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Generalization of Hilbert-Hardy integral inequalities

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Abstract

In this paper, Hilbert's integral inequalities with some parameters are considered, by using new methods in the proof. Several results of Hardy and Yang are special cases of the new given inequality. As an application, we give some applied examples that illustrate our results.

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1. Introduction

First, let us recall the well-known standard Hilbert's integral inequality and Hardy-Hilbert's integral inequality as follows:(Hardy *et al*., 1964).

Let f and g are real functions, such that

$$0 < \int_0^\infty f^2(x)dx < \infty, 0 < \int_0^\infty g^2(y)\,dy < \infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \pi\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(y)\,dy\right)^{\frac{1}{2}} \quad (1)$$

where the constant factor π is the best possible.

The inequality (1) is extended by Hardy *et al*., 1964 as follows:

Let $p > 1,\ q > 1, \frac{1}{p} + \frac{1}{q} = 1,\ f, g > 0$ satisfy

$$0 < \int_0^\infty f^p(x)dx < \infty, 0 < \int_0^\infty g^q(y)\,dy < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \quad (2)$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible . The inequality (2) is well known as Hardy-Hilbert's integral inequality. Hilbert's inequalities (1) and (2) are important in mathematical analysis and its applications. In the past 100 years, a large number of mathematicians have investigated the subject of Hilbert's inequalities as well as Hilbert-type inequalities in a very broad context and proved a variety of several inequalities (El-Marouf & AL-Oufi, 2012; El-Marouf, 2013; Marouf, 2014; El-Marouf, 2015; Mingzhe, 1997; Salem, 2006; Peachey, 2003 and Weijian & Mingzhe 2006). An excellent account related to the above inequalities and many important applications in analysis can be found in Bashan *et al*. 2015 and Garg & Chanchlani, 2013.

In this paper, we introduce some new integral inequalities of Hilbert's type or similar to Hilbert inequality, which contains some additional parameters . Our results are generalization of each of the results of **H**ardy *et al*. 1964; **Y**ang, 2000 and **Y**ang, 2004. As an application, we give some examples on some partial differential equations that illustrate our results.

2. Lemmas and main results

First, we introduce the following two lemmas .

Lemma 2.1. For $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, and $\propto \ge 0$ define the weight function $w_1(x)$ as

$$w_1(x) = \int_0^\infty \frac{1}{\left(x^a y^b + x^c y^d\right)^\propto}\left(\frac{x}{y}\right)^{\frac{1}{q}} dy\ , \quad (3)$$

where $a > c,\ d > b$ and $a, c, b, d \ge 0$.

Then we get

$$w_1(x) = \frac{1}{d-b}x^{\frac{a-c}{d-b}\left(\frac{1}{p}-\propto b\right)+1-\frac{1}{p}-\propto a} B\left(\frac{\frac{1}{p}-b\propto}{d-b}, \propto - \left(\frac{\frac{1}{p}-b\propto}{d-b}\right)\right), \quad (4)$$

where $B(a,b)$ is the well known Beta function, $a > 0,\ b > 0$.

Proof. From the weight function (3), we have

$$w_1(x)=\int_0^\infty \frac{1}{(x^a y^b)^\propto (1+x^{c-a}y^{d-b})^\propto}\left(\frac{x}{y}\right)^{\frac{1}{q}} dy \,. \tag{5}$$

Setting $u=x^{c-a}y^{d-b}$ then we get $y=x^{\frac{a-c}{d-b}}u^{\frac{1}{d-b}}$ and $dy=\frac{1}{d-b}x^{\frac{a-c}{d-b}}u^{\frac{1}{d-b}-1}du$ and $0\le u<\infty$.

Substituting by y and dy in (5), we obtain

$$w_1(x)=\int_0^\infty \frac{1}{\left(x^a\left(x^{\frac{a-c}{d-b}}u^{\frac{1}{d-b}}\right)^b\right)^\propto (1+u)^\propto}\left(\frac{x}{x^{\frac{a-c}{d-b}}u^{\frac{1}{d-b}}}\right)^{\frac{1}{q}}\frac{1}{d-b}x^{\frac{a-c}{d-b}}u^{\frac{1}{d-b}-1}du\,,$$

where $\frac{1}{d-b}$ is constant .

Since x is constant,then we obtain

$$w_1(x)=\frac{1}{d-b}x^{\frac{a-c}{d-b}\left(1-\frac{1}{q}-\propto b\right)+\frac{1}{q}-\propto a}\int_0^\infty \frac{u^{\frac{1}{d-b}\left(1-\frac{1}{q}-b\propto\right)-1}}{(1+u)^\propto}du. \tag{6}$$

Use $\frac{1}{p}+\frac{1}{q}=1,$ we get

By the well-known Beta function (**A**ndrews, 1985) and equation (6), we get

$$a=\frac{\frac{1}{p}-b\propto}{d-b}\,. \tag{7}$$

Also, from $a+b=\propto$, we get

$$b=\propto-\left(\frac{\frac{1}{p}-b\propto}{d-b}\right). \tag{8}$$

Properties of Beta function, (7) and (8), give

$$\int_0^\infty \frac{u^{\frac{1}{d-b}\left(\frac{1}{p}-b\propto\right)-1}}{(1+u)^\propto}du=B\left(\frac{\frac{1}{p}-b\propto}{d-b},\propto-\left(\frac{\frac{1}{p}-b\propto}{d-b}\right)\right). \tag{9}$$

Substituting from (9) in 6), we get

$$w_1(x)=\frac{1}{d-b}x^{\frac{a-c}{d-b}\left(\frac{1}{p}-\propto b\right)+1-\frac{1}{p}-\propto a}B\left(\frac{\frac{1}{p}-b\propto}{d-b},\propto-\left(\frac{\frac{1}{p}-b\propto}{d-b}\right)\right).$$

Hence the lemma is proved.

By similar manner as in the proof of Lemma 2.1., we can proof the following lemma.

Lemma 2.2. For $p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$, and $\propto\ge 0$ define the weight function $w_2(y)$ as

$$w_2(y)=\int_0^\infty \frac{1}{(x^a y^b+x^c y^d)^\propto}\left(\frac{y}{x}\right)^{\frac{1}{p}}dx\,, \tag{10}$$

where $a>c,\ d>b$ and $a,c,b,d\ge 0$.

Then we have

$$w_2(y)=\frac{1}{a-c}y^{\frac{d-b}{a-c}\left(\frac{1}{q}-\propto c\right)+1-\frac{1}{q}-\propto d}B\left(\frac{\frac{1}{q}-c\propto}{a-c},\propto-\left(\frac{\frac{1}{q}-c\propto}{a-c}\right)\right), \tag{11}$$

where *B (a,b)* is the well known Beta function, $a>0,\ b>0$.

Theorem 2.3. Let *f,g* be real – valued functions defined on $[0,\infty)$ such that

$$0<\int_0^\infty x^{\frac{a-c}{d-b}\left(\frac{1}{p}-\propto b\right)+1-\frac{1}{p}-\propto a}f^p(x)dx<\infty\,,$$

and

$$0<\int_0^\infty y^{\frac{d-b}{a-c}\left(\frac{1}{q}-c\propto\right)+1-\frac{1}{q}-d\propto}g^q(y)dy<\infty.$$

Then we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^a y^b+x^c y^d)^\propto}dxdy\le$$

$$\left\{\frac{1}{d-b}B\left(\frac{\frac{1}{p}-b\propto}{d-b},\propto-\left(\frac{\frac{1}{p}-b\propto}{d-b}\right)\right)\int_0^\infty x^{\frac{a-c}{d-b}\left(\frac{1}{p}-\propto b\right)+1-\frac{1}{p}-\propto a}f^p(x)dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\frac{1}{a-c}B\left(\frac{\frac{1}{q}-c\propto}{a-c},\propto-\left(\frac{\frac{1}{q}-c\propto}{a-c}\right)\right)\int_0^\infty y^{\frac{d-b}{a-c}\left(\frac{1}{q}-c\propto\right)+1-\frac{1}{q}-d\propto}g^q(y)dy\right\}^{\frac{1}{q}}, \tag{12}$$

where $>c,\ d>b,\ \ a,b,c,d\ge 0,\ \propto\ge 0,\ p>1,\ q>1,$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof .We can set the left hand side of (12) in the form

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^a y^b+x^c y^d)^\propto}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)}{(x^a y^b+x^c y^d)^{\frac{\propto}{p}}}\left(\frac{x}{y}\right)^{\frac{1}{pq}}\frac{g(y)}{(x^a y^b+x^c y^d)^{\frac{\propto}{q}}}\left(\frac{y}{x}\right)^{\frac{1}{pq}}dxdy\,. \tag{13}$$

Using Holder's inequality (**M**itrinovic, 1970) on (13), then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^a y^b+x^c y^d)^\propto}dxdy\le$$

$$\left(\int_0^\infty\int_0^\infty \frac{f^p(x)}{(x^a y^b+x^c y^d)^\propto}\left(\frac{x}{y}\right)^{\frac{1}{q}}dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty \frac{g^q(y)}{(x^a y^b+x^c y^d)^\propto}\left(\frac{y}{x}\right)^{\frac{1}{p}}dxdy\right)^{\frac{1}{q}}$$

$$=\left(\int_0^\infty f^p(x)\left(\int_0^\infty \frac{1}{(x^a y^b+x^c y^d)^\propto}\left(\frac{x}{y}\right)^{\frac{1}{q}}dy\right)dx\right)^{\frac{1}{p}}$$

$$\times$$

$$\left(\int_0^\infty g^q(y)\left(\int_0^\infty \frac{1}{(x^a y^b+x^c y^d)^\propto}\left(\frac{y}{x}\right)^{\frac{1}{p}}dx\right)dy\right)^{\frac{1}{q}}.$$

From (3) and (10), we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^a y^b + x^c y^d)^\propto}dxdy \le \left(\int_0^\infty f^p(x)w_1(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)w_2(y)\,dy\right)^{\frac{1}{q}}. \quad (14)$$

Applying the results of Lemma 2.1 and Lemma 2.2 in (14), we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^a y^b + x^c y^d)^\propto}dxdy \le$$

$$\left(\int_0^\infty f^p(x)\frac{1}{d-b}x^{\frac{a-c}{d-b}\left(\frac{1}{p}-\propto b\right)+1-\frac{1}{p}-\propto a}B\left(\frac{\frac{1}{p}-b\propto}{d-b},\propto-\left(\frac{\frac{1}{p}-b\propto}{d-b}\right)\right)dx\right)^{\frac{1}{p}}$$

$$\times$$

$$\left(\int_0^\infty g^q(y)\frac{1}{a-c}y^{\frac{d-b}{a-c}\left(\frac{1}{q}-\propto c\right)+1-\frac{1}{q}-\propto d}B\left(\frac{\frac{1}{q}-c\propto}{a-c},\propto-\left(\frac{\frac{1}{q}-c\propto}{a-c}\right)\right)dy\right)^{\frac{1}{q}},$$

and

$$\left(\int_0^\infty f^p(x)\frac{1}{d-b}x^{\frac{a-c}{d-b}\left(\frac{1}{p}-\propto b\right)+1-\frac{1}{p}-\propto a}B\left(\frac{\frac{1}{p}-b\propto}{d-b},\propto-\left(\frac{\frac{1}{p}-b\propto}{d-b}\right)\right)dx\right)^{\frac{1}{p}}$$

$$\times$$

$$\left(\int_0^\infty g^q(y)\frac{1}{a-c}y^{\frac{d-b}{a-c}\left(\frac{1}{q}-\propto c\right)+1-\frac{1}{q}-\propto d}B\left(\frac{\frac{1}{q}-c\propto}{a-c},\propto-\left(\frac{\frac{1}{q}-c\propto}{a-c}\right)\right)dy\right)^{\frac{1}{q}},$$

Hence the theorem is proved.

Remarks 2.1.

1. Let $a=d=1$, $b=c=0$, and $\propto=1$ in (12) ,then we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \left\{B\left(\frac{1}{p},1-\frac{1}{p}\right)\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{B\left(\frac{1}{q},1-\frac{1}{q}\right)\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}}.$$

Since $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \left\{B\left(\frac{1}{p},1-\frac{1}{p}\right)\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{B\left(1-\frac{1}{p},\frac{1}{p}\right)\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.$$

It well known that

$B(a\,,b)=B(b\,,a)$, and $B(a\,,1-a)=\frac{\pi}{\sin a\pi}$. Therefore

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \quad (15)$$

which is Hardy - Hilbert's inequality (Hardy, et. al., 1964).

2. If we put $p=q=2$ in (15), we obtain

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \pi\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(y)\,dy\right)^{\frac{1}{2}},$$

which is the standard Hilbert's integral inequality (Hardy et. al., 1964).

3. Let $a=d=1$, $b=c=0$ in (12), we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\propto}dxdy \le \left[B\left(\frac{1}{p},\propto-\frac{1}{p}\right)\right]^{\frac{1}{p}}\left[B\left(\frac{1}{q},\propto-\frac{1}{q}\right)\right]^{\frac{1}{q}}$$

$$\times\left\{\int_0^\infty x^{1-\propto}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{1-\propto}g^q(y)\,dy\right\}^{\frac{1}{q}}, \quad (16)$$

which is Yang Bicheng's inequality (Yang, 2000 and Yang, 2004).

Putting $p=q=2$ in (16) , then we obtain the following inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\propto}dxdy \le B\left(\frac{1}{2},\propto-\frac{1}{2}\right)\left\{\int_0^\infty x^{1-\propto}f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\propto}g^2(y)dy\right\}^{\frac{1}{2}}.$$

4. Setting $a=d=\beta$, $b=c=0$, and $\propto=1$ in (12) , we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\beta+y^\beta}dxdy \le \frac{1}{\beta}\left[B\left(\frac{1}{\beta p},1-\frac{1}{\beta p}\right)\right]^{\frac{1}{p}}$$

$$\times\left[B\left(\frac{1}{\beta q},1-\frac{1}{\beta q}\right)\right]^{\frac{1}{q}}\left\{\int_0^\infty x^{1-\beta}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{1-\beta}g^q(y)\,dy\right\}^{\frac{1}{q}}, \quad (17)$$

which is considered as special case of Hilbert-Hardy inequality.

5. Putting $\beta=2$ in (17), we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^2+y^2}dxdy \le \frac{1}{2}\left[B\left(\frac{1}{2p},1-\frac{1}{2p}\right)\right]^{\frac{1}{p}}$$

$$\times\left[B\left(\frac{1}{2q},1-\frac{1}{2q}\right)\right]^{\frac{1}{q}}\left\{\int_0^\infty \frac{f^p(x)}{x}dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \frac{g^q(y)}{y}dy\right\}^{\frac{1}{q}}. \quad (18)$$

Setting $p=q=2$ in (18), then we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^2+y^2}dxdy \le \frac{1}{2}B\left(\frac{1}{4},1-\frac{1}{4}\right)\left\{\int_0^\infty \frac{f^2(x)}{x}dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \frac{g^2(y)}{y}dy\right\}^{\frac{1}{2}}$$

$$=\frac{\pi}{2\sin\frac{\pi}{4}}\left\{\int_0^\infty \frac{f^2(x)}{x}dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \frac{g^2(y)}{y}dy\right\}^{\frac{1}{2}},$$

or

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^2+y^2}dxdy \le \frac{\pi}{\sqrt{2}}\left\{\int_0^\infty \frac{f^2(x)}{x}dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \frac{g^2(y)}{y}dy\right\}^{\frac{1}{2}},$$

which is a generalization of Hilbert-Hardy inequality .

6. Let $a=2, b=0, d=c=1$ in (17), then we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^2+xy)^\alpha}dxdy \le \left\{B\left(\frac{1}{p},\alpha-\frac{1}{p}\right)\int_0^\infty x^{1-2\alpha}f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{B\left(\frac{1}{q}-\alpha, 2\alpha-\frac{1}{q}\right)\int_0^\infty y^{1-2\alpha}g^q(y)dy\right\}^{\frac{1}{q}}.$$

Also, this is a generalization of Hilbert-Hardy integral inequality.

Lemma 2.4. For $p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$,

define the weight function $w_1(x)$ as

$$w_1(x)=\int_0^\infty \frac{1}{(ax^\alpha+by^\beta)^\gamma}\left(\frac{x}{y}\right)^{\frac{1}{q}}dy,$$

where $a,\ b,\ \alpha,\ \beta>0$ and $\gamma\ge 0$. (19)

Then we get

$$w_1(x)=\frac{1}{\beta a^{\gamma-\frac{1}{p\beta}}b^{\frac{1}{p\beta}}}x^{\frac{1}{p}\left(\frac{\alpha}{\beta}-1\right)-\alpha\gamma+1}B\left(\frac{1}{p\beta},\gamma-\frac{1}{p\beta}\right), \quad (20)$$

where $B(a,b)$ is the well-known Beta function with $a>0,\ b>0$.

Proof. From the weight function (19), we have

$$w_1(x)=\int_0^\infty \frac{1}{(ax^\alpha)^\gamma\left(1+\frac{by^\beta}{ax^\alpha}\right)^\gamma}\left(\frac{x}{y}\right)^{\frac{1}{q}}dy. \quad (21)$$

Setting $u=\frac{by^\beta}{ax^\alpha}$, then $y=\left(\frac{a}{b}\right)^{\frac{1}{\beta}}x^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}}$ we get

$y=\left(\frac{a}{b}\right)^{\frac{1}{\beta}}x^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}}$ and $dy=\frac{1}{\beta}\left(\frac{a}{b}\right)^{\frac{1}{\beta}}x^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}-1}du,\ 0\le u<\infty$.

By substituting by y and dy in (21), we obtain

$$w_1(x)=\int_0^\infty\frac{1}{(ax^\alpha)^\gamma(1+u)^\gamma}\left(\frac{x}{\left(\frac{a}{b}\right)^{\frac{1}{\beta}}x^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}}}\right)^{\frac{1}{q}}\frac{1}{\beta}\left(\frac{a}{b}\right)^{\frac{1}{\beta}}x^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}-1}du,$$

where $\frac{1}{\beta}$ is constant.

Therefore

$$w_1(x)=\frac{1}{\beta}\int_0^\infty\frac{1}{(ax^\alpha)^\gamma(1+u)^\gamma}\left(x^{1-\frac{\alpha}{\beta}}\left(\frac{a}{b}\right)^{-\frac{1}{\beta}}u^{\frac{-1}{\beta}}\right)^{\frac{1}{q}}\left(\frac{a}{b}\right)^{\frac{1}{\beta}}x^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}-1}du$$

$$=\frac{1}{\beta a^\gamma}\left(\frac{a}{b}\right)^{\frac{1}{\beta}-\frac{1}{\beta q}}x^{\frac{\alpha}{\beta}+\frac{1}{q}\left(1-\frac{\alpha}{\beta}\right)-\alpha\gamma}\int_0^\infty\frac{u^{\frac{1}{\beta}-1-\frac{1}{\beta q}}}{(1+u)^\gamma}du$$

$$=\frac{1}{\beta a^\gamma}\left(\frac{a}{b}\right)^{\frac{1}{\beta}\left(1-\frac{1}{q}\right)}x^{\frac{\alpha}{\beta}\left(1-\frac{1}{q}\right)+\frac{1}{q}-\alpha\gamma}\int_0^\infty\frac{u^{\frac{1}{\beta}\left(1-\frac{1}{q}\right)-1}}{(1+u)^\gamma}du.$$

But sine $\frac{1}{p}+\frac{1}{q}=1$, thus we have

$$w_1(x)=\frac{1}{\beta a^{\gamma-\frac{1}{p\beta}}b^{\frac{1}{p\beta}}}x^{\frac{1}{p}\left(\frac{\alpha}{\beta}-1\right)-\alpha\gamma+1}\int_0^\infty\frac{u^{\frac{1}{p\beta}-1}}{(1+u)^\gamma}du. \quad (22)$$

From (22) and definition Beta function, we get

$$a=\frac{1}{p\beta}, \quad (23)$$

$$b=\gamma-\frac{1}{p\beta}. \quad (24)$$

From Beta function, (23) and (24) we obtain

$$\int_0^\infty\frac{u^{\frac{1}{p\beta}-1}}{(1+u)^\gamma}du=B\left(\frac{1}{p\beta},\gamma-\frac{1}{p\beta}\right). \quad (25)$$

Substituting from (25) in (22), we have

$$w_1(x)=\frac{1}{\beta a^{\gamma-\frac{1}{p\beta}}b^{\frac{1}{p\beta}}}x^{\frac{1}{p}\left(\frac{\alpha}{\beta}-1\right)-\alpha\gamma+1}B\left(\frac{1}{p\beta},\gamma-\frac{1}{p\beta}\right).$$

Then the proof is completed.

In similar way, as in the proof of Lemma 2.4., we can prove the following lemma.

Lemma 2.5. For $p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$,

define the weight function $w_2(y)$ as

$$w_2(y)=\int_0^\infty\frac{1}{(ax^\alpha+by^\beta)^\gamma}\left(\frac{y}{x}\right)^{\frac{1}{p}}dx, \quad (26)$$

Then we get

$$w_2(y)=\frac{1}{\alpha a^{\frac{1}{q\alpha}}b^{\gamma-\frac{1}{q\alpha}}}y^{\frac{1}{q}\left(\frac{\beta}{\alpha}-1\right)-\beta\gamma+1}B\left(\frac{1}{q\alpha},\gamma-\frac{1}{q\alpha}\right), \quad (27)$$

where $B(a,b)$ is the well known Beta function, $a>0,\ b>0$.

Theorem 2.6. Let $f,\ g$ be real – valued functions defined on $[0,\infty)$ such that

$$0<\int_0^\infty x^{\frac{1}{p}\left(\frac{\alpha}{\beta}-1\right)-\alpha\gamma+1}f^p(x)\,dx<\infty,$$

and

$$0<\int_0^\infty y^{\frac{1}{q}\left(\frac{\beta}{\alpha}-1\right)-\beta\gamma+1}g^q(y)dy<\infty.$$

Then we get

$$\int_0^\infty\int_0^\infty\frac{f(x)\,g(y)}{(ax^\alpha+by^\beta)^\gamma}dx\,dy\le$$

$$\left\{\frac{1}{\beta a^{\gamma-\frac{1}{p\beta}}b^{\frac{1}{p\beta}}}B\left(\frac{1}{p\beta},\gamma-\frac{1}{p\beta}\right)\int_0^\infty x^{\frac{1}{p}\left(\frac{\alpha}{\beta}-1\right)-\alpha\gamma+1}f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\frac{1}{\alpha a^{\frac{1}{q\alpha}}b^{\gamma-\frac{1}{q\alpha}}}B\left(\frac{1}{q\alpha},\gamma-\frac{1}{q\alpha}\right)\int_0^\infty y^{\frac{1}{q}\left(\frac{\beta}{\alpha}-1\right)-\beta\gamma+1}g^q(y)dy\right\}^{\frac{1}{q}}, \quad (28)$$

where $a\,,\ b\,,\ \alpha\,,\ \beta > 0\,,\ \gamma \geq 0\,,\ p > 1\,,\ q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Setting the left hand side of (28) in the following form

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(ax^\alpha + by^\beta)^\gamma}\,dx\,dy = \int_0^\infty \int_0^\infty \frac{f(x)}{(ax^\alpha + by^\beta)^{\frac{\gamma}{p}}} \left(\frac{x}{y}\right)^{\frac{1}{pq}} \frac{g(y)}{(ax^\alpha + by^\beta)^{\frac{\gamma}{q}}} \left(\frac{y}{x}\right)^{\frac{1}{pq}} dx\,dy. \tag{29}$$

Using Holder's inequality on (29), we get

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(ax^\alpha + by^\beta)^\gamma}\,dx\,dy \leq$$

$$\left(\int_0^\infty \int_0^\infty \frac{f^p(x)}{(ax^\alpha + by^\beta)^\gamma} \left(\frac{x}{y}\right)^{\frac{1}{q}} dx\,dy\right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^\infty \frac{g^q(y)}{(ax^\alpha + by^\beta)^\gamma} \left(\frac{y}{x}\right)^{\frac{1}{p}} dx\,dy\right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty f^p(x) \left(\int_0^\infty \frac{1}{(ax^\alpha + by^\beta)^\gamma} \left(\frac{x}{y}\right)^{\frac{1}{q}} dy\right) dx\right)^{\frac{1}{p}} \times$$

$$\left(\int_0^\infty g^q(y) \left(\int_0^\infty \frac{1}{(ax^\alpha + by^\beta)^\gamma} \left(\frac{y}{x}\right)^{\frac{1}{p}} dx\right) dy\right)^{\frac{1}{q}}.$$

From (28) and (29) , we have

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(ax^\alpha + by^\beta)^\gamma}\,dx\,dy \leq \left(\int_0^\infty f^p(x) w_1(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) w_2(y)\,dy\right)^{\frac{1}{q}}. \tag{30}$$

Applying the results of Lemma 2.4 and Lemma 2.5 in (30), we have

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(a\,x^\alpha + b\,y^\beta)^\gamma}\,dx\,dy \leq$$

$$\left(\int_0^\infty f^p(x) \frac{1}{\beta\, a^{\gamma - \frac{1}{p\beta}}\, b^{\frac{1}{p\beta}}} x^{\frac{1}{p}\left(\frac{\alpha}{\beta} - 1\right) - \alpha\gamma + 1} B\left(\frac{1}{p\beta}, \gamma - \frac{1}{p\beta}\right) dx\right)^{\frac{1}{p}}$$

$$\times$$

$$\left(\int_0^\infty g^q(y) \frac{1}{\alpha\, a^{\frac{1}{q\alpha}}\, b^{\gamma - \frac{1}{q\alpha}}} y^{\frac{1}{q}\left(\frac{\beta}{\alpha} - 1\right) - \beta\gamma + 1} B\left(\frac{1}{q\alpha}, \gamma - \frac{1}{q\alpha}\right) dy\right)^{\frac{1}{q}},$$

where *B(a,b)* is constant function and $\frac{1}{\beta\, a^{\gamma-\frac{1}{p\beta}}\, b^{\frac{1}{p\beta}}}, \frac{1}{\alpha\, a^{\frac{1}{q\alpha}}\, b^{\gamma-\frac{1}{q\alpha}}}$ also is constant.

Then, we obtain

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{(a\,x^{\propto}+b\ y^{\beta})^{\gamma}}\,dx\,dy \le$$

$$\left\{\frac{1}{\beta\, a^{\gamma-\frac{1}{p\beta}}\ b^{\frac{1}{p\beta}}} B\left(\frac{1}{p\,\beta}, \gamma-\frac{1}{p\,\beta}\right)\int_0^\infty x^{\frac{1}{p}\left(\frac{\alpha}{\beta}-1\right)-\alpha\gamma+1} f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\frac{1}{\alpha\ a^{\frac{1}{q\alpha}}\, b^{\gamma-\frac{1}{q\alpha}}}\ B\left(\frac{1}{q\alpha}, \gamma-\frac{1}{q\alpha}\right)\int_0^\infty y^{\frac{1}{q}\left(\frac{\beta}{\alpha}-1\right)-\beta\gamma+1} g^q(y)dy\right\}^{\frac{1}{q}}.$$

Then the theorem is proved.

Remarks 2.2.

1. Let $a=b=\propto=\beta=\gamma=1$ in (28), then we get the Hilbert's integral inequality as follow

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy \le \left\{B\left(\frac{1}{p},1-\frac{1}{p}\right)\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{B\left(\frac{1}{q},1-\frac{1}{q}\right)\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.$$

Since $\frac{1}{p}+\frac{1}{q}=1,$ then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}dx\,dy \le \left\{B\left(\frac{1}{p},1-\frac{1}{p}\right)\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{B\left(1-\frac{1}{p},\frac{1}{p}\right)\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.$$

According to the properties of Beta function ,we have

$B(a\,,b)=B(b\,,a)$,and $B(a\,,1-a)=\frac{\pi}{\sin a\pi}$, therefore

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}},$$

which is Hardy-Hilbert's inequality (**M**itrinovic, 1970).

2. Putting $\propto=\beta=1$in (28), then we have

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{(a\,x\ \ +by)^{\gamma}}\,dx\,dy \le\left\{\frac{1}{a^{\gamma-\frac{1}{p}}\ b^{\frac{1}{p}}}B\left(\frac{1}{p},\gamma-\frac{1}{p}\right)\int_0^\infty x^{1-\gamma} f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\frac{1}{a^{\frac{1}{q}}\,b^{\gamma-\frac{1}{q}}}\ B\left(\frac{1}{q},\gamma-\frac{1}{q}\right)\int_0^\infty y^{1-\gamma}\,g^q(y)dy\right\}^{\frac{1}{q}}, \tag{31}$$

this is a generalization of Hilbert-Hardy integral inequality.

Setting $\gamma=1$ in (31),we deduce that

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{a\,x \quad + b\,y}\,dx\,dy \le \left\{\frac{1}{a^{\ 1-\frac{1}{p}}\ \ b^{\frac{1}{p}}} B\left(\frac{1}{p},1-\frac{1}{p}\right)\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\frac{1}{a^{\frac{1}{q}}\,b^{\ 1-\frac{1}{q}}}\ B\left(\frac{1}{q},1-\frac{1}{q}\right)\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.$$

Since $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{a\,x \quad + b\,y}\,dx\,dy \le$$

$$\left\{\frac{1}{a^{\frac{1}{q}}\ \ b^{\frac{1}{p}}} B\left(\frac{1}{p},1-\frac{1}{p}\right)\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\frac{1}{a^{\frac{1}{q}}\,b^{\frac{1}{p}}}\ B\left(1-\frac{1}{p},\frac{1}{p}\right)\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}},$$

from the properties of Beta function, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{a\,x + b\,y}dx\,dy \le \frac{\pi}{a^{\frac{1}{q}}b^{\frac{1}{p}}\ sin\frac{\pi}{p}}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}. \qquad (32)$$

Set $p = q = 2$ in (32), we get

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{a\,x \quad + b\,y}\,dx\,dy \le \frac{\pi}{(\ a \quad b)^{\frac{1}{2}}}\left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(y)\,dy\right)^{\frac{1}{2}},$$

which is the same result of **Y**ang, 2000.

3. If we put $a = b = 1$in (32), then we obtain the inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+\ y)^\gamma}\,dx\,dy \le \left\{B\left(\frac{1}{p},\gamma-\frac{1}{p}\right)\int_0^\infty x^{1-\gamma} f^p(x)dx\right\}^{\frac{1}{p}}$$

$$\times\left\{B\left(\frac{1}{q},\gamma-\frac{1}{q}\right)\int_0^\infty y^{1-\gamma}\,g^q(y)dy\right\}^{\frac{1}{q}}, \qquad (33)$$

which is a special case of Hilbert-Hardy integral inequality.

Let $p = q = 2$ and $\gamma = 1$ in (33) , we have

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy \le \pi\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(y)\,dy\right)^{\frac{1}{2}},$$

which it is the standard Hilbert's integral inequality **(**El**-**Marouf, 2013).

4. Taking $a = b = 1$ and $\propto = \beta = \delta$ in(28), we obtain the inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{(x^{\delta} + y^{\delta})^{\gamma}}\,dx\,dy \le \frac{1}{\delta}\left\{B\left(\frac{1}{p\,\delta},\gamma - \frac{1}{p\,\delta}\right)\int_0^\infty x^{1-\delta\gamma} f^{p}(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{B\left(\frac{1}{q\delta},\gamma - \frac{1}{q\delta}\right)\int_0^\infty y^{1-\delta\gamma} g^{q}(y)dy\right\}^{\frac{1}{q}}. \qquad (34)$$

This is similar to El-Marouf, 2103 integral inequality.

Putting $\gamma = 1$ and $p = q = 2$ in (34), we deduce that

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{x^{\delta} + y^{\delta}}\,dx\,dy \le \frac{1}{\delta}\left\{B\left(\frac{1}{2\,\delta},1 - \frac{1}{2\,\delta}\right)\int_0^\infty x^{1-\delta} f^{2}(x)\,dx\right\}^{\frac{1}{2}}$$

$$\times\left\{B\left(\frac{1}{2\delta},1 - \frac{1}{2\delta}\right)\int_0^\infty y^{1-\delta} g^{2}(y)dy\right\}^{\frac{1}{2}}.$$

From the properties of Beta function,we have

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{x^{\delta} + y^{\delta}}\,dx\,dy \le \frac{\pi}{\delta\sin\frac{\pi}{2\delta}}\left\{\int_0^\infty x^{1-\delta} f^{2}(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\delta} g^{2}(y)dy\right\}^{\frac{1}{2}}. \qquad (35)$$

It is a generalization of Hilbert-Hardy integral inequality.

Set $\delta = 2$ in (35), we have

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{x^{2} + y^{2}}\,dx\,dy \le \frac{\pi}{2\sin\frac{\pi}{4}}\left\{\int_0^\infty \frac{f^{2}(x)}{x}\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \frac{g^{2}(y)}{y}dy\right\}^{\frac{1}{2}},$$

or

$$\int_0^\infty\int_0^\infty \frac{f(x)\ g(y)}{x^{2} + y^{2}}\,dx\,dy \le \frac{\pi}{\sqrt{2}}\left\{\int_0^\infty \frac{f^{2}(x)}{x}\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \frac{g^{2}(y)}{y}dy\right\}^{\frac{1}{2}}.$$

Also this is a special case of Hilbert-Hardy integral inequality.

4. Applications

In this section we introduce some examples of partial differential equation, which is considered as applications of our obtained results.

Example1. Consider the following partial differential equation

$$u_{xy} = \frac{f(x)\, g(y)}{1+xy}, \tag{36}$$

where

$$u_x(x,0) = a(x), \tag{37}$$

$$u(0,y) = b(y). \tag{38}$$

From (36) and (37) by integrating from 0 to y , we get

$$u_x(x,y) - u_x(x,0) = \int_0^y \frac{f(x)\, g(t)}{1+xt}\, dt .$$

Hence

$$u_x(x,y) = a(x) + \int_0^y \frac{f(x)\, g(t)}{1+xt}\, dt . \tag{39}$$

From (38) , (39) by integrating from 0 to x , we have

$$u(x,y) - u(0,y) = \int_0^x a(s)\, ds + \int_0^x \int_0^y \frac{f(s)\, g(t)}{1+st}\, ds\, dt,$$

i.e.

$$u(x,y) = b(y) + \int_0^x a(s)\, ds + \int_0^x \int_0^y \frac{f(s)\, g(t)}{1+st}\, ds\, dt. \tag{40}$$

Thus if

$$\int_0^x \int_0^y \frac{f(s)\, g(t)}{1+st}\, ds\, dt \le \int_0^\infty \int_0^\infty \frac{f(s)\, g(t)}{1+st}\, ds\, dt , \tag{41}$$

Then it follows from (40) and (41) that

$$u(x,y) \le b(y) + \int_0^x a(s)\, ds + \int_0^\infty \int_0^\infty \frac{f(s)\, g(t)}{1+st}\, ds\, dt$$

$$\le b(y) + \int_0^x a(s)\, ds + \frac{\pi}{\sin\frac{\pi}{p}} \left\{ \int_0^\infty s^{1-\frac{2}{p}} f^p(s)\, ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{1-\frac{2}{q}} g^q(t) dt \right\}^{\frac{1}{q}}, \tag{42}$$

which is application of Hilbert's integral inequality.

Example 2. Consider the following partial differential equation

$$u_{xy} = \frac{f(x)\, g(y)}{x^\beta + y^\beta}, \tag{43}$$

where

$$u_x(x,0) = a(x), \tag{44}$$