculated values of the invariants obtained by the present method with those obtained in Ref. (Roshan, 2012; Khalifa *et al*., 2008; Haq *et al*., 2010; Ali, 2009; Karakoc *et al*., 2013). It is seen that the obtained values of the invariants remain almost constant during the computer run. Figure (4) shows the development of the interaction of two solitary waves. It is clear from the figure that, at $t=0$ the wave with larger amplitude is to the left of the second wave with smaller amplitude. Since the taller wave moves faster than the shorter one, it catches up and collides with the shorter one at $t=8$ and then moves away from the shorter one as time increases. At $t=20$, the amplitude of larger waves is 1.9922913 at the point $x=127.6$ whereas the amplitude of the smaller one is 0.9954384 at the point $x=20$. It is found that the absolute difference in amplitude is 4.5×10^{-3} for the smaller wave and 7.7×10^{-3} for the larger wave for this algorithm.

Table 4. Comparison of invariants for the interaction of two solitary waves with results from (Haq *et al.*, 2010) with $h = 0.02$, $k = 0.025$ in the region $0 \le x \le 250$

	Present method			(Haq *et al.*, 2010)		
t	I_1	I_2	I_3	I_1	I_2	I_3
	11.4676542	14.6290766	22.8804898	11.467698	14.629277	22.880432
	11.4675751	14.6291789	22.8802823	11.467698	14.624259	22.860365
	11.4674004	14.6287111	22.8783932	11.467698	14.619226	22.840279
	11.4672351	14.6282736	22.8766213	11.467699	14.614169	22.820069
	11.4685470	14.6360654	22.9020513	11.467700	14.606821	22.787857
	11.4681751	14.6349679	22.9024807	11.467700	14.603687	22.771773
	11.4663725	14.6257527	22.8717460	11.467699	14.603056	22.775766
	11.4664929	14.6260926	22.8702439	11.467699	14.598059	22.756029
	11.4664794	14.6260304	22.8686659	11.467700	14.593048	22.736127
	11.4663697	14.6257202	22.8668953	11.467700	14.588061	22.716289
	11.4662207	14.6253125	22.8650456	11.467701	14.583089	22.696510
(Roshan, 2012)	11.4677	14.6299	22.8806			
(Khalifa *et al.*, 2008)	11.4677	14.6292	22.8809			
(Ali, 2009)MQ	11.467698	14.583052	22.696539			
(Ali, 2009)TPS	11.467742	14.582424	22.694269			
(Karakoc *et al.*, 2013)	11.4691886	14.6331334	22.8764330			

INTERACTION OF THREE SOLITARY WAVES

As a last problem, we study the behavior of the interaction of three solitary waves having different amplitudes and travelling in the same direction. So, we consider (3) with initial condition given by the linear sum of three well-seperated solitary waves of different amplitudes

$$U(x,0) = \sum_{j=1}^{3} A_j \sec h(p_j[x - x_j]), \tag{25}$$

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{\dfrac{c_j}{\mu(c_j+1)}}$, $j = 1,2,3,$ c_j and x_j are arbitrary constants.

The analytical values of the conservation laws are found from (22) as

$$I_1 = \sum_{j=1}^{3} \frac{\pi\sqrt{c_j}}{p_j} = 14.9801,$$

$$I_2 = \sum_{j=1}^{3} \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3} \right) = 15.8218, \tag{26}$$

$$I_3 = \sum_{j=1}^{3} \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3} \right) = 22.9923.$$

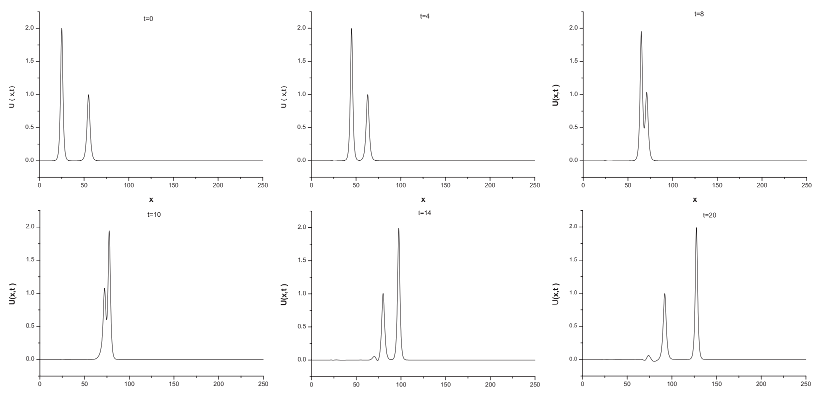

Fig. 4. Interaction of two solitary waves with t = 0 , 4 , 8 , 10 , 14 , 20.

For the computational work, parameters $\mu = 1, h = 0.2, k = 0.025, c_1 = 4, c_2 = 1,$ $c_3 = 0.25,$ $x_1 = 15, x_2 = 45, x_3 = 60$ are taken over the interval $0 \le x \le 250.$ Simulations are done up to time $t = 45.$ Table 5 displays a comparison of the values of the invariants obtained by the present method with those obtained in Ref. (Khalifa *et al.*, 2008; Haq *et al.*, 2010; Ali, 2009; Karakoc *et al.*, 2013). It is seen from the table that the obtained values of the invariants remain almost during the computer run which are all in good agreement with their analytical values given by (26). The absolute difference between the values of the conservative constants obtained by the present method at times $t = 0$ and $t = 45$ are $\Delta I_1 = 4.8\times10^{-2},$ $\Delta I_2 = 9.5\times10^{-3}, \Delta I_3 = 4.1\times10^{-2}.$ Figure (5) shows the interaction of these solitary waves at different times. As it is seen from Figure (5) interaction started about time $t = 10,$ overlapping processes occured between time $t = 15$ and $t = 40$ and waves started to resume their original shapes after the time $t = 40.$

Table 5. Comparison of invariants for the interaction of two solitary waves with results from (Haq *et al.,* 2010) with $h = 0.02$, $k = 0.025$ in the region $0 \leq x \leq 250$

	Present method			(Haq *et al*., 2010)		
t	I_1	I_2	I_3	I_1	I_2	I_3
	14.9800750	15.8373533	23.0083122	14.980099	15.837528	23.008136
	14.9799710	15.8373541	23.0058377	14.980105	15.824928	22.957891
	14.9850842	15.8652441	23.0903527	14.980109	15.807025	22.877972
	14.9809869	15.8409759	23.0051096	14.980106	15.807032	22.885947
	14.9790729	15.8352645	22.9959403	14.980106	15.795022	22.837454
	14.9781209	15.8326417	22.9898148	14.980107	15.782840	22.788852
	14.9776464	15.8313166	22.9849266	14.980107	15.770634	22.740419
	14.9772377	15.8301663	22.9802178	14.980108	15.758480	22.692279
	14.9768320	15.8290288	22.9755365	14.980108	15.746389	22.644448
	14.9316345	15.8277899	22.9664579	14.968030	15.734374	22.596591
(Khalifa *et al*., 2008)	13.7043	15.6563	22.9303			
(Ali, 2009)MQ	14.96814	15.73434	22.596625			
(Ali, 2009)TPS	14.96824	15.73376	22.594494			
(Karakoc *et al*., 2013)	14.7145273	15.4927592	23.3529062			

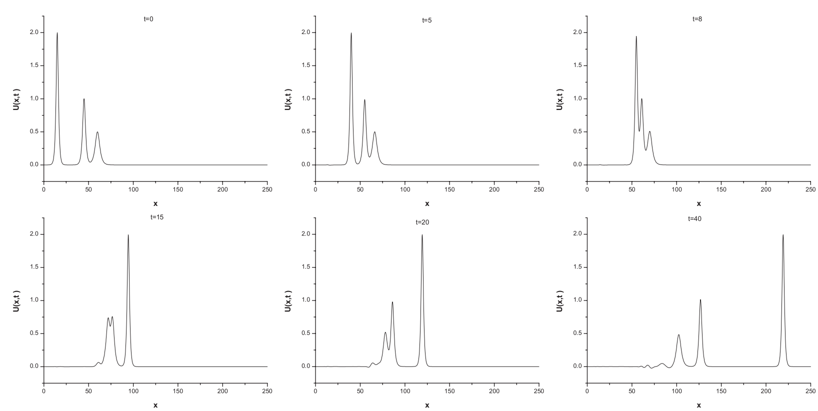

Fig. 5. Interaction of three solitary waves with t = 0 , 5, 8 , 15, 20, 40

CONCLUSION

In this paper, a lumped Galerkin method based on cubic B-splines has been successfully applied to the MRLW equation to examine the motion of a single solitary wave, whose analytical solution is known and extended the scheme to the study of two and three solitary waves, whose analytical solution is unknown during the interaction. To show how good and accurate the numerical solutions of the test problems, we have calculated the error norms L_2 and ∞ and the invariant quantities I_1, I_2 and It has been observed that the error norms are satisfactorily small and the invariants are well conserved. The method successfully models the motion and interaction of the solitary waves. The obtained results indicate that the present method is more accurate than some earlier results found in the literature. Moreover, since the method uses piece-wise approximation due to its nature, non-homogenous problems and the problems defined over irregular shapes can also be solved by the present method. These are the most important merits of the method.Therefore, this method can be a reliable method for obtaining the numerical solutions of the physically important non-linear partial differential equations.

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حلول عددية لمعادلة MRLW بواسطة طريقة غارلنكن للشريحة التكعيبية المنتهية العنصر

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الملخص

نقوم في هذا البحث بإيجاد حل عمودي لمعادلة الموجة الطويلة المعدلة المنظومة (MRLW) وذلك بواسطة تقنية عددية مستندة إلى طريقة غارلنكن التي تستخدم الشريحة التكعيبية للعناصر المنتهية. ثم نقوم بدراسة الحركة الموجية المنفردة، والتفاعل بين الموجات المنفردة الثنائية والثلاثية وذلك للتحقق من صوابية طريقتنا المقترحة. كما نقوم بحساب اللامتغيرات الثلاثية (I_1، I_2، I_3) للحركة وذلك للحصول على خصائص الحفظ للمخطط. ونستخدم معياري الخطأ L_2، L_∞ لقياس الفروقات بين الحلول العددية والحلول الدقيقة. وأخيراً، نقترح تحليلاً خطياً لاستقرار المخطط.