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On certain convergence of S-iteration scheme in CAT(0) spaces

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ABSTRACT

The aim of this paper is to study convergence behaviour of S-iteration scheme in CAT(0) spaces for generalized nonexpansive mappings. In process, several relevant results of the existing literature are generalized and improved.

Keywords and phrases: CAT(0) spaces; Condition (C); Δ-convergence; fixed point and Opial's property.

INTRODUCTION

A self-mapping T defined on a bounded, closed and convex subset K of a Banach space X is said to be nonexpansive if (for all $x, y \in K$)

$$\|Tx - Ty\| \leq \|x - y\|.$$

It is well known that sequence of Picard (1890) iteration defined as (for any $x_1 \in K$)

$$x_{n+1} = T^n x, \quad n \in \mathbb{N} \tag{1.1}$$

need not be convergent in respect of a nonexpansive mapping, e.g., the sequence of iterates $x_{n+1} = Tx_n$ for the mapping $T : [-1,1] \to [-1,1]$ defined by $Tx = -x$ does not converges to 0 for any nonzero $x \in [-1,1]$ which is indeed the fixed point of T. In an attempt to construct a convergent sequence of iterates in respect of a nonexpansive mapping, Mann (1953) defined an iteration method as: (for any $x_1 \in K$)

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \in \mathbb{N} \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$.

In 1974, with a view to approximate fixed point of pseudo-contractive compact mappings in Hilbert spaces Ishikawa (1974) introduced a new iteration procedure as follows: (for $x_1 \in K$)

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \ n \in \mathbb{N} \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.

For a comparison of the two iterative schemes in the one-dimensional case, we refer the reader to Rhoades (1976), wherein it is shown that under suitable conditions (see part-(a) of Theorem 3) rate of convergence of Ishikawa iteration is better than that of Mann iteration procedure. Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been investigated by various authors (see; e.g., Ishikawa (1974, 1976); Riech (1979); Tan & Xu (1993) and Razani & Salahifard (2011)) using the Mann iteration scheme or the Ishikawa iteration scheme. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an upto date account of literature on this theme, we refer the readers to Berinde (2007). For the different aspect of fixed point theory one can consult Badshah & Farkhunda (2000); Abd-Rabou (2012); Uddin *et al*. (2014) and Uddin & Imdad (2015).

Most recently, Agarwal *et al*. (2007) introduced the following iterative process and give the name as S-iteration process: for any $x_1 \in K$

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)Tx_n + \beta_n T y_n, \ n \in \mathbb{N} \end{cases} \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.

The S-iteration process given by Agarwal *et al*. (2007) is independent of the Mann and Ishikawa iteration processes. Agarwal *et al*. (2007) showed that (1.4) converges at a rate same as that of Picard iteration and faster than Mann iteration for contractions and it is not hard to see on similar lines that S-iteration scheme (1.4) also converges faster than the Ishikawa iteration scheme.

On the other hand, in 2008, Suzuki (2008) introduced a new class of mappings, which is larger than the class of nonexpansive mappings and name the defining condition as Condition (C) (sometimes also referred as generalized nonexpansive mapping) which runs follows:

A mapping T defined on a subset K of a Banach space X is said to satisfy

Condition (C) if (for all $x, y \in K$)

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \Rightarrow \|Tx - Ty\| \le \|x - y\|.$$

Using Condition (C), Suzuki (2008) proved the following fixed point theorem:

Theorem 1.1 (Suzuki, 2008) Let T be a mapping defined on a convex subset K of a Banach space X which enjoys Condition(C). Also, assume that either of the following holds:

1. K is compact or
2. K is weakly compact and X has the Opial property.

Then T has a fixed point in K.

The purpose of this paper is to study the convergence of S-iteration scheme (1.4) for generalized nonexpansive mapping in CAT(0) spaces which enable us to enlarge the classes of mappings as well as classes of spaces.

PRELIMINARIES

To make our presentation self contained, we collect relevant definitions and relevant results. In a metric space (X,d), a geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0,r] \subset R$ to X such that $c(0)=x, c(r)=y$ and $d(c(t),c(s)) = |s-t|$ for all $s,t \in [0,r]$. In particular, the mapping c is an isometry and $d(x,y)=r$. The image of c is called a geodesic segment joining x and y which is denoted by $[x,y]$, whenever such a segment exists uniquely. For any $x, y \in X$, we denote the point $z \in [x,y]$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \le \alpha \le 1$ if $d(x,z) = \alpha d(x,y)$ and $d(z,y) = (1-\alpha)d(x,y)$. The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of X is called convex if C contains every geodesic segment joining any two points in C.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points of X (as the vertices of Δ) and a geodesic segment between each pair of points (as the edges of Δ). A comparison triangle for $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$. A point $\overline{x} \in [\overline{x}_1, \overline{x}_2]$ is said to be comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x}_1, \overline{x})$. Comparison points on $[\overline{x}_2, \overline{x}_3]$ and $[\overline{x}_3, \overline{x}_1]$ are defined in the same way.

A geodesic metric space *X* is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom namely: CAT(0) inequality

Let Δ be a geodesic triangle in *X* and let $\overline{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

If x, y_1 and y_2 are points of CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

The above inequality is known as (CN) inequality and was given by Bruhat & Tits (1972). A geodesic space is a CAT(0) space if and only if it satisfies (CN) inequality.

Towards certain classes of examples, one may recall that every convex subset of Euclidean space $\mathbb{R}^n$ endowed with the induced metric is a CAT(0) space. Also, the class of Hilbert spaces are examples of CAT(0) space. Moreover, if any real normed space X is CAT(0) space, then it is a pre-Hilbert space. Furthermore, if X_1 and X_2 are CAT(0) spaces, then so is $X_1 \times X_2$. For further details on CAT(0) spaces, one can consult Bruhat & Tits (1972), Bridson & Haefliger (1999); Brown (1989) and Burago *et al*. (2001).

Now, we collect some basic geometric properties, which are instrumental throughout the discussions. Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, set:

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\},$$

and the asymptotic center $A((x_n))$ of (x_n) is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\})\}.$$

It is well known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see Proposition 5 of Dhompongsa *et al*. (2006)).

In 2008, Kirk & Panyanak (2008) gave a concept of convergence in CAT(0) spaces which is analogue of weak convergence in Banach spaces and restriction of Lim's concept of convergence Lim (1976) to CAT(0) spaces.

Definition 2.1 (Kirk & Panyanak, 2008) A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is the Δ-limit of $\{x_n\}$.

Notice that for a given $\{x_n\} \subset X$ which Δ-converges to x and for any $y \in X$ with $y \neq x$ (owing to uniqueness of asymptotic center), we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property. Now, we collect some basic facts about CAT(0) spaces which will be frequently used throughout the text.

Lemma 2.1 (Kirk & Panyanak, 2008) Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.

Lemma 2.2 (Dhompongsa *et al.*, 2007) If C is closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Lemma 2.3 (Dhompongsa & Panyanak, 2008) Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0,1]$, there exists a unique $z \in [x, y]$ such that

$$d(x,z) = td(x,y) \quad and \quad d(y,z) = (1-t)d(x,y).$$

Notice that we use the notation $(1-t)x \oplus ty$ for the unique point z of the above lemma.

Lemma 2.4 (Dhompongsa & Panyanak, 2008) For $x, y, z \in X$ and $t \in [0,1]$ we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 2.5 (Dhompongsa & Panyanak, 2008) Let X be a CAT (0) space. Then

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Next, we state two results for generalized nonexpansive mapping in the setting of CAT(0) space which are very useful to prove our main results.

Theorem 2.1 (Nanjaras *et al.*, 2010) Let C be a nonempty bounded, closed and convex subset of a complete CAT(0) space X. If $T : C \to C$ is a generalized nonexpansive mapping, then T has a fixed point in C. Moreover, $F(T)$ is closed and convex.

Lemma 2.6 Let C be a subset of a CAT(0) space X and $T : C \to C$ be a generalized nonexpansive mapping. Then for all $x, y \in C$ the following holds:

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y).$$

Now, we adopt the iteration procedure of Agarwal *et al*. (2007) in the framework of CAT(0) spaces wherein (for any $x_1 \in K$)

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)Tx_n \oplus \beta_n T y_n, \ n \in \mathbb{N} \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\} \in (0,1)$.

In this paper, we prove some convergence theorems for Iteration Process (1.5) for generalized nonexpansive mapping in CAT(0) spaces. Our results generalize and extend the corresponding relevant results in Agarwal *et al*. (2007) and Khan & Abbas (2011).

RESULTS

Lemma 3.1 Let C be a nonempty bounded closed convex subset of complete CAT(0) space X. Let $T: C \to C$ be a generalized nonexpansive mapping and let $\{x_n\}$ be an iteration process described by (1.5). If $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0 < a \le \alpha_n, \beta_n \le b < 1$ for some $a, b \in (0,1)$, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

Proof. By Theorem 2.1, $F(T) \neq \varnothing$ so that there exists a $p \in F(T)$.

$$\frac{1}{2} d(p, Tp) = 0 \le d(p, x_n),$$

which due to Condition (C) gives rise $d(Tx_n, Tp) \le d(x_n, p)$. Similarly, using Condition (C), we have $d(Ty_n, Tp) \le d(y_n, p)$. Now, using Lemma 2.4, we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Tx_n \oplus \alpha_n T y_n, p) \\ &\le (1-\alpha_n) d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n d(y_n, p), \end{aligned} \tag{3.1}$$

while

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T x_n, p) \\ &\le (1-\beta_n) d(x_n, p) + \beta_n d(Tx_n, p) \\ &\le (1-\beta_n) d(x_n, p) + \beta_n d(x_n, p) = d(x_n, p), \end{aligned} \tag{3.2}$$

so that in view of Equations (3.1) and (3.2), we have

$$d(x_{n+1}, p) \leq d(x_n, p) \tag{3.3}$$

which shows that $\{d(x_n, p)\}$ is a decreasing sequence of non-negative reals. Thus in all, sequence $\{d(x_n, p)\}$ is bounded below and decreasing, therefore remains convergent.

Lemma 3.2 Let C be a nonempty bounded, closed and convex subset of complete CAT(0) space X. Let $T : C \to C$ be a generalized nonexpansive mapping and let $\{x_n\}$ be an iteration process described by (1.5). If $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in (0,1)$, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = c.$ By Equation (3.1), we have

$$d(x_{n+1}, p) \leq (1 - \alpha_n) d(x_n, p) + \alpha_n d(y_n, p)$$

or

$$\alpha_n d(x_n, p) \leq d(x_n, p) + \alpha_n d(y_n, p) - d(x_{n+1}, p)$$

or

$$\begin{aligned} d(x_n, p) &\leq d(y_n, p) + \frac{1}{\alpha_n}\{d(x_n, p) - d(x_{n+1}, p)\} \\ &\leq d(y_n, p) + \frac{1}{a}\{d(x_n, p) - d(x_{n+1}, p)\}. \end{aligned}$$

On taking liminf of both the sides, we have

$$\liminf_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(y_n, p) + \frac{1}{a} \liminf_{n\to\infty} \{d(x_n, p) - d(x_{n+1}, p)\}$$

so that,

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.4}$$

By Equation (3.2), we have

$$\limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c. \tag{3.5}$$

Owing to Equations (3.4) and (3.5), we get

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{3.6}$$

Now, by Lemma 2.5,

$$\begin{aligned} d(y_n,p)^2 &= d((1-\beta_n)x_n \oplus \beta_n Tx_n,p)^2 \\ &\leq (1-\beta_n)d(x_n,p)^2+\beta_n d(Tx_n,p)^2-\beta_n(1-\beta_n)d(x_n,Tx_n)^2 \\ &\leq d(x_n,p)^2-\beta_n(1-\beta_n)d(x_n,Tx_n)^2 \end{aligned}$$

so that

$$\beta_n(1-\beta_n)d(x_n,Tx_n)^2 \leq d(x_n,p)^2-d(y_n,p)^2.$$

Also, we have

$$\begin{aligned} d(x_n,Tx_n)^2 &\leq \frac{1}{\beta_n(1-\beta_n)}[d(x_n,p)^2-d(y_n,p)^2] \\ &\leq \frac{1}{a(1-b)}[d(x_n,p)^2-d(y_n,p)^2]. \end{aligned}$$

By taking $\limsup_{n\to\infty}$ of both sides and using Equations (3.3) and (3.5), we infer that

$$\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

Now, we are equipped to state and prove our main result:

Theorem 3.1 Let C be a nonempty bounded, closed and convex subset of a complete CAT(0) space X. Let $T:C\to C$ be a generalized nonexpansive mapping and let $\{x_n\}$ be an iteration process described by (1.5). If $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0<a\leq\alpha_n,\beta_n\leq b<1$, then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. By Lemma 3.2, we observe that $\{x_n\}$ is a bounded sequence with

$$\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

Let $W_\omega(\{x_n\})=:\cup A(\{u_n\})$, where union is taken over all subsequence $\{u_n\}$ over $\{x_n\}$. In order to show that the Δ-convergence of $\{x_n\}$ to a fixed point of T, firstly we will prove $W_\omega(\{x_n\})\subset F(T)$ and thereafter argue that $W_\omega(\{x_n\})$ is a singleton set. To show $W_\omega(\{x_n\})\subset F(T)$, let $y\in W_\omega(\{x_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A(\{y_n\})=y$. By Lemmas 2.1 and 2.2, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta-\lim_n z_n=z$ and $z\in C$. Since $\lim_{n\to\infty} d(z_n,Tz_n)=0$ and T satisfies Condition (C), therefore by Lemma 2.6, we have

$$d(z_n, Tz) \leq 3d(z_n, Tz_n) + d(z_n, z).$$

By taking $\limsup$ of both the side, we have

$$\limsup_{n\to\infty} d(z_n, Tz) \leq \limsup_{n\to\infty}\{3d(z_n, Tz_n) + d(z_n, z)\}$$
$$\leq \limsup_{n\to\infty} d(z_n, z).$$

As $\Delta - \lim_n z_n = z$, by Opial property, we have

$$\limsup_{n\to\infty} d(z_n, z) \leq \limsup_{n\to\infty} d(z_n, Tz).$$

Hence $Tz = z$, i.e. $z \in F(T)$. Now, we assert that $z = y$. If not, by Lemma 3.1, $\lim_n d(x_n, z)$ exists and owing to the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, y) \\
&\leq \limsup_{n\to\infty} d(y_n, y) \\
&< \limsup_{n\to\infty} d(y_n, z) \\
&= \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z),
\end{aligned}$$

which is a contradiction so that $y = z$. To show that $W_\omega(\{(x_n\})$ is a singleton, let $\{y_n\}$ be a subsequence of $\{x_n\}$. In view of Lemmas 2.1 and 2.2, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim z_n = z$. Let $A(\{y_n\}) = y$ and $A(\{x_n\}) = x$. Earlier, we have shown that $y = z$, therefore it is enough to show $z = x$. If $z \neq x$, by Lemma 3.1 $\{d(x_n, z)\}$ is convergent. By uniqueness of asymptotic centers

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z)
\end{aligned}$$

which is a contradiction so that the conclusion follows.

Theorem 3.2 Let C be a nonempty bounded, closed and convex subset of complete CAT(0) space X. Let $T : C \to C$ be a generalized nonexpansive mapping and let $\{x_n\}$ be an iteration process described by (1.5). If $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0 < a \le \alpha_n, \beta_n \le b < 1$ for some $a, b \in (0,1)$, then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(z, F(T)) = \inf\{d(z,p) : p \in F(T)\}$.

Proof. If $\{x_n\}$ converges to a fixed point p of T, then

$$\liminf_{n\to\infty} d(x_n, p) = 0$$

so that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, let $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ In view of Equation (3.3) for all $p \in F(T)$, we have

$$d(x_{n+1}, p) \le d(x_n, p),$$

so that

$$\inf_{p\in F(T)} d(x_{n+1}, p) \le \inf_{p\in F(T)} d(x_n, \mathrm{p}),$$

which amounts to say that

$$d(x_{n+1}, F(T)) \le d(x_n, F(T))$$

and hence $\lim_{n\to\infty} d(x_n, F(T))$ exists so that by assumption we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, for any $\varepsilon > 0$, there exists a positive integer k such that (for all $n \ge k$)

$$d(x_n, F(T)) < \frac{\varepsilon}{4},$$

or

$$\inf\{d(x_k, p) : p \in F(T)\} < \frac{\varepsilon}{4}$$

so that there exists a $p \in F(T)$ such that

$$d(x_k, F(T)) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \geq k$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, p) + d(p, x_n) \\ &\leq 2d(x_k, p) \\ &< 2(\frac{\varepsilon}{2}) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C and converges to some x in C. As $\lim_{n\to\infty} d(x_n, F(T) = 0$ which implies that $d(x, F(T) = 0$ In view of Theorem 2.1, $F(T)$ is closed so that $x \in F(T)$.

Senter & Dotson (1974) introduced the Condition (A) as follows: A mapping $T: C \to C$ is said to satisfy the Condition (A) if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, Tx) = f(d(x, F(T)))$ for all $x \in C$.

Theorem 3.3 Let C be a nonempty bounded, closed and convex subset of complete CAT(0) space X. Let $T: C \to C$ be a generalized nonexpansive mapping and let $\{x_n\}$ be an iteration process described by (1.5). Moreover, if T satisfies Condition (A), $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in (0,1)$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and let us take to be c. If $c = 0$, then there is nothing to prove. If $c > 0$, then as argued in Theorem 3.2, $\lim_{n\to\infty} d(x_n, F(T)$ exists. Owing to Condition (A) there exists a nondecreasing function f such that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

so that $\lim_{n\to\infty} f(d(x_n, F(T)))$ =0. Since, f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F(T)$ =0. Now, in view of Theorem 3.2, we are through.

CONCLUSION

In this paper we proved some Δ-convergence as well as strong convergence theorems in CAT(0) spaces for generalized nonexpansive mappings. Our results generalized the results of Agarwal *et al*. (2007) and Khan & Abbas (2011).

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REFERENCES

Abd-Rabou, Khaled 2012. Common fixed point theorems for six mappings with some weaker conditions in 2-metric spaces. Kuwait Journal of Science & Engineering, **39(1A)**: 99–112.

Agarwal, R. P., O' Regan, D & Sahu, D. R. 2007. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, Journal of Nonlinear and Convex Analysis, **8**: 61–79.

Badshah, V. H. & Farkhunda, S 2000. Random fixed points of random multivalued operators on Polish spaces. Kuwait Journal of Science & Engineering, **27(2)**: 201–208.

Berinde, V. 2007. Iterative Approximation of fixed points, in: Lecture Notes in Mathematics, vol. 1912, Springer, Berlin.

Bridson, M. & Haefliger, A. 1999. Metric spaces of nonpositive curvature, Springer-Verlag, Berlin.

Brown, K. S. 1989. Buildings, Springer-Verlag, New York.

Burago, D., Burago, Y. & Ivanov, S. 2001. A course in metric geometry, in: Graduate Studies in Math., vol. 33: American Mathematical Society Providence, RI.

Bruhat, F. & Tits, J. 1972. Groupes reductifs sur un corps local. I. Donnees radicielles valuees Inst Hautes Etudes Sci. Publ Math. 41.

Dhompongsa, S., Kirk, W. A. & Sims, B. 2006. Fixed points of uniformly lipschitzian mappings, Nonlinear Analysis: Theory, Methods & Applications, **65**: 762-772.

Dhompongsa, S., Kirk, W. A. & Panyanak, B. 2007. Nonexpansive set-valued mappings in metric and Banach spaces. Journal and Nonlinear Convex Analysis, 8: 35–45.

Dhompongsa, S. & Panyanak, B. 2008. On پ Δ-convergence theorems in CAT(0) space, Computers & Mathematics with Applications, **56**: 2572–2579.

Ishikawa, S. 1974. Fixed points by a new iteration method. Proceeding of American Mathematical Society, **44**: 147–150.

Ishikawa, S. 1976. Fixed point and iteration of a nonexpansive mapping in a Banach space. Proceeding of American Mathematical Society, **59**: 65–71.

Kirk, W. A. & Panyanak, B. 2008. A concept of convergence in geodesic spaces. Nonlinear Analysis: Theory, Methods & Applications, **68**: 3689–3696.

Khan, S. H. & Abbas, M. 2011. Strong and Δ-convergence of some iterative schemes in CAT(0) spaces. Computers & Mathematics with Applications, **61**: 109–116.

Lim, T. C. 1976. Remarks on some fixed point theorems. Proceeding of American Mathematical Society, **60**: 179–182.

Mann, W. R. 1953. Mean value methods in iterations. Proceeding of American Mathematical Society, **4**: 506–510.

Nanjaras, B., Panyanaka, B. & Phuengrattana, W. 2010. Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces, Nonlinear Analysis: Hybrid Systems **4**: 25–31.

Picard, E. 1890. Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. Journal des Mathématiques Pures et Appliquées, **6**: 145–210.

Razani, A. & Salahifrd, H. 2011. Approximating fixed points of generalized non-expansive mappings, The Bulletin of the Iranian Mathematical Society, **37**: 235–246.

Reich, S. 1979. Weak convergence theorems for nonexpansive mappings in Banach spaces, Journal of Mathematical Analysis and Applications, **67**: 274–276.

Rhoades, B. E. 1976. Comments on two fixed point iteration methods. Journal of Mathematical Analysis and Applications, **56**: 741–750.

Senter, H. F. & Dotson, W. G. 1974. Approximating fixed points of nonexpansive mappings, Proceeding of American Mathematical Society, **44**: 375–380.

Suzuki, T. 2008. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. Journal of Mathematical Analysis and Applications, 340: 1088–1095.

Tan, K. K. & Xu, H. K. 1993. Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, Journal of Mathematical Analysis and Applications, **178**: 301–308.

Uddin, Izhar, Abdou, Afrah A. N. & Imdad, Mohammad 2014. A new iteration scheme for a hybrid pair of generalized nonexpansive mappings, Fixed Point theory and Applications, **2014:205**, 13 pp.

Uddin, Izhar & Imdad, Mohammad 2015. Some convergence theorems for a hybrid pair of generalized nonexpansive mappings in CAT(0) spaces, Journal of Nonlinear and Convex Analysis. **Vol. 16** (3), 447–457.

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حول تقارب معين لمخطط تكراري في فضاءات (0) CAT

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خلاصة

قام أغاروول (2007) مؤخراً بإدخال عملية تكرار جديد مستقلة عن التركرارين التقليديين: مخطط تكرار مان ومخطط تكرار إيشيكاوا. قام أغاروول باستخدام طريقتهم لإثبات مبرهنة تقارب لتطبيقات غير ناشرة في فضاءات بناخ. الهدف من هذا البحث هو مد هذه المبرهنة إلى فضاءات (ATCO) للتطبيقات المعممة وغير الناشرة. ونقوم من خلال ذلك بتعميم وتحسين نتائج أغاروول (2007) نتائج خان وعباس (2011).