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A second-order difference equation with sign-alternating coefficients

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Abstract

We provide an explicit solution for the terms of the sequence $(x_{n,k})$ defined by

$$x_{n,k} = x_{n-1,k} - (-1)^{\lfloor (n-2)/k \rfloor} x_{n-2,k} \, ,$$

for $n \geqslant 3$, setting $x_{1,k} = 1$ and $x_{2,k} = 0$. Several particular examples are considered.

Keywords: Chebyshev polynomials of the second kind; difference equations; Fibonacci numbers; integer sequences; tridiagonal matrices

1. Introduction

Second order linear difference equations emerge in distinct areas of mathematics. For example, solutions of constant coefficient homogeneous equations consist of many sequences of numbers, such as Fibonacci. In many instances, finding explicit forms of homogeneous second order equations is a ceaseless problem in research. Moreover, many results are focused exclusively to this aim. For several relevant references the reader is refereed to (Koshy, T., 2018; Kızılateş, C., 2021; da Fonseca, C.M., 2014).

Recently, in (Andelić, M. *et al.*, 2020) it was proposed to establish an explicit expression for the sequence $(x_{n,k})$ where

$$x_{n,k} = x_{n-1,k} - (-1)^{\lfloor (n-2)/k \rfloor} x_{n-2,k} \, , \quad \text{for } n \geqslant 3, \tag{1}$$

setting $x_{1,k} = 1$ and $x_{2,k} = 0$. This problem was motivated by similar questions originally proposed in (Trojovský, P., 2017), namely when the recurrence relations

$$y_{n,k} = (-1)^{\lfloor (n-1)/k \rfloor} y_{n-1,k} - y_{n-2,k} \, ,$$

and

$$z_{n,k} = (-1)^{\lfloor (n-1)/k \rfloor} z_{n-1,k} - (-1)^{\lfloor (n-2)/k \rfloor} z_{n-2,k}$$

are satisfied. The solutions of these cases were obtained in terms of the Fibonacci sequence, defined by the standard recurrence relation $F_{n+2} = F_{n+1} + F_n$, for $n \geqslant 0$, with $F_0 = 0$ and $F_1 = 1$.

In this note, we provide a close formula for Equation (1) and discuss some particular cases. In the next section, we recall the formula of the determinant of a tridiagonal k-Toeplitz matrix. Then using a similar approach we have seen for example in (Andelić, M. *et al.*, 2020; Andelić, M *et al.*, 2011; Du, Z. *et al.*, 2022), we establish the requested formula. We also consider in particular the cases $k = 2, 3$ in detail. In the final section, we discuss several general instances.

2. The formula

From (Rózsa, P., 1969), we know that the determinant of the tridiagonal k-Toeplitz matrix

$$A_n = \begin{pmatrix} a_1 & b_1 & & & & & & & \\ 1 & \ddots & \ddots & & & & & & \\ & \ddots & a_k & b_k & & & & & \\ & & 1 & a_1 & b_1 & & & & \\ & & & 1 & \ddots & \ddots & & & \\ & & & & \ddots & a_k & b_k & & \\ & & & & & 1 & a_1 & b_1 & \\ & & & & & & 1 & \ddots & \ddots \\ & & & & & & & \ddots & \end{pmatrix}_{n\times n}, \tag{2}$$

is given by

$$\det A_n = (\sqrt{b_1}\cdots\sqrt{b_k})^q \left(\Delta_{1,\dots,r}\, U_q(x) + \frac{\sqrt{b_k}\sqrt{b_1}\cdots\sqrt{b_r}}{\sqrt{b_{r+1}}\cdots\sqrt{b_{k-1}}}\, \Delta_{r+2,\dots,k-1}\, U_{q-1}(x) \right),$$

where

$$x = \frac{\Delta_{1,\dots,k} - b_k \Delta_{2,\dots,k-1}}{2\sqrt{b_1}\cdots\sqrt{b_k}},$$

with $\Delta_{1,\dots,k} = \det A_k$ and where $U_q(x)$ is the Chebyshev polynomials of the second kind and $n = qk+r$, with $0 \leqslant r \leqslant k-1$. In general, by $\Delta_{i,\dots,j}$ we understand the determinant of the submatrix obtained A_k with rows and columns indexed by $\{i,\dots,j\}$. An independent approach can be found in (Fonseca, C.M. & Petronilho, J., 2005).

Now, let us define now

$$T_{n,k} = \begin{pmatrix} 1 & 1 & & & & & & & \\ 1 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & 1 & & & & & \\ & & 1 & \ddots & -1 & & & & \\ & & & 1 & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & -1 & & \\ & & & & & 1 & \ddots & 1 & \\ & & & & & & 1 & \ddots & \ddots \\ & & & & & & & \ddots & \end{pmatrix}_{n\times n}, \tag{3}$$

where the superdiagonal is of the form

$$(\underbrace{1,\dots,1}_{k\,\times}, \underbrace{-1,\dots,-1}_{k\,\times}, \underbrace{1,\dots,1}_{k\,\times}, \underbrace{-1,\dots,-1}_{k\,\times}, 1,\dots)\,. \tag{4}$$

If we set

$$(b_1,\dots,b_k,b_{k-1}\dots,b_{2k}) = (\underbrace{1,\dots,1}_{k\,\times}, \underbrace{-1,\dots,-1}_{k\,\times})$$

and replace k by $2k$ in (2), the matrix defined in Equation (3) can be seen as a tridiagonal $2k$-Toeplitz matrix. Hence, we have explicitly

$$\det T_{n,k} = x_{n,k} = i^{qk} \left(\Delta_{1,\ldots,r}\, U_q(x) + \frac{\sqrt{b_{2k}}\sqrt{b_1}\cdots\sqrt{b_r}}{\sqrt{b_{r+1}}\cdots\sqrt{b_{2k-1}}}\, \Delta_{r+2,\ldots,2k-1}\, U_{q-1}(x) \right) ,$$

where

$$x = \frac{\Delta_{1,\ldots,2k} + \Delta_{2,\ldots,2k-1}}{2i^k} , \tag{5}$$

with $n = 2qk + r$, for $0 \leqslant r \leqslant 2k - 1$.

Notice that Equation (5) can be rewritten in terms of Chebyshev polynomials of the second kind. Namely, since

$$\Delta_{1,\ldots,2k} = i^k U_k\left(\frac{1}{2}\right) U_k\left(-\frac{i}{2}\right) - i^{k-1} U_{k-1}\left(\frac{1}{2}\right) U_{k-1}\left(-\frac{i}{2}\right) ,$$

one can conclude that

$$x = \frac{1}{2}\left(U_k\left(\frac{1}{2}\right) U_k\left(-\frac{i}{2}\right) + U_{k-2}\left(\frac{1}{2}\right) U_{k-2}\left(-\frac{i}{2}\right) \right) .$$

3. The cases $k = 2$ and $k = 3$

As a first example, if we set $k = 2$, considering the table

r	0	1	2	3
Δ_r	1	1	0	-1
$\tilde{\Delta}_r$	0	1	1	0
ϵ_r	1	1	1	-1

we obtain

$$\det T_{n,k} = (-1)^q (\Delta_r U_q(1/2) + \epsilon_r \tilde{\Delta}_r U_{q-1}(1/2)) ,$$

where $n = 4q + r$, with $0 \leqslant r \leqslant 3$.

The term $x_{4q+r,2}$ is given by

	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$x_{4q+r,2}$	$U_q(-1/2)$	$U_q(-1/2) - U_{q-1}(-1/2)$	$-U_{q-1}(-1/2)$	$-U_q(-1/2)$

Recall that

$$U_\ell\left(-\frac{1}{2}\right) = \begin{cases} 1 & \text{if } \ell \equiv 0 \pmod 3 \\ -1 & \text{if } \ell \equiv 1 \pmod 3 \\ 0 & \text{if } \ell \equiv 2 \pmod 3 \end{cases} .$$

Another simple example is the case when $k = 3$. Let us consider the following table:

r	0	1	2	3	4	5
Δ_r	1	1	0	-1	-1	-2
$\tilde{\Delta}_r$	-1	1	2	1	1	0
ϵ_r	$-i$	$-i$	$-i$	$-i$	i	$-i$

We obtain

$$\det T_{n,k} = (-i)^q(\Delta_r U_q(-2i) + \epsilon_r \tilde{\Delta}_r U_{q-1}(-2i))\,,$$

where $n = 6q + r$, with $0 \leqslant r \leqslant 5$.

Since $F_{3(n+1)} = 2i^n U_n(-2i)$ (cf. (Zhang, W., 2002)), the term $x_{6q+r,3}$ is given by

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$(-1)^q x_{6q+r,3}$	F_{3q+1}	F_{3q+2}	F_{3q}	$-F_{3q+1}$	$-F_{3q+2}$	$-F_{3q+3}$

4. Some general examples

In this section we obtain some other particular solutions for Equation (1). Throughout the rest of the note, we will use the Iverson bracket for a given statement S define as

$$[S] = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

Our first result is straightforward.

Theorem 4.1. *For $n - k \leqslant 1$ and $n \geqslant 3$, the solution of Equation (1) is:*

$$x_{n,k} = \begin{cases} (-1)^m & \text{if } n \not\equiv 2 \pmod 3 \\ 0 & \text{otherwise} \end{cases},$$

where $n = 3m + t$ and $0 \leqslant t \leqslant 2$.

Proof. We will use induction on m. For $n = 3$, $t = 0$ and $m = 1$, we have

$$x_{3,k} = x_{2,k} - (-1)^{\left\lfloor \frac{1}{k} \right\rfloor} x_{1,k} = -1 = (-1)^1 = (-1)^m\,.$$

Assume that our assertion holds for $n = 3m + t$. In this case, using the inductive hypothesis, we have

$$\begin{aligned} x_{3m,k} &= (-1)^m\,, \quad \text{if } t = 0, \\ x_{3m+1,k} &= (-1)^m\,, \quad \text{if } t = 1, \\ x_{3m+2,k} &= 0\,, \quad \text{if } t = 2. \end{aligned}$$

Then, for $n = 3\,(m + 1) + t$, we have

$$\begin{aligned} x_{3m+3,k} &= x_{3m+2,k} - (-1)^{\left\lfloor \frac{3m+1}{k} \right\rfloor} x_{3m+1,k} = (-1)^{m+1}\,, \quad \text{if } t = 0, \\ x_{3m+4,k} &= x_{3m+3,k} - (-1)^{\left\lfloor \frac{3m+2}{k} \right\rfloor} x_{3m+2,k} = (-1)^{m+1}\,, \quad \text{if } t = 1, \\ x_{3m+5,k} &= x_{3m+4,k} - (-1)^{\left\lfloor \frac{3m+3}{k} \right\rfloor} x_{3m+3,k} = 0\,, \quad \text{if } t = 2. \end{aligned}$$

The proof is now completed. □

The following theorem provides general solutions for (1) when $n - k > 1$ and $k \equiv 3 \pmod 6$.

Theorem 4.2. *Let $k \geqslant 1$ be an integer such that $k \equiv 3 \pmod 6$. Then, for all $n \geqslant 3$, the solution of Equation* (1) *has these forms:*

1. *For $n \equiv 0 \pmod 6$ and $n \equiv a \pmod k$,*

$$x_{n,k} = \begin{cases} (-1)^{\frac{n(n+1)}{2}+[a\equiv 6 \pmod{12}]} F_{\frac{n-a}{2}+1} & \textit{if } a \equiv 0 \pmod 6 \\ (-1)^{\left(\begin{array}{c} \frac{n(n+1)}{2} + [k \equiv 3 \pmod{12} \textit{ and } a \equiv 9 \pmod{12}] \\ + [k \equiv 9 \pmod{12} \textit{ and } a \equiv 3 \pmod{12}] \end{array}\right)} F_{\frac{n-k+a}{2}+1} & \textit{if } a \equiv 3 \pmod 6 \end{cases}.$$

2. *For $n \equiv 1 \pmod 6$ and $n \equiv a \pmod k$,*

$$x_{n,k} = \begin{cases} (-1)^{\frac{n(n+1)}{2}+1-[a\equiv 7 \pmod{12}]} F_{\frac{n-a}{2}+2} \\ \textit{if } a \equiv 1 \pmod 6 \\ (-1)^{\left(\begin{array}{c} \frac{n(n+1)}{2} + 1 - [k \equiv 3 \pmod{12} \textit{ and } a \equiv 10 \pmod{12}] \\ - [k \equiv 9 \pmod{12} \textit{ and } a \equiv 4 \pmod{12}] \end{array}\right)} F_{\frac{n-k+a}{2}+1} \\ \textit{if } a \equiv 4 \pmod 6 \end{cases}.$$

3. *For $n \equiv 2 \pmod 6$ and $n \equiv a \pmod k$,*

$$x_{n,k} = \begin{cases} (-1)^{\frac{n(n+1)}{2}+1-[a\equiv 8 \pmod{12}]} F_{\frac{n-a}{2}} \\ \textit{if } a \equiv 2 \pmod 6 \\ (-1)^{\left(\begin{array}{c} \frac{n(n+1)}{2} + 1 - [k \equiv 3 \pmod{12} \textit{ and } a \equiv 11 \pmod{12}] \\ - [k \equiv 9 \pmod{12} \textit{ and } a \equiv 5 \pmod{12}] \end{array}\right)} F_{\frac{n-k+a}{2}+1} \\ \textit{if } a \equiv 5 \pmod 6 \end{cases}.$$

4. *For $n \equiv 3 \pmod 6$ and $n \equiv a \pmod k$,*

$$x_{n,k} = \begin{cases} (-1)^{\left(\begin{array}{c} \frac{n(n+1)}{2} + [k \equiv 3 \pmod{12} \textit{ and } a \equiv 0 \pmod{12}] \\ + [k \equiv 9 \pmod{12} \textit{ and } a \equiv 6 \pmod{12}] \end{array}\right)} F_{\frac{n-k+a}{2}+1} & \textit{if } a \equiv 0 \pmod 6 \\ (-1)^{\frac{n(n+1)}{2}+[a\equiv 3 \pmod{12}]} F_{\frac{n-a}{2}+1} & \textit{if } a \equiv 3 \pmod 6 \end{cases}.$$

5. *For $n \equiv 4 \pmod 6$ and $n \equiv a \pmod k$,*

$$x_{n,k} = \begin{cases} (-1)^{\left(\begin{array}{c} \frac{n(n+1)}{2} + 1 - [k \equiv 3 \pmod{12} \textit{ and } a \equiv 7 \pmod{12}] \\ - [k \equiv 9 \pmod{12} \textit{ and } a \equiv 1 \pmod{12}] \end{array}\right)} F_{\frac{n-k+a}{2}+1} \\ \textit{if } a \equiv 1 \pmod 6 \\ (-1)^{\frac{n(n+1)}{2}+1-[a \equiv 10 \pmod{12}]} F_{\frac{n-a}{2}+2} \\ \textit{if } a \equiv 4 \pmod 6 \end{cases}.$$

6. *For* $n \equiv 5 \pmod 6$ *and* $n \equiv a \pmod k$*,*

$$x_{n,k} = \begin{cases} (-1)^{\left(\begin{array}{c}\frac{n(n+1)}{2} + [k \equiv 3 \pmod{12} \text{ and } a \equiv 8 \pmod{12}] \\ + [k \equiv 9 \pmod{12} \text{ and } a \equiv 2 \pmod{12}]\end{array}\right)} F_{\frac{n-k+a}{2}+1} & \text{if } a \equiv 2 \pmod 6 \\ (-1)^{\frac{n(n+1)}{2} + [a \equiv 11 \pmod{12}]} F_{\frac{n-a}{2}} & \text{if } a \equiv 5 \pmod 6 \end{cases}.$$

Proof. Although there are several cases to be considered, we will prove here only one of them to avoid unnecessary repetitions. We shall only prove item (1). Assume that $n \equiv 0 \pmod 6$ and $n \equiv a \pmod k$. The proof will be done by strong induction on n. Let us consider the case $n = 24$, $k = 9$, namely: $n \equiv 0 \pmod 6$, $a \equiv 0 \pmod 6$ and $a \equiv 6 \pmod{12}$. Taking into account that

$$n - 1 \equiv 5 \pmod 6, \quad a - 1 \equiv 5 \pmod 6,$$

and

$$n - 2 \equiv 4 \pmod 6, \quad a - 2 \equiv 4 \pmod 6,$$

we obtain

$$x_{24,9} = (-1)^{\frac{24.25}{2}+1} F_{\frac{24-6}{2}+1} = -F_{10},$$
$$x_{23,9} = (-1)^{\frac{23.24}{2}} F_{\frac{23-5}{2}} = F_9,$$
$$x_{22,9} = (-1)^{\frac{22.23}{2}+1} F_{\frac{22-4}{2}+2} = F_{11}.$$

Thus

$$x_{24,9} = x_{23,9} - (-1)^{\lfloor \frac{22}{9} \rfloor} x_{22,9} = F_9 - F_{11} = -F_{10}.$$

Assume that our assertion holds for $n = t$. In this case, using the inductive hypothesis, we have

$$x_{t,k} = (-1)^{\frac{t(t+1)}{2}+1} F_{\frac{t-a}{2}+1},$$

$$x_{t-1,k} = (-1)^{\frac{t(t-1)}{2}} F_{\frac{t-a}{2}},$$

and

$$x_{t-2,k} = (-1)^{\frac{(t-2)(t-1)}{2}+1} F_{\frac{t-a}{2}+2}.$$

Then, for $n = t + 1$, we have

$$\begin{aligned} x_{t,k} - (-1)^{\lfloor \frac{t-1}{k} \rfloor} x_{t-1,k} &= (-1)^{\frac{t(t+1)}{2}+1} F_{\frac{t-a}{2}+1} - (-1)^{\lfloor \frac{t-1}{k} \rfloor} (-1)^{\frac{t(t-1)}{2}} F_{\frac{t-a}{2}} \\ &= (-1)^{\frac{(t+1)(t+2)}{2}} \left((-1)^{-t} F_{\frac{t-a}{2}+1} - (-1)^{\lfloor \frac{t-1}{k} \rfloor} (-1)^{-2t-1} F_{\frac{t-a}{2}} \right) \\ &= (-1)^{\frac{(t+1)(t+2)}{2}} \left(F_{\frac{t-a}{2}+1} + (-1)^{\lfloor \frac{t-1}{k} \rfloor} F_{\frac{t-a}{2}} \right) \\ &= (-1)^{\frac{(t+1)(t+2)}{2}} F_{\frac{t-a}{2}+2} \\ &= x_{t+1,k}, \end{aligned}$$

where

$$t + 1 \equiv 1 \pmod 6, \quad a + 1 \equiv 1 \pmod 6, \quad a + 1 \equiv 7 \pmod{12},$$
$$t \equiv 0 \pmod 6, \quad a \equiv 0 \pmod 6, \quad a \equiv 6 \pmod{12},$$
$$t - 1 \equiv 5 \pmod 6, \quad a - 1 \equiv 5 \pmod 6, \quad a - 1 \equiv 5 \pmod{12},$$

and $\left\lfloor \frac{t-1}{k} \right\rfloor$ is even. So, the first part of item (1) is proved. We now prove the second part. For this, we will again use the induction on n. Let us consider the case $n = 48$, $k = 27$, namely: $n \equiv 0 \pmod 6$, $a \equiv 3 \pmod 6$, $a \equiv 9 \pmod{12}$. Using

$$n-1 \equiv 5 \pmod 6, \quad a-1 \equiv 2 \pmod 6, \quad a-1 \equiv 8 \pmod{12},$$

and

$$n-2 \equiv 4 \pmod 6, \quad a-2 \equiv 1 \pmod 6, \quad a-2 \equiv 7 \pmod{12},$$

we have

$$x_{48,27} = (-1)^{\frac{48.49}{2}+1} F_{\frac{48-27+21}{2}+1} = -F_{22},$$
$$x_{47,27} = (-1)^{\frac{47.48}{2}+1} F_{\frac{47-27+20}{2}+1} = -F_{21},$$
$$x_{46,27} = (-1)^{\frac{46.47}{2}} F_{\frac{46-27+19}{2}+1} = -F_{20}.$$

So, we get

$$x_{47,27} - (-1)^{\left\lfloor \frac{46}{27} \right\rfloor} x_{46,27} = -F_{21} - F_{20} = -F_{22} = x_{48,27}.$$

Assume that our claim holds for $n = t$. In this case, using the induction hypothesis, we obtain

$$x_{t,k} = (-1)^{\left(\frac{t(t+1)}{2}+1\right)} F_{\frac{t-k+a}{2}+1},$$
$$x_{t-1,k} = (-1)^{\left(\frac{(t-1)t}{2}+1\right)} F_{\frac{t-k+a}{2}},$$
$$x_{t-2,k} = (-1)^{\left(\frac{(t-2)(t-1)}{2}\right)} F_{\frac{t-k+a}{2}-1}.$$

Then, for $n = t+1$, we get

$$\begin{aligned}
x_{t,k} - (-1)^{\left\lfloor \frac{t-1}{k} \right\rfloor} x_{t-1,k} &= (-1)^{\left(\frac{t(t+1)}{2}+1\right)} F_{\frac{t-k+a}{2}+1} - (-1)^{\left\lfloor \frac{t-1}{k} \right\rfloor} (-1)^{\left(\frac{(t-1)t}{2}+1\right)} F_{\frac{t-k+a}{2}} \\
&= (-1)^{\frac{(t+1)(t+2)}{2}} \left((-1)^{-t} F_{\frac{t-k+a}{2}+1} - (-1)^{\left\lfloor \frac{t-1}{k} \right\rfloor} (-1)^{-2t} F_{\frac{t-k+a}{2}} \right) \\
&= (-1)^{\frac{(t+1)(t+2)}{2}} \left(F_{\frac{t-k+a}{2}+1} - (-1)^{\left\lfloor \frac{t-1}{k} \right\rfloor} F_{\frac{t-k+a}{2}} \right) \\
&= (-1)^{\frac{(t+1)(t+2)}{2}} \left(F_{\frac{t-k+a}{2}+1} + F_{\frac{t-k+a}{2}} \right) \\
&= (-1)^{\frac{(t+1)(t+2)}{2}} F_{\frac{t-k+a}{2}+2} \\
&= x_{t+1,k},
\end{aligned}$$

where

$$t+1 \equiv 1 \pmod 6, \quad a+1 \equiv 4 \pmod 6, \quad a+1 \equiv 10 \pmod{12},$$
$$t \equiv 0 \pmod 6, \quad a \equiv 3 \pmod 6, \quad a \equiv 9 \pmod{12},$$
$$t-1 \equiv 5 \pmod 6, \quad a-1 \equiv 2 \pmod 6, \quad a-1 \equiv 8 \pmod{12},$$

and $\left\lfloor \frac{t-1}{k} \right\rfloor$ is odd. Therefore, our proof is completed. □

Remark 4.1. In Section 3, we obtained the solutions of Equation (1) for $n = 6q + r$ and $k = 3$ based on the determinant of the matrix $T_{n,k}$. Now, if we take $n = 6q + r$ and $k = 3$ in Theorem 4.2, we can obtain the solutions of Equation (1) in a different method. Namely, if q is even, we have

Also, if q is odd, we get

Remark 4.2. For $k \not\equiv 0 \pmod 3$, the solutions of Equation (1) do not necessarily belong to the Fibonacci sequence. For example

$$x_{34,7} = 169,$$

or

$$x_{99,32} = 3010349,$$

are not Fibonacci numbers.

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$x_{6q+r,3}$	F_{3q+1}	F_{3q+2}	F_{3q}	$-F_{3q+1}$	$-F_{3q+2}$	$-F_{3q+3}$

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$-x_{6q+r,3}$	F_{3q+1}	F_{3q+2}	F_{3q}	$-F_{3q+1}$	$-F_{3q+2}$	$-F_{3q+3}$

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