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Approximation of the KdVB equation by the quintic B-spline differential quadrature method

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ABSTRACT

In this paper, the Korteweg-de Vries-Burgers' (KdVB) equation is solved numerically by a new differential quadrature method based on quintic B-spline functions. The weighting coefficients are obtained by semi-explicit algorithm including an algebraic system with five-band coefficient matrix. The L_2 and L_∞ error norms and lowest three invariants I_1, I_2 and I_3 have computed to compare with some earlier studies. Stability analysis of the method is also given. The obtained numerical results show that the present method performs better than the most of the methods available in the literature.

Keywords: KdVB equation; differential quadrature method; quintic B-splines; partial differential equation; stability.

INTRODUCTION

Many physical phenomena in the nature can accurately be described by the Korteweg-de Vries-Burgers'(KdVB) equation which has the general form

$$U_t + \varepsilon U U_x - \upsilon U_{xx} + \mu U_{xxx} = 0, \tag{1}$$

where ε, υ and μ are positive constant coefficients and the subscripts t and x denote differentiation.

The KdVB equation was first introduced by Su & Gardner (1969). The equation presents an appropriate model equation for a wide range of nonlinear systems in the weak nonlinearity and long wavelength approximations, since it contains both damping and dispersion. The equation possesses steady-state solution, which has been demonstrated to model weak plasma shocks propagating perpendicular to a magnetic field (Grad & Hu, 1967). When diffusion dominates dispersion, the steady-state solutions of the KdVB equation are monotonic shocks and when dispersion dominates diffusion, then the shocks are oscillatory. The equation has been used in the study of wave propagation through liquid filled elastic tubes and for a description of shallow water waves on a viscous fluid (Johnson, 1970; 1972). Some numerical works have been carried out to solve the equation. Canosa & Gazdag (1977), who discussed the evolution of non-analytic initial data into a monotonic shock, have given brief details of a numerical solution of the KdVB equation using the accurate space derivative method. Ali *et al*. (1992; 1993) have produced a B-spline finite element scheme using Galerkin's method with quadratic B-spline interpolation function over the finite elements. KdVB equation has also been solved by using various numerical techniques such as finite element scheme (Zaki & Zaki, 2000a; 2000b, Saka & Dağ, 2007; 2009), tanh method (Sahu & Roychoudhury, 2003), hyperbolic tangent method, an exponential rational function approach (Demiray, 2004), finite difference scheme (Helal & Mehanna, 2006) and decomposition method (Kaya, 1999; 2004).

If $\upsilon = 0$, the equation (1) turns into KdV equation of the form

$$U_t + \varepsilon U U_x + \mu U_{xxx} = 0. \tag{2}$$

If $\mu = 0$, the equation (1) turns into Burgers' equation of the form

$$U_t + \varepsilon U U_x - \upsilon U_{xx} = 0. \tag{3}$$

Bellman *et al*. (1972) first introduced differential quadrature method (DQM) in 1972 for solving partial differential equations. The method has widely become popular in recent years, thanks to its simplicity for application. The fundamental idea behind the method is to find out the weighting coefficients of the functional values at nodal points by using base functions, of which derivatives are already known at the same nodal points over the entire region. Numerous researchers have developed different types of DQMs by utilizing various test functions. Bellman *et al*. (1972 1976) have used Legendre polynomials and spline functions in order to get weighting coefficients. Quan & Chang (1989a; 1989b) have introduced an explicit formulation for determining the weighting coefficients using Lagrange interpolation polynomials. Shu & Richards (1992) have presented an explicit formulae including both Lagrange interpolation polynomials. Moreover, Shu & Xue (1997) have used the Lagrange interpolated trigonometric polynomials to determine weighting coefficients in an explicit manner. Zhong (2004), Guo & Zhong (2004) and Zhong & Lan (2006) have introduced another efficient DQM as spline based DQM and applied to numerous problems. Cheng *et al*. (2005) have used Hermite polynomials for finding out the weighting coefficients required for DQM. Shu & Wu (2007) have introduced some of the implicit formulations of weighting coefficients with the help of radial basis functions. The weighting coefficients have also been found out by Striz *et al*. (1995) using harmonic functions implicitly. Sinc functions have been used as basis functions in order to find the weighting coefficients by Bonzani (1997). Thanks to its production of accurate numerical solutions and easy application for the solution process of numerous physical fields such as engineering, chemistry and physics problems, several DQMs have been used by Civalek (2004; 2006), Zhu *et al*. (2004), Lee *et* *al*. (2004), Korkmaz (2010a), Korkmaz & Dağ (2009; 2010b; 2011a; 2011b; 2012; 2013a; 2013b), Saka *et al*. (2008), Tomasiello (2010), Mittal & Jiwari (2009; 2011; 2012), Arora (2013).

In the present study, Quintic B-spline Differential Quadrature Method (QBDQM) is applied to obtain approximate solutions of the KdVB equation. Cubic B-spline DQM used for solving third order differential equation like KdV equation need transforming for solution (Korkmaz & Dağ, 2010b). But, QBDQM do not need transforming for solving the third order differential equations like KdV, KdVB and in order to make the stability analysis of the method there should not be a reduction such as splitting in the solution process. Therefore, in order to be able to make stability analysis of the third order non-linear KdVB equation we have preferred the quintic B-spline basis functions.The differential quadrature method has many advantages over the classical techniques, mainly, it prevents linearization and perturbation in order to find better solutions of given nonlinear equations.

QUINTIC B-SPLINE DIFFERENTIAL QUADRATURE METHOD

DQM can be defined as an approximation to a derivative of a given function by using the linear summation of its values at specific discrete nodal points over the solution domain of a problem. Let's take the grid distribution $a = x_1 < x_2 < \cdots < x_N = b$ of a finite interval $[a,b]$ into consideration. Provided that any given function $U(x)$ is enough smooth over the solution domain, its derivatives with respect to x at a nodal point x_i can be approximated by a linear summation of all the functional values in the solution domain, namely,

$$U_x^{(r)}(x_i) = \frac{d^{(r)}U}{dx^{(r)}}\Big|_{x_i} = \sum_{j=1}^{N} w_{ij}^{(r)} U(x_j), \quad i = 1,2,...,N, \quad r = 1,2,...,N-1 \tag{4}$$

where *r* denotes the order of the derivative, $w_{ij}^{(r)}$ represent the weighting coefficients of the *r - th* order derivative approximation, and *N* denotes the number of nodal points in the solution domain. Here, the index *j* represents the fact that $w_{ij}^{(r)}$ is the corresponding weighting coefficient of the functional value $U(x_j)$. In this study, we need first, second and third order derivative of the function $U(x)$. So, we will find value of the equation(4) for the $r = 1,2,3$.

If we consider Eq.(4), then it is seen that the fundamental process for approximating the derivatives of any given function through DQM is to find out the corresponding weighting coefficients $w_{ij}^{(r)}$. The main idea behind DQM approximation is to find out the corresponding weighting coefficients $w_{ij}^{(r)}$ by means of a set of base functions spanning the problem domain. While determining the corresponding weighting coefficients different basis may be used. In the present study, we will compute weighting coefficients with quintic B-spline basis.

Let $Q_m(x)$, be the quintic B-splines with knots at the points x_i where the uniformly distributed N nodal points are taken as $a = x_1 < x_2 < \cdots < x_N = b$ on the ordinary real axis. The B-splines $\{Q_{-1}, Q_0, \ldots, Q_{N+2}\}$ form a basis for functions defined over $[a,b]$. The quintic B-splines $Q_m(x)$ are defined by the relationships:

$$Q_m(x) = \frac{1}{h^5}\begin{cases} (x-x_{m-3})^5, & x \in x_{m-3}, x_{m-2}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5, & x \in x_{m-2}, x_{m-1}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5, & x \in x_{m-1}, x_m], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - \\ 20(x-x_m)^5, & x \in x_m, x_{m+1}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - \\ 20(x-x_m)^5 + 15(x-x_{m+1})^5, & x \in x_{m+1}, x_{m+2}], \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - \\ 20(x-x_m)^5 + 15(x-x_{m+1})^5 - 6(x-x_{m+2})^5, & x \in x_{m+2}, x_{m+3}], \\ 0, & otherwise. \end{cases}$$

where $h = x_m - x_{m-1}$ for all *m*.

Table 1. The value of quintic B-splines and derivatives functions at the grid points

x	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
Q_m	0	1	26	66	26	1	0
Q_m'	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$-\frac{50}{h}$	$-\frac{5}{h}$	0
Q_m''	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$-\frac{120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
Q_m'''	0	$\frac{60}{h^3}$	$-\frac{120}{h^3}$	0	$\frac{120}{h^3}$	$-\frac{60}{h^3}$	0
$Q_m^{(4)}$	0	$\frac{120}{h^4}$	$\frac{480}{h^4}$	$\frac{720}{h^4}$	$-\frac{480}{h^4}$	$\frac{120}{h^4}$	0

Using the quintic B-splines as test functions in the fundamental DQM equation (4) leads to the equation

$$\frac{\partial^{(r)}Q_m(x_i)}{\partial x^{(r)}}=\sum_{j=m-2}^{m+2}w_{i,j}^{(r)}Q_m(x_j),\quad m=-1,0,\ldots,N+2, i=1,2,...,N. \tag{5}$$

An arbitrary choice of i leads to an algebraic equation system

$$\begin{bmatrix} Q_{-1,-3} & Q_{-1,-2} & Q_{-1,-1} & Q_{-1,0} & Q_{-1,1} & & & & \\ & Q_{0,-2} & Q_{0,-1} & Q_{0,0} & Q_{0,1} & Q_{0,2} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & Q_{N+1,N-1} & Q_{N+1,N} & Q_{N+1,N+1} & Q_{N+1,N+2} & Q_{N+1,N+3} & & \\ & & & Q_{N+2,N} & Q_{N+2,N+1} & Q_{N+2,N+2} & Q_{N+2,N+3} & Q_{N+2,N+4} & \end{bmatrix}W_1=\Phi_1 \tag{6}$$

where $Q_{i,j}$ denotes $Q_i(x_j)$,

$$W_1=\begin{bmatrix} w_{i,-3}^{(r)} & w_{i,-2}^{(r)} & \cdots & w_{i,N+3}^{(r)} & w_{i,N+4}^{(r)} \end{bmatrix}^T \tag{7}$$

and

$$\Phi_1=\begin{bmatrix} \frac{\partial^{(r)}Q_{-1}(x_i)}{\partial x^{(r)}} & \frac{\partial^{(r)}Q_0(x_i)}{\partial x^{(r)}} & \cdots & \frac{\partial^{(r)}Q_{N+1}(x_i)}{\partial x^{(r)}} & \frac{\partial^{(r)}Q_{N+2}(x_i)}{\partial x^{(r)}} \end{bmatrix}^T. \tag{8}$$

The weighting coefficients $w_{i,j}^{(r)}$ related to the i - th grid point are determined by solving equation system (6) The system (6) consists of $N+8$ unknowns and $N+4$ equations. To have a unique solution of the system, it is required to add four additional equations to the system. By the addition of the equations

$$\frac{\partial^{(r+1)}Q_{-1}(x_i)}{\partial x^{(r+1)}}=\sum_{j=-3}^{1}w_{i,j}^{(r)}Q_{-1}'(x_j), \tag{9}$$

$$\frac{\partial^{(r+1)}Q_0(x_i)}{\partial x^{(r+1)}}=\sum_{j=-2}^{2}w_{i,j}^{(r)}Q_0'(x_j), \tag{10}$$

$$\frac{\partial^{(r+1)}Q_{N+1}(x_i)}{\partial x^{(r+1)}}=\sum_{j=N-1}^{N+3}w_{i,j}^{(r)}Q_{N+1}'(x_j), \tag{11}$$

$$\frac{\partial^{(r+1)}Q_{N+2}(x_i)}{\partial x^{(r+1)}}=\sum_{j=N}^{N+4}w_{i,j}^{(r)}Q_{N+2}'(x_j) \tag{12}$$

to the system (6) becomes

$$M_1W_1=\Phi_2, \tag{13}$$

where

$$M_1=\begin{bmatrix} Q_{-1,-3} & Q_{-1,-2} & Q_{-1,-1} & Q_{-1,0} & Q_{-1,1} & & & & & \\ Q'_{-1,-3} & Q'_{-1,-2} & Q'_{-1,-1} & Q'_{-1,0} & Q'_{-1,1} & & & & & \\ & Q_{0,-2} & Q_{0,-1} & Q_{0,0} & Q_{0,1} & Q_{0,2} & & & & \\ & Q'_{0,-2} & Q'_{0,-1} & Q'_{0,0} & Q'_{0,1} & Q'_{0,2} & & & & \\ & & Q_{1,-1} & Q_{1,0} & Q_{1,1} & Q_{1,2} & Q_{1,3} & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & Q_{N+1,N-1} & Q_{N+1,N} & Q_{N+1,N+1} & Q_{N+1,N+2} & Q_{N+1,N+3} & & \\ & & & Q'_{N+1,N-1} & Q'_{N+1,N} & Q'_{N+1,N+1} & Q'_{N+1,N+2} & Q'_{N+1,N+3} & & \\ & & & & Q_{N+2,N} & Q_{N+2,N+1} & Q_{N+2,N+2} & Q_{N+2,N+3} & Q_{N+2,N+4} & \\ & & & & Q'_{N+2,N} & Q'_{N+2,N+1} & Q'_{N+2,N+2} & Q'_{N+2,N+3} & Q'_{N+2,N+4} & \end{bmatrix}$$

and

$$W_1=\left[w^{(r)}_{i,-3} \quad w^{(r)}_{i,-2} \quad \cdots \quad w^{(r)}_{i,N+3} \quad w^{(r)}_{i,N+4}\right]^T,$$

and

$$\Phi_2 = [\frac{\partial^{(r)}Q_{-1}(x_i)}{\partial x^{(r)}} \quad \frac{\partial^{(r+1)}Q_{-1}(x_i)}{\partial x^{(r+1)}} \quad \frac{\partial^{(r)}Q_{0}(x_i)}{\partial x^{(r)}} \quad \frac{\partial^{(r+1)}Q_{0}(x_i)}{\partial x^{(r+1)}} \quad \frac{\partial^{(r)}Q_{1}(x_i)}{\partial x^{(r)}}$$
$$\cdots \quad \frac{\partial^{(r)}Q_{N+1}(x_i)}{\partial x^{(r)}} \quad \frac{\partial^{(r+1)}Q_{N+1}(x_i)}{\partial x^{(r+1)}} \quad \frac{\partial^{(r)}Q_{N+2}(x_i)}{\partial x^{(r)}} \quad \frac{\partial^{(r+1)}Q_{N+2}(x_i)}{\partial x^{(r+1)}}]^T.$$

After the using the values of quintic B-splines at the grid points and eliminating $w^{(r)}_{i,-3}$, $w^{(r)}_{i,-2}$, $w^{(r)}_{i,N+3}$ and $w^{(r)}_{i,N+4}$ from system, we obtain an algebraic equation system having 5-banded coefficient matrix of the form

$$M_2W_2=\Phi_3, \tag{14}$$

where

$$M_2=\begin{bmatrix} 37 & 82 & 21 & & & & & & \\ 8 & 33 & 18 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{bmatrix} \text{ and } W_2=\begin{bmatrix} w^{(r)}_{i,-1} \\ w^{(r)}_{i,0} \\ \vdots \\ w^{(r)}_{i,i-2} \\ w^{(r)}_{i,i-1} \\ w^{(r)}_{i,i} \\ w^{(r)}_{i,i+1} \\ w^{(r)}_{i,i+2} \\ \vdots \\ w^{(r)}_{i,N+1} \\ w^{(r)}_{i,N+2} \end{bmatrix}.$$

The non-zero entries of the load vector Φ_3 are given as,

$$\Phi_{-1} = \frac{1}{30}\left[-5Q_{-1}^{(r)}(x_i) + hQ_{-1}^{(r+1)}(x_i) + 40Q_0^{(r)}(x_i) + 8hQ_0^{(r+1)}(x_i)\right],$$

$$\Phi_0 = \frac{1}{10}\left[5Q_0^{(r)}(x_i) - hQ_0^{(r+1)}(x_i)\right],$$

$$\Phi_{i-2} = Q_{i-2}^{(r)}(x_i),$$

$$\Phi_{i-1} = Q_{i-1}^{(r)}(x_i),$$

$$\Phi_i = Q_i^{(r)}(x_i),$$

$$\Phi_{i+1} = Q_{i+1}^{(r)}(x_i),$$

$$\Phi_{i+2} = Q_{i-2}^{(r)}(x_i),$$

$$\Phi_{N+1} = \frac{1}{10}\left[5Q_{N+1}^{(r)}(x_i) + hQ_{N+1}^{(r+1)}(x_i)\right],$$

$$\Phi_{N+2} = \frac{-1}{30}\left[-40Q_{N+1}^{(r)}(x_i) + 8hQ_{N+1}^{(r+1)}(x_i) + 5Q_{N+2}^{(r)}(x_i) + hQ_{N+2}^{(r+1)}(x_i)\right]. \quad (15)$$

For example, if we apply the test functions Q_m, $m = -1, 0, \ldots, N+2$ at the first grid point x_1 for first order derivative approximation by the selection of $i = 1$ and $r = 1$ at Equation (15).

$$\Phi_{-1} = \frac{1}{30}\left[-5Q_{-1}^{(1)}(x_1) + hQ_{-1}^{(2)}(x_1) + 40Q_0^{(1)}(x_1) + 8hQ_0^{(2)}(x_1)\right],$$

$$\Phi_{-1} = \frac{1}{30}\left[-5\left(\frac{-5}{h}\right) + h\left(\frac{20}{h^2}\right) + 40\left(\frac{-50}{h}\right) + 8h\left(\frac{40}{h^2}\right)\right] = \frac{-109}{2h},$$

$$\Phi_0 = \frac{1}{10}\left[5Q_0^{(1)}(x_1) - hQ_0^{(2)}(x_1)\right],$$

$$\Phi_0 = \frac{1}{10}\left[5\left(\frac{-50}{h}\right) - h\left(\frac{40}{h^2}\right)\right] = \frac{-29}{h},$$

$$\Phi_1 = Q_1^{(1)}(x_1) = 0,$$

$$\Phi_2 = Q_2^{(1)}(x_1) = \frac{50}{h},$$

$$\Phi_3 = Q_3^{(1)}(x_1) = \frac{5}{h},$$

$$\Phi_{N+1} = \frac{1}{10}\left[5Q_{N+1}^{(1)}(x_1) + hQ_{N+1}^{(2)}(x_1)\right],$$

$$\Phi_{N+1} = \frac{1}{10}\left[5.0 + h.0\right] = 0,$$

$$\Phi_{N+2} = \frac{-1}{30}\left[-40Q_{N+1}^{(r)}(x_i) + 8hQ_{N+1}^{(r+1)}(x_i) + 5Q_{N+2}^{(r)}(x_i) + hQ_{N+2}^{(r+1)}(x_i)\right],$$

$$\Phi_{N+2} = \frac{-1}{30}\left[-40.0 + 8h.0 + 5.0 + h.0\right] = 0$$

is obtained and written at matrix form as:

$$\begin{bmatrix} 37 & 82 & 21 & & & & & & & \\ 8 & 33 & 18 & 1 & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & 1 & 26 & 66 & 26 & 1 & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{bmatrix} \begin{bmatrix} w_{1,-1}^{(1)} \\ w_{1,0}^{(1)} \\ w_{1,1}^{(1)} \\ w_{1,2}^{(1)} \\ w_{1,3}^{(1)} \\ w_{1,4}^{(1)} \\ \vdots \\ w_{1,N+1}^{(1)} \\ w_{1,N+2}^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{109}{2h} \\ -\frac{29}{h} \\ 0 \\ \frac{50}{h} \\ \frac{5}{h} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

By the same idea, for the determine weighting coefficients $w_{k,j}^{(1)}, j = -1,0,\ldots,N+2$ at grid points x_k, $2 \le k \le N-1$ we got the algebraic equation system:

$$\begin{bmatrix} 37 & 82 & 21 & & & & & & & \\ 8 & 33 & 18 & 1 & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & 1 & 26 & 66 & 26 & 1 & & \\ & & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{bmatrix} \begin{bmatrix} w^{(1)}_{k,-1} \\ \vdots \\ w^{(1)}_{k,k-3} \\ w^{(1)}_{k,k-2} \\ w^{(1)}_{k,k-1} \\ w^{(1)}_{k,k} \\ w^{(1)}_{k,k+1} \\ w^{(1)}_{k,k+2} \\ w^{(1)}_{k,k+3} \\ \vdots \\ w^{(1)}_{k,N+2} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{5}{h} \\ -\frac{50}{h} \\ 0 \\ \frac{50}{h} \\ \frac{5}{h} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For the last grid point of the domain x_N with same idea, determine weighting coefficients $w^{(1)}_{N,j}, \quad j=-1,0,\ldots,N+2$ we got the algebraic equation system:

$$\begin{bmatrix} 37 & 82 & 21 & & & & & & & \\ 8 & 33 & 18 & 1 & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & 1 & 26 & 66 & 26 & 1 & & \\ & & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{bmatrix} \begin{bmatrix} w^{(1)}_{N,-1} \\ w^{(1)}_{N,0} \\ \vdots \\ w^{(1)}_{N,N-3} \\ w^{(1)}_{N,N-2} \\ w^{(1)}_{N,N-1} \\ w^{(1)}_{N,N} \\ w^{(1)}_{N,N+1} \\ w^{(1)}_{N,N+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{5}{h} \\ -\frac{50}{h} \\ 0 \\ \frac{29}{h} \\ \frac{109}{2h} \end{bmatrix}.$$

We can obtain the second and third order derivative approximations with a same calculation. So the system (14) is solved by 5-banded Thomas algorithm.

NUMERICAL DISCRETIZATIONS

Here, we consider the KdV, Burgers' and KdVB equations.

DISCRETIZATION OF KdV EQUATION

As it is said before, If $\upsilon = 0$, the equation (1) turns into KdV equation of the form

$$U_t + \varepsilon U U_x + \mu U_{xxx} = 0,$$

with the following boundary conditions taken from

$$U(a,t) = g_1(t), \qquad U(b,t) = g_2(t), \quad t \in (0,T]. \tag{16}$$

and the following initial condition

$$U(x,0) = f_1(x), \quad a \le x \le b, \tag{17}$$

is rewritten as,

$$U_t = -\varepsilon U U_x - \mu U_{xxx}. \tag{18}$$

Then, the differential quadrature derivative approximations given in the Equation (4), have been used in Equation (18) for the value of $r = 1$ and $r = 3$. The application of the boundary conditions results in

$$\frac{dU(x_i)}{dt} = -\varepsilon U(x_i,t)\sum_{j=2}^{N-1} w_{i,j}^{(1)} U(x_j,t) - \mu \sum_{j=2}^{N-1} w_{i,j}^{(3)} U(x_j,t) + B(U), \quad i = 2,3,\ldots,N-1 \tag{19}$$

where

$$B(U) = -\varepsilon U(x_i,t)\left[w_{i,1}^{(1)} g_1(t) + w_{i,N}^{(1)} g_2(t)\right] - \mu\left[w_{i,1}^{(3)} g_1(t) + w_{i,N}^{(3)} g_2(t)\right].$$

DISCRETIZATION OF BURGERS' TYPE EQUATION

As it is mentioned before, If $\mu = 0$, the Equation (1) turns into Burgers' equation of the form

$$U_t + \varepsilon U U_x - \upsilon U_{xx} = 0,$$

with boundary conditions chosen from

$$U(a,t) = g_3(t), \qquad U(b,t) = g_4(t), \quad t \in (0,T]. \tag{20}$$

and initial condition

$$U(x,0) = f_2(x), \quad a \le x \le b, \tag{21}$$

is rewritten as,

$$U_t = -\varepsilon U U_x + \upsilon U_{xx}. \tag{22}$$

Then, the differential quadrature derivative approximations given in the Equation (4), have been used in Equation (22) for the value of $r=1$ and $r=2$. The application of the boundary conditions yield

$$\frac{dU(x_i)}{dt}=-\varepsilon U(x_i,t)\sum_{j=2}^{N-1}w_{i,j}^{(1)}U(x_j,t)+\upsilon\sum_{j=2}^{N-1}w_{i,j}^{(2)}U(x_j,t)+C(U),\quad i=2,3,...,N-1 \tag{23}$$

where

$$C(U)=-\varepsilon U(x_i,t)\left[w_{i,1}^{(1)}g_3(t)+w_{i,N}^{(1)}g_4(t)\right]+\upsilon\left[w_{i,1}^{(2)}g_3(t)+w_{i,N}^{(2)}g_4(t)\right].$$

DISCRETIZATION OF KdVB EQUATION AND STABILITY ANALYSIS

If $\upsilon,\mu\neq 0$, Equation (1) of the form

$$U_t+\varepsilon UU_x-\upsilon U_{xx}+\mu U_{xxx}=0,$$

with the following boundary conditions taken from

$$U(a,t)=g_5(t),\qquad U(b,t)=g_6(t),\quad t\in(0,T]. \tag{24}$$

and the following initial condition

$$U(x,0)=f_3(x),\qquad a\le x\le b, \tag{25}$$

is rewritten as,

$$U_t=-\varepsilon UU_x+\upsilon U_{xx}-\mu U_{xxx}. \tag{26}$$

The differential quadrature derivative approximations given in the Equation (4), have been used in Equation (26) for the value of $r=1,2$ and 3. The application of the boundary conditions results in

$$\frac{dU(x_i)}{dt}=-\varepsilon U(x_i,t)\sum_{j=2}^{N-1}w_{i,j}^{(1)}U(x_j,t)+\upsilon\sum_{j=2}^{N-1}w_{i,j}^{(2)}U(x_j,t)$$
$$-\mu\sum_{j=2}^{N-1}w_{i,j}^{(3)}U(x_j,t)+D(U),\qquad i=2,3,...,N-1 \tag{27}$$

where

$$D(U)=-\varepsilon U(x_i,t)\left[w_{i,1}^{(1)}g_5(t)+w_{i,N}^{(1)}g_6(t)\right]+\upsilon\left[w_{i,1}^{(2)}g_5(t)+w_{i,N}^{(2)}g_6(t)\right]$$
$$-\mu\left[w_{i,1}^{(3)}g_5(t)+w_{i,N}^{(3)}g_6(t)\right].$$

Then, the ordinary differential equation given by (27) is integrated in time by means of any appropriate method. Here, we have preferred fourth-order Runge-Kutta method since its advantages such as accuracy, stability and memory allocation properties.

The stability of a time-dependent problem:

$$\frac{\partial U}{\partial t}=l(U) \tag{28}$$

with proper initial and boundary conditions, where l is a spatial differential operator. After discretization with DQM, equation (28) is reduced into a set of ordinary differential equations in time:

$$\frac{d\{u\}}{dt} = [A]\{u\} + \{b\} \tag{29}$$

where $\{u\}$ is an unknown vector of the functional values at the grid points except left and right boundary points, $\{b\}$ is a vector containing the non-homogenous part and the boundary conditions. and A is the coefficient matrix. The stability of a numerical scheme for numerical integration of equation (29) depends on the stability of the ordinary differential equation (29). If the ordinary differential equation (29) is not stable, numerical methods may not generate converged solutions. The stability of equation (29) is related to the eigenvalues of the matrix A, since its exact solution is directly determined by the eigenvalues of the matrix A. When all $R_e(\lambda_i) \leq 0$ for all i is enough to show the stability of the exact solution of $\{u\}$ as $t \rightarrow \infty$ where Re denotes the real part of the eigenvalues λ_i of the matrix A. The matrix A at Equation (29) is determined as $A_{ij} = -\alpha_i w_{i,j}^{(1)} + \upsilon w_{i,j}^{(2)} - \mu w_{i,j}^{(3)}$ where $\alpha_i = U(x_i, t)$

The stable solution of $\{u\}$ as $t \rightarrow \infty$ requires:

1 If all eigenvalues are real, $-2.78 < \Delta t.\lambda_i < 0$,

2 If all eigenvalues have only complex components, $-2\sqrt{2} < \Delta t.\lambda_i < 2\sqrt{2}$,

3 If eigenvalues have only complex, $\Delta t.\lambda_i$ should be in the region, Figure 1.

When the eigenvalues are complex, there exist some tolerance that the real parts of the eigenvalues may be small positive numbers (Jain, 1983).

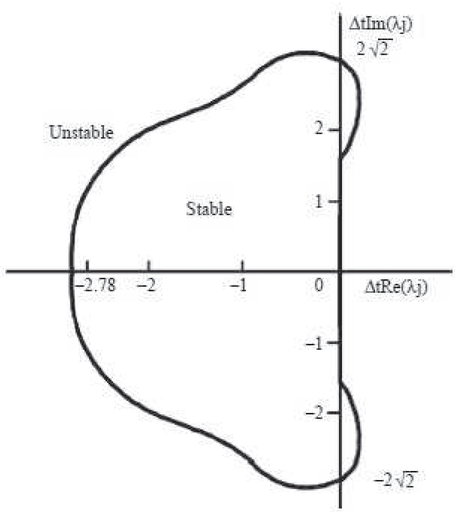

Fig. 1. Stability region of complex eigenvalues

The accuracy of the numerical method is checked using the error norms L_2 and L_∞ respectively:

$$L_2 = \sqrt{h\sum_{J=1}^{N}\left|U_j^{exact} - (U_N)_j\right|^2}, \qquad L_\infty = \max_j \left|U_j^{exact} - (U_N)_j\right|. \quad (30)$$

The following lowest three invariants corresponding to conservation of mass, momentum and energy will be computed.

$$I_1 = \int_a^b U dx, \qquad I_2 = \int_a^b U^2 dx, \qquad I_3 = \int_a^b \left[U^3 - \frac{3\mu}{\varepsilon}\left(U'\right)^2\right]dx. \quad (31)$$

NUMERICAL EXAMPLES

In this section, the numerical solutions of the KdV, Burgers' and KdVB equations are obtained by the proposed method.

KdV EQUATION

The initial condition:

$$U(x,0) = 3C\,\mathrm{sec}\,h^2\left(AX + D\right), \quad (32)$$

here A, C and D are constants given by the boundary conditions $U(0,t) = U(2,t) = 0$ for all times.

Table 2. Comparison of L_2 and L_∞ error norms at various times

					Time		
$L_2 X 10^6$ error norms at various times	ε	**$\mu \times 10^4$**	N	Δt	**1.0**	**2.0**	**3.0**
QBDQM (Present)	1	4.84	101	0.001	227.1	354.5	485.2
LPDQ (Korkmaz, 2010a)	1	4.84	100	0.001	1185.0	1290.0	1381.0
Galerkin Quad-spline (Gardner *et al*. 1991)	1	4.84	200	0.005	600.0	860.0	107.0
RBF Coll IMQ (Dağ *et al*. 2008)	1	4.84	200	0.005			2751.0
RBF Coll IQ (Dağ *et al*. 2008)	1	4.84	200	0.005			1013.0
RBF Coll TPS (Dağ *et al*. 2008)	1	4.84	200	0.005			2606.0
Septic spline Coll.(Soliman, 2004)	1	4.84	200	0.005	22100.0		
					Time		
$L_\infty \times 10^5$ error norms at various times	ε	**$\mu \times 10^4$**	N	Δt	**1.0**	**2.0**	**3.0**
QBDQM (Present)	1	4.84	101	0.001	73.8	108.6	142.8
LPDQ (Korkmaz, 2010a)	1	4.84	100	0.001	274.5	224.0	242.2
RBF Coll IMQ (Dağ *et al*.ğ *et al*. 2008)	1	4.84	200	0.005			501.8
RBF Coll IQ (Dağ *et al*. 2008)	1	4.84	200	0.005			200.0
RBF Coll TPS (Dağ *et al*. 2008)	1	4.84	200	0.005			634.5

For this condition, the KdV equation has an analytic solution given in the form of

$$U(x,t) = 3C\,\mathrm{sech}^2\left(AX - Bt + D\right), \tag{33}$$

provided that

$$A = \frac{1}{2}\left(\varepsilon C/\mu\right)^{1/2} \text{ and } B = \frac{1}{2}\varepsilon C\left(\varepsilon C/\mu\right)^{1/2}, \tag{34}$$

so that Equation (33) yields a probable initial condition when $A = \frac{1}{2}\left(\varepsilon/\mu\right)^{1/2}$ and really simulates a single soliton that moves toward the right having the velocity εC.

Table 3. Invariants for single soliton: $\Delta t = 0.001$ and $N = 101$.

	QBDQM (Present)			LPDQ (Korkmaz, 2010a)		
t	I_1 x 10^1	I_2 x 10^2	I_3 x 10^2	I_1 x 10^1	I_2 x 10^2	I_3 x 10^2
0.0	1.44598100	8.67592700	4.68502700	1.44597627	8.67592530	4.68499446
1.0	1.44591200	8.67592400	4.68502400	1.44229897	8.67613393	4.68501205
2.0	1.44600600	8.67592600	4.68502600	1.44245451	8.67615517	4.68501312
3.0	1.44609700	8.67592900	4.68502800	1.44461700	8.67617981	4.68501755

To be able to make a comparison with earlier studies, $\upsilon = 0$, $\varepsilon = 1$, $\mu = 4.84\times10^{-4}$, $C = 0.3$, $D = -6$, $\Delta t = 0.001$ and $\Delta x = 0.02$ will be used. For the present case, the obtained solution is going to move toward the right having a speed of εC. If we plot the graphs of the numerical solution and the exact solution, their curves will be indistinguishable. The agreement is very good. To make a comparison quantitatively, we have also computed the error norms L_2 and L_∞ as well as the first three invariants I_1, I_2 and I_3, in Table 2 and Table 3 until $t = 3.0$, respectively.

In Table 2, L_2 norm is less than 2.3×10^{-4} while the L_∞ norm is less than 7.4×10^{-4} at time $t = 1.0$ and so are enough small to accept. As it is obviously seen from Table 3, all of the computed three invariants are satisfactory constant. The results of the present study compares with earlier works.

BURGERS' TYPE EQUATION

For solving the KdVB equation (1) as a Burgers' type equation ($\mu = 0$), considering the initial condition the function as follows

$$U(x,t) = \frac{x/t}{1 + \left(t/t_0\right)\exp\left(x^2/4\upsilon t\right)}, \tag{35}$$

will be very appropriate. Here $t_0 = exp(\frac{1}{8\upsilon})$, evaluated at $t = 1$. The solution of the system of equations for different values of υ with the following boundary conditions

$$U(a,t) = U(b,t) = 0, \qquad \forall t \geq 1, \tag{36}$$

will be sought. The initial condition (35) will be preferred because of the fact that the resulting analytic solution can be expressed in a closed form allowing the easy computation of the L_2 and L_∞ error norms for any given value of υ. We will consider the value $\upsilon = 0.05$ for comparison with earlier works. Figure 2, illustrate the development of the initial condition (35) with time for the values of $\nu = 0.005$, $\varepsilon = 1$, $\mu = 0$, $\Delta t = 0.01$ and $\Delta x = 0.02$ for $0 \leq x \leq 1$. The program has been run until the time $t = 3.1$. The top curve has been recorded at $t = 1.0$ whereas the bottom curve has been recorded at $t = 3.1$. In order to evaluate the convergence, the error norms are tabulated in Table 4 with the comparison of earlier works. For comparison the results of Quintic B-spline DQ and Cubic B-spline DQ we selected $\nu = 0.005$, $\varepsilon = 1$, $\mu = 0$ and $\Delta t =$ 0.001 for $0 \leq x \leq 1.2$. Then, the error norms for each approximation are tabulated in Table 5. As it is seen from the Table that our results are better than the those previous papers. Error norms for $\nu = 0.005$, $\varepsilon = 1$, $\mu = 0$, $\Delta t = 0.01$ and $N = 51$ for $0 \leq x \leq 1$ at $t = 3.1$ and also $\nu = 0.005$, $\varepsilon = 1$, $\mu = 0$ and $N = 201$ for $0 \leq x \leq 1.2$ at $t = 3.6$ plotted at Figure 3 and Figure 4, respectively.

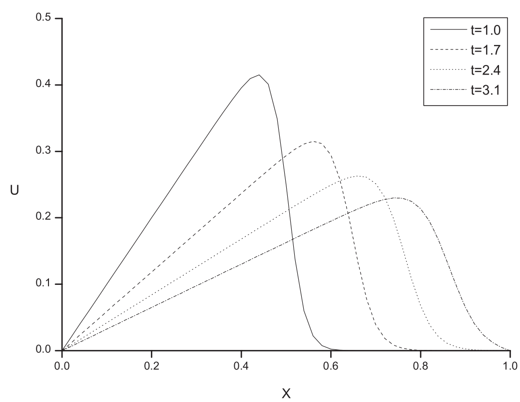

Fig. 2. $\upsilon = 0.005$, $\varepsilon = 1$, $\Delta t = 0.01$ and $\Delta x = 0.02$.

Table 4. L_2 and L_∞ error norms at the $0 \le x \le 1$ for $\upsilon = 0.005,$ and $\varepsilon = 1$ $\Delta t = 0.01$.

	Present		Ali *et al.* (1992)		Saka and Dağ (2007)		Saka and Dağ (2007)	
	$\Delta x = 0.02$		$\Delta x = 0.02$		$\Delta x = 0.005$		$\Delta x = 0.005$	
t	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
1.7	0.069	0.433	0.857	2.576	0.017	0.061	0.358	1.211
2.4	0.056	0.312	0.423	1.242	0.012	0.058	0.251	0.807
3.1	0.430	2.635	0.230	0.688	0.601	4.434	0.630	4.790

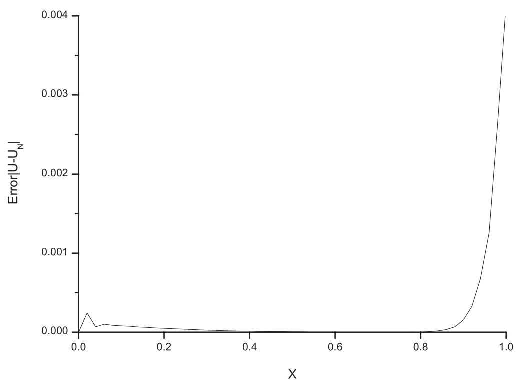

Fig. 3. Error norms for $\upsilon = 0.005,$ $\varepsilon = 1,$ $\Delta t = 0.01$ $N =$ 51 at $t = 3.1$

Table 5. Error norms for $\upsilon = 0.005$, $\varepsilon = 1$, $\mu = 0$ and $\Delta t = 0.001$ for $0 \le x \le 1.2$

	Present		Korkmaz and Dağ (2013a)					
	QBDQM		Method1		Method2		Method3	
N	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
21	0.71	2.00	1.64	3.10	1.41	3.29	7.05	11.6
31	0.42	1.31	1.00	2.13	0.79	2.22	0.94	1.73
41	0.30	0.97	0.70	1.61	0.57	1.68	0.92	1.48
61	0.19	0.62	0.44	1.07	0.37	1.12	0.26	0.95
81	0.13	0.44	0.31	0.77	0.27	0.83	0.20	0.76
101	0.09	0.33	0.23	0.59	0.21	0.64	0.16	0.63
121	0.07	0.25	0.18	0.46	0.16	0.52	0.14	0.54
151	0.04	0.15	0.12	0.32	0.12	0.39	0.11	0.45
161	0.03	0.13	0.11	0.28	0.11	0.35	0.10	0.43
201	0.01	0.08	0.06	0.16	0.07	0.24	0.09	0.36

KdVB EQUATION

Now, we have examined the behavior of the KdVB equation (1) and have studied the effect of using different values of μ and υ onto the solution vector. To carry out such a work, first of all we need to use as an initial condition (Ali *et al*. 1993)

$$U(x,0) = 0.5\left[1 - \tanh\frac{|x| - x_0}{d}\right], \tag{37}$$

and boundary conditions

$$U(-50,t) = U(150,t) = 0, \tag{38}$$

where $-50 \le x \le 150$, $d = 5$ and $x_0 = 25$ will be considered in all simulations.

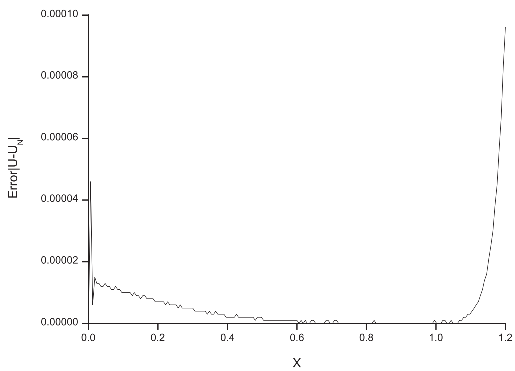

Fig. 4. Error norms for $\upsilon = 0.005,\ \varepsilon = 1,\ \Delta t = 0.001$ and $N = 201$ at $t = 3.6$

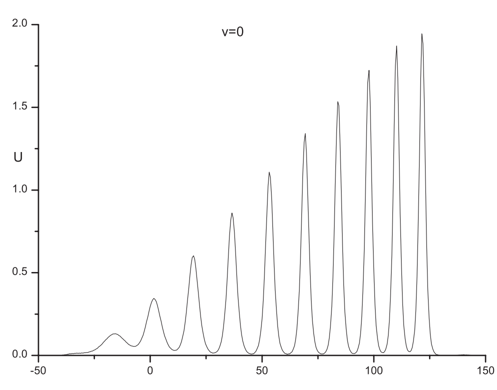

Fig. 5. KdVB type solution taken at time $t = 800$ with $\upsilon = 0$, $\varepsilon = 0.2$, $\mu = 0.1$, $\Delta t = 0.4$ and $N = 373$.

Table 6. Three invariants for $\nu = 0$, $\varepsilon = 0.2$, $\mu = 0.1$, $\Delta t = 0.4$ and $N = 373$.

	QBDQM $\Delta t = 0.4$ and $N = 373$			Zaki (2000a) $\Delta t = 0.4$ and $h = 0.01$		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	50.00013	45.00046	42.30068	50.00021	45.00055	42.30074
100	50.00031	45.00048	42.29989	50.00034	45.00003	42.30028
200	50.00072	45.00054	42.29736	50.00058	44.99962	42.30098
300	50.00568	45.00058	42.29604	50.00612	44.99999	42.30227
400	50.00259	45.00057	42.29560	50.00237	44.99921	42.30135
500	49.99523	45.00054	42.29548	49.99435	44.99850	42.30030
600	49.97926	45.00049	42.29546	49.97857	44.99820	42.29995
700	49.96699	45.00054	42.29548	49.96607	44.99815	42.29979
800	49.96415	45.00052	42.29552	49.96331	44.99803	42.29974

Table 7. Three invariants for $\nu = 0$, $\varepsilon = 0.2$, $\mu = 0.1$, $\Delta t = 0.05$ and $h = 0.4$.

	QBDQM $\Delta t = 0.05$, $h = 0.4$			Ali *et al*. (1993) $\Delta t = 0.05$, $h = 0.4$			Zaki (2000b) $\Delta t = 0.05$, $h = 0.2$		
t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0	50.00012	45.00045	42.30068	42.30068	50.00	42.301	42.301	45.00041	42.30065
100	50.00042	45.00046	42.30042	42.30042	50.00	42.257	42.257	45.00242	42.30354
200	49.99980	45.00047	42.29957	42.29957	50.01	42.110	42.110	45.00441	42.30647
300	50.00722	45.00049	42.29913	42.29913	50.01	42.041	42.041	45.00672	42.30942
400	50.00568	45.00047	42.29897	42.29897	50.00	42.033	42.033	45.00995	42.31197
500	50.00089	45.00046	42.29895	42.29895	49.99	42.038	42.038	45.01577	42.31489
600	49.98500	45.00037	42.29891	42.29891	49.98	42.049	42.049	45.01577	42.31489
700	49.96844	45.00045	42.29895	42.29895	49.99	42.057	42.057	45.02153	42.31489
800	49.95939	45.00053	42.29900	42.29900	50.02	42.064	42.064	45.02899	42.32111

Solution vector after a very long run time $t = 800$ with $\Delta t = 0.4$, $\nu = 0$, $\varepsilon = 0.2$, $\mu = 0.1$ and $N = 373$ has been shown in Figure 5. In this case Equation (1) is a KdV type equation and a train of 10 solitons have been formed. The invariants I_1, I_2 and I_3 are recorded and compared with Zaki (2000a) in Table 6 for the present case. It is obviously seen from Table 6 that by using less number of grid points the invariants change by less than 0.072%, 0.00027% and 0.013%, respectively, with respect to their original values during this very long run and therefore they can be considered almost constant.

We have utilized all the data as the same except that $h = 0.4$ to compare with Ali *et al*. (1993) and Zaki (2000b) in Table 7. The invariants change by less than 0.082%,

0.00018% and 0.0042%, respectively. So the quantities in the invariants remain almost constant during the computer run. It is clearly seen from Figure 6 that when viscosity is too small $(\upsilon = 0.0001)$ the solution of KdVB behaves similarly to a KdV solution ($\upsilon = 0$). In fact, the graphs given at Figure 6 are indistinguishable similar to those obtained for the KdV equation using the same parameters. Again, a train of 10 solitons have been obtained.

In Figure $7(b)$, the solution vector at time $t = 800$ with the same set of data of Figure $7(a)$ except that υ has been increased to the new value $\upsilon = 0.0001$ very small viscosity has been graphed. In fact, this graph is indistinguishable from that of Figure $7(a)$. Also a train of 10 solitons is formed.

We have used all the data as the same except that υ takes the increasing values 0, 0.0001, 0.001, 0.005, 0.01, 0.03, 0.05, 0.1 and 0.2 in order to study the effect of increasing the viscosity and hence the dispersion term on the solution vector. Figure (7 $a)-(i)$ represent the solution profiles for these cases at time $t = 800$, respectively. It is clear from these graphs that the more we increase the υ the solution vector for the KdVB Equation (1) tends to behave more like a solution of Burgers' equation $(\mu = 0)$. This fact can be seen clearly in Figure $7(i)$, where the solution vectors end up behaving like traveling waves for which the amplitudes are damped.

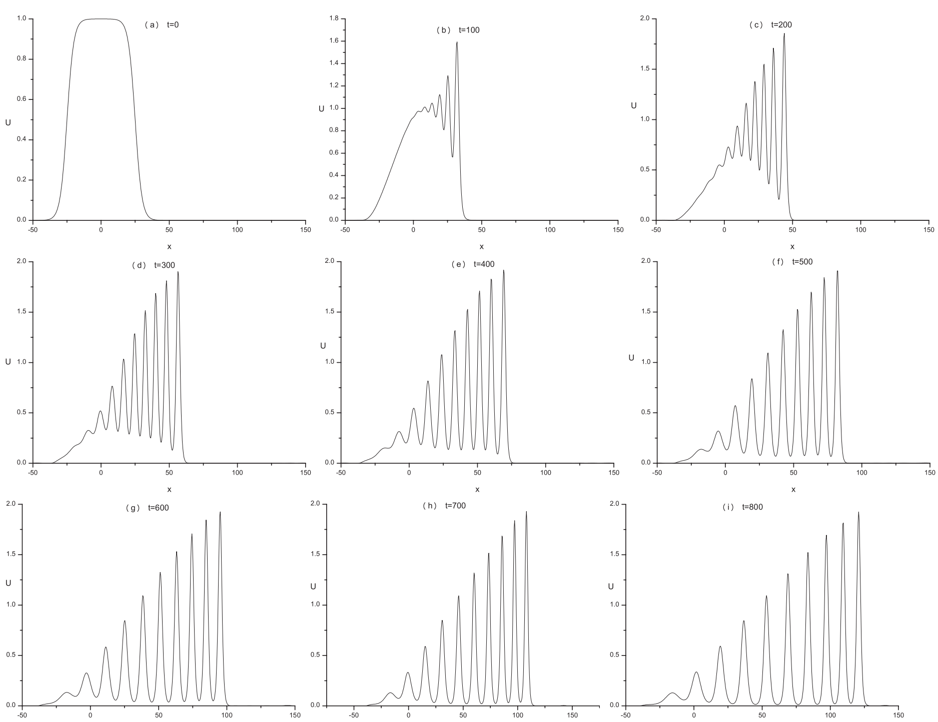

Fig. 6. KdVB type solutions taken at time from $t = 0$ to $t = 800$ with $\upsilon = 0.0001$, $\varepsilon = 0.2$, $\mu = 0.1$, $\Delta t = 0.05$ and $h = 0.4$.

Table 8. Maximum absolute value of eigenvalues at various number of grid points.

	QBDQM					
Grid Number	**11**	**21**	**31**	**41**	**61**	**81**
$\lvert R_e(\lambda_{\max})\rvert$	9.0 x 10^{-3}	2.5 x 10^{-2}	3.9 x 10^{-2}	5.5 x 10^{-2}	9.3 x 10^{-2}	1.3 x 10^{-1}
$\lvert Im(\lambda_{\max})\rvert$	3.0 x 10^{-4}	9.1 x 10^{-4}	2.1 x 10^{-3}	4.0 x 10^{-3}	1.6 x 10^{-2}	4.4 x 10^{-2}

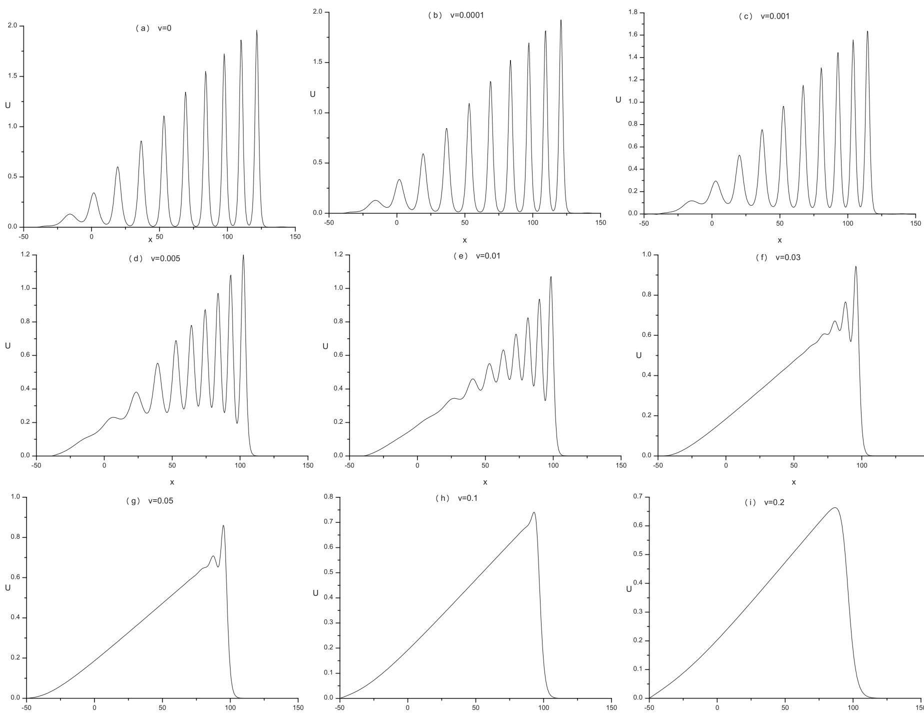

Fig. 7. KdVB type solutions taken at time $t = 800$, $\varepsilon = 0.2$, $\mu = 0.1$, $\Delta t = 0.05$ and $h = 0.4$ with different value of υ .

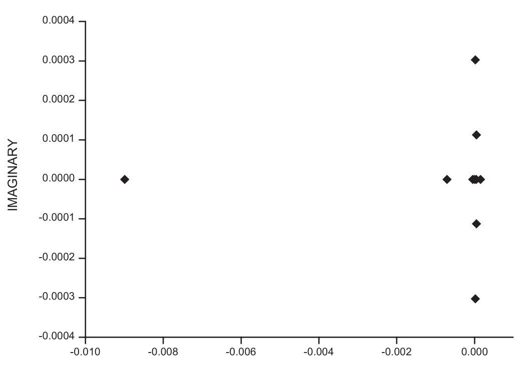

Fig. 8. Eigenvalues for $N = 11.$

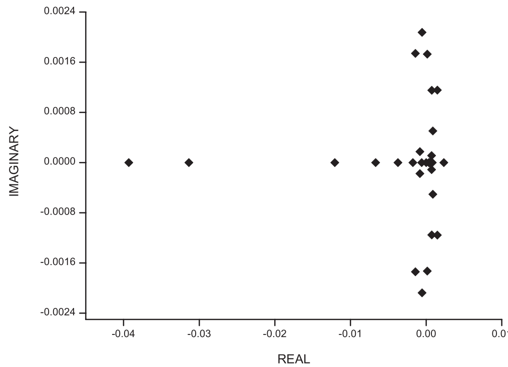

Fig. 9. Eigenvalues for $N = 31$.

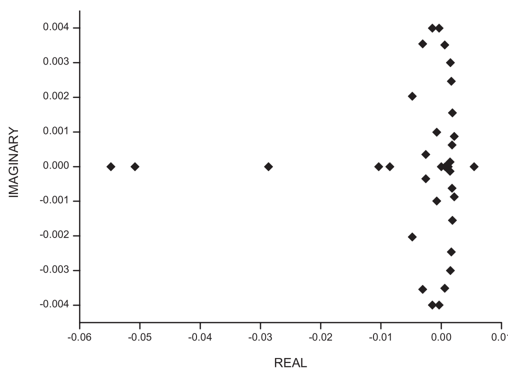

Fig. 10. Eigenvalues for $N = 41$.

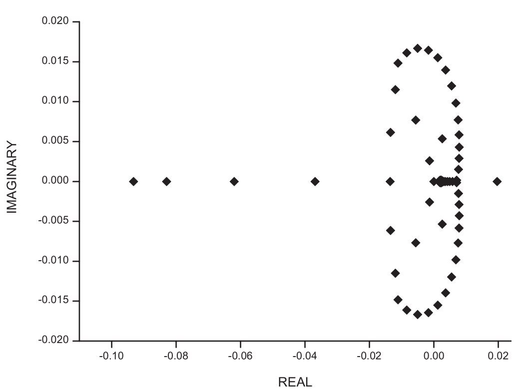

Fig. 11. Eigenvalues for $N = 61$.

A matrix stability analysis is also done for the QBDQM. We used the matlab program to obtain the eigenvalues of the coefficient matrix. Eigenvalues of suggested method for various number of nodals are shown in Figure 8-11. As the eigenvalues for $N = 1$, $N = 31$, $N = 41$ and $N = 61$ have imaginary parts. Furthermore, for $N = 1$, $N = 31$, $N = 41$ and $N = 61$, the maximum and the nonnegative real parts of eigenvalues determined as $4.8\times10^{-5}, 2.3\times10^{-3}, 5.5\times10^{-3}, 1.9\times10^{-2}$,respectively. Also, maximum absolute value of eigenvalues at various number of grid points tabulated in Table 8. All the eigenvalues are convenience with stability criteria (Jain, 1983).

CONCLUSION

In this study, we have constructed the quintic B-spline differential quadrature method to obtain numerical solution of the KdVB equation. The weighting coefficients of the derivative approximations are determined by solving linear algebraic systems, which included five-banded coefficients matrix. After the weighting coefficients are determined, KdVB equation is discretized in space by using the differential quadrature method approximations, so, the ordinary differential equation system is obtained. By using fourth-order Runge-Kutta method the ordinary differential equation system is integrated in time. To show the validity of the method and compare with earlier works we choose the appropriate test problem and observe the solutions under the different values of υ and μ. It is shown that our scheme is stable. When $\upsilon = 0$ the KdV equation has proved that the method is conservative through the recorded values of I_1, I_2 and I_3, as expected, all the results obtained using the KdVB equation with different values of υ and μ have indicated the physics of the problem. It has been concluded that the numerical solutions tend to behave like Burgers' equation when diffusion dominates whereas KdV type behavior has been obtained when dispersion dominates. Our scheme for KdV equation and Burgers' equation is more accurate than other earlier schemes in the literature. The numerical method has been shown for the long have assured us that the present method can be effectively used for long runs of the KdVB equation. The obtained numerical results show that the present method is a remarkably successful numerical technique for solving the KdVB equation and also useful for a wide range of applications, where continuity of derivatives is essential.

REFERENCES

Ali, A. H. A., Gardner, L. R. T. & Gardner, G. A. 1993. Numerical study of the KdVB equation using B-spline finite elements, J. Math.Phys. Sci. **27**: 37-53.

Ali, A. H. A., Gardner, G. A. Gardner & L. R. T. 1992. A collocation solution for Burgers' equation using cubic B-spline finite elements, Computer Methods in Applied Mechanics and Engineering, **100**: 325-337.

Arora, G. & Singh, B. K. 2013. Numerical solution of Burgers equation with modified cubic B-spline differential quadrature method, Appl. Math. Comput., **224**: 166–177.

Bellman, R., Kashef, B. & Casti, J. 1972. Differential quadrature: a tecnique for the rapid solution of nonlinear differential equations, Journal of Computational Physics, **10**: 40-52.

Bellman, R., Kashef, B., Lee, E. S. & Vasudevan, R. 1976. Differential Quadrature and Splines, Computers and Mathematics with Applications, Pergamon, Oxford, **1**: 371-376.

Bonzani, I. 1997. Solution of non-linear evolution problems by parallelized collocation-interpolation methods, Computers & Mathematics and Applications, **34**: 71-79.

Canosa, J. & Gazdag, J. 1977. The Korteweg–de Vries–Burgers' equation, J. Comp. Phys. **23**: 393-403.

Cheng, J., Wang, B. & Du, S. 2005. A theoretical analysis of piezoelectric/composite laminate with larger-amplitude deflection effect, Part II: hermite differential quadrature method and application, International Journal of Solids and Structures, 42: 6181-6201.

Civalek, O. 2004. Application of differential quadrature (DQ) and harmonic differential quadrature (HDQ) for buckling analysis of thin isotropic plates and elastic columns, An International Journal of Engineering Structures, **26**: 171-186.

Civalek, O. 2006. Harmonic differential quadrature-finite differences coupled approaches for geometrically nonlinear static and dynamic analysis of rectangular plates on elastic foundation, Journal of Sound and Vibration , **294**: 966-980.

Demiray, H. 2004. A travelling wave solution to the KdV–Burgers equation, Appl. Math. Comput. **154**: 665-670.

Dağ, I. & Dereli, Y. 2008. Numerical solutions of KdV equation using radial basis functions, Applied Mathematical Modelling, **32**: 535-546.

Gardner, L. R. T., Gardner, G. A. & Ali, A. H. A. 1991. Simulations of solitons using quadratic spline finite elements, Comput. Methods Appl. Mech. Engrg. **92**: 231.

Grad, H. & Hu, P. N. 1967. Unified shock profile in a plasma, Phys. Fluids, **10**: 2596-2601.

Guo, Q. & Zhong, H. 2004. Non-linear vibration analysis of beams by a spline-based differential quadrature method, Journal of Sound and Vibration , **269**: 413-420.

Helal, M. A. & Mehanna, M. S. 2006. A comparison between two different methods for solving KdV–Burgers equation, Chaos Soliton. Fract., **28**: 320-326.

Jain, M. K. 1983. Numerical Solution of Differential Equations, 2nd ed.,Wiley, New York, NY.

Johnson, R. S. 1970. A non-linear equation incorporating damping and dispersion, J. Fluid Mech. **42**: 49-60.

Johnson, R. S. 1972. Shallow water waves in a viscous fluid, the undular bore, Phys. Fluids **15**: 1958-1988.

Kaya, D. 1999. On the solution of a Korteweg–de Vries like equation by the decomposition method, Int. J. Comput. Math., **72**: 531-539.

Kaya, D. 2004. An application of the decomposition method for the KdVB equation, Appl. Math. Comput., **152**: 279-288.

Korkmaz, A. (2010a). Numerical algorithms for solutions of Korteweg-de Vries Equation, Numerical methods for partial differential equations, **26**: 1504-1521. Korkmaz, A. & Dağ, I. 2009. Solitary wave simulations of complex modified Korteweg-de Vries equation using differential quadrature method, omputers Physics Communications, **180**: 1516-1523.

Korkmaz, A. & Dağ, I. 2010b. Numerical solutions of some one dimensional partial differential equations using B-spline differential quadrature method, Doctoral Dissertation, Eskisehir Osmangazi University.

Korkmaz, A. & DağI. 2011a. Shock wave simulations using Sinc Differential Quadrature Method, International Journal for Computer-Aided Engineering and Software, **28**: 654-674.

Korkmaz, A. & Dağ, I. 2011b. Polynomial based differential quadrature method for numerical solution of nonlinear Burgers' equation, Journal of the Franklin Institute, **348**: 2863-2875.

Korkmaz, A. & Dağ, I. 2012. Cubic B-spline differential quadrature methods for the advection-diffusion equation, International Journal of Numerical Methods for Heat & Fluid Flow, **22**: 1021-1036.

Korkmaz, A. & Dağ, I. 2013a. Cubic B-spline differential quadrature methods and stability for Burgers' equation, International Journal for Computer-Aided Engineering and Software, **30**: 320-344.

Korkmaz, A. & Dağ, I. 2013b. Numerical Simulations of Boundary-Forced RLW Equation with Cubic B-spline based differential quadrature methods , Arab. J. Sci. Eng., **38**: 1151-1160.

Lee, T. S., Hu, G. S. & Shu, C. 2004. Application of GDQ method for study of mixed convection in horizontal eccentric annuli, International Journal of Computational Fluid Dynamics, **18**: 71-79.

Mittal, R. C. & Jiwari, R. 2009. Differential quadrature method for two-dimensional Burgers' equations, International Journal for Computational Methods in Engineering Science and Mechanics, **10**: 450-459.

Mittal, R. C. & Jiwari, R. 2011. Numerical solution of two-dimensional reaction–diffusion Brusselator system, Appl. Math. Comput. **217**: 5404-5415.

Mittal, R. C. & Jiwari, R. 2012. A differential quadrature method for numerical solutions of Burgers-type equations, International Journal of Numerical Methods for Heat & Fluid Flow, **22**: 880-895.

Sahu, B. & Roychoudhury, R. 2003. Travelling wave solution of Korteweg-de Vries–Burger's equation, Czech. J. Phys. **53**: 517-527.

Saka, B., Dağ, I. Dereli, Y. & Korkmaz, A. 2008. Three different methods for numerical solutions of the EW equation, Engineering Analysis with Boundary Elements, **32**: 556-566.

Saka, B. & Dağ, I. 2009. Quartic B-spline Galerkin approach to the numerical solution of the KdVB equation, Appl. Math. Comput. **215**: 746-758.

Saka, B. & Dağ, I. 2007. Quartic B-spline collocation method to the numerical solutions of the Burgers equation, Chaos Soliton. Fract., **32**: 1125-1137.

Shu, C. & Xue, H. 1997. Explicit computation of weighting coefficients in the harmonic differential quadrature, Journal of Sound and Vibration, **204**: 549-555.

Shu, C. & Richards, B. E. 1992. Application of generalized differential quadrature to solve two dimensional incompressible Navier-Stokes equations, Int. J. Numer. Meth. Fluids, **15**: 791-798.

Shu, C. & Wu, Y. L. 2007. Integrated radial basis functions-based differential quadrature method and its performance, Int. J. Numer. Meth. Fluids, 53: 969-984. Soliman, A. A. 2004. Collocation solution of the Korteweg-de Vries Equation using septic splines, Int. J. Comput. Math., **81**: 325-331.

Striz, A. G., Wang, X. & Bert, C. W. 1995. Harmonic differential quadrature method and applications to analysis of structural components, Acta Mechanica,**111**: 85-94.

Su, C. H. & Gardner, C. S. 1969. Derivation of the Korteweg-de Vries and Burgers equation, J. Math. Phys., Vol. **10**: 536-539.

Quan, J. R. & Chang, C. T. 1989a. New sightings in involving distributed system equations by the quadrature methods-I, Comput. Chem. Eng., 13: 779-788. Quan, J. R. & Chang, C. T. 1989b. New sightings in involving distributed system equations by the quadrature methods-II, Comput. Chem. Eng., 13: 717-724. Tomasiello, S. 2010. Numerical solutions of the Burger-Huxley equation by the IDQ method, Int. J. Comput. Math., 87: 129-140.

Zaki, S. I. 2000a. A quintic B-spline finite elements scheme for the KdVB equation, Comput. Meth. Appl. Mech. Engrg. 188: 121-134.

Zaki, S. I. 2000b. Solitary waves of the Korteweg-de Vries–Burgers' equation, Comput. Phys. Commun.,**126**: 207-218.

Zhong, H. 2004. Spline-based differential quadrature for fourth order equations and its application to Kirchhoff plates, Applied Mathematical Modelling , **28**: 353-366.

Zhong, H. & Lan, M. 2006. Solution of nonlinear initial-value problems by the spline-based differential quadrature method, Journal of Sound and Vibration, **296**: 908-918.

Zhu, Y. D., Shu, C., Qiu, J. & Tani, J. 2004. Numerical simulation of natural convection between two elliptical cylinders using DQ method, International Journal of Heat and Mass Transfer, **47**: 797-808.

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تقريب لمعادلة KdVB بواسطة طريقة طريقة الشريحة الخماسية التفاضلية المكتملة

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خلاصة

نقوم في هذا البحث بحل معادلة برغر عددياً بواسطة طريقة تفاضلية مكتملة جديدة تستند إلى دوال الشريحة الخماسية. المعاملات الوازنة يمكن الحصول عليها بواسطة خوارزمية شبه صريحة تشتمل على نظام جبري له له مصفوفة خرام خماسي. حسبنا معياري الخطأ L_2 و L_∞ وكذلك أصغر ثلاثة لا متغيرات وذلك لمقارنتها بنتائج دراسات سابقة. كما نعطي كذلك تحليل الاستقرار لطريقتنا الجديدة. وتبين من المقارنة أن أداء طريقتنا هو أفضل من أداء معظم الطرق المعروفة.