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Recovery of coefficients of a heat equation by Ritz collocation method

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Abstract

In this work, we discuss a one dimensional inverse problem for the heat equation where the unknown functions are solely time-dependent lower order coefficient and multiplicative source term. We use as data two integral overdetermination conditions along with the initial and Dirichlet boundary conditions. In the first step, the lower order term is eliminated by applying a transformation and the problem is converted to an equivalent inverse problem of determining a heat source with initial and boundary conditions, as well as a nonlocal energy over-specification. Then, we propose a Ritz approximation as the solution of the unknown temperature distribution and consider a truncated series as the approximation of unknown time-dependent coefficient in the heat source. The collocation method is utilized to reduce the inverse problem to the solution of a linear system of algebraic equations. Since the problem is ill-posed, numerical discretization of the reformulated problem may produce ill-conditioned system of equations. Therefore, the Tikhonov regularization technique is employed in order to obtain stable solutions. For the perturbed measurements, we employ the mollification method to derive stable numerical derivatives. Numerical simulations while solving two test examples are presented to show the applicability of the proposed method.

Keywords: Inverse coefficient problem; mollification method; parabolic equation; Ritz approximation; Tikhonov regularization

1. Introduction

In this paper, we consider the inverse problem of finding $\Big(u(x,t), c(t), d(t)\Big)$ in the parabolic equation (Shekarpaz & Azari, 2018)

$$u_t - a(x,t)u_{xx} + b(x,t)u_x + c(t)u = d(t)g(x,t), \quad (x,t) \in Q, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad -L < x < L, \tag{2}$$

boundary conditions

$$u(-L,t) = u(L,t) = 0, \quad 0 < t < T, \tag{3}$$

and subject to the integral over-specifications of the functions $\omega_1(x)u(x,t)$ and $\omega_2(x)u(x,t)$ over the spatial domain (energy over-specifications)

$$\int_{-L}^{L} \omega_1(x)u(x,t)dx = \mu_1(t), \quad t \in [0,T], \tag{4}$$

$$\int_{-L}^{L} \omega_2(x)u(x,t)dx = \mu_2(t), \quad t \in [0,T], \tag{5}$$

where $Q = [-L, L] \times [0, T]$ and $a(x,t),\ b(x,t),\ g(x,t),\ \mu_1(t),\ \mu_2(t),\ u_0(x),\ \omega_1(x),\ \omega_2(x)$ are given functions with appropriate conditions. The additional Equations 4-5 are interpreted as the measurements of function $u(x,t)$ by sensor averaging over the segment $[-L, L]$ of space variable. Furthermore, we assume that the following compatibility conditions hold:

$$u_0(-L) = u_0(L) = 0, \quad \int_{-L}^{L} \omega_1(x) u_0(x) dx = \mu_1(0), \quad \int_{-L}^{L} \omega_2(x) u_0(x) dx = \mu_2(0). \tag{6}$$

Integral overdetermination conditions are employed to establish an integral or integro-differential equation of the Fredholm or Volterra type and then the analysis of the existence, uniqueness and continuous dependence of the solution is given for the new reformulated problem. The properties of the kernel functions $\omega_1(x)$ and $\omega_2(x)$ included in the integral boundary conditions can directly affect the solvability constraints of the problem and further may complicate the application of the numerical techniques to obtain accurate solutions.

As a special class of the inverse problems, the inverse coefficient problems (ICPs) appear in studying various physical phenomena in order to determine some unknown properties of a region in parabolic and hyperbolic equations. The unknown coefficients can be a function of only time variable if the spatial change in the solution of the direct problem is small in comparison with the change in time (see (Dehghan & Shamsi, 2006; Shamsi & Dehghan, 2012) and (Shamsi & Dehghan, 2006) and many references therein). Moreover, if the property of the medium under study does not change rapidly, the unknown coefficient can be space-wise dependent solely (Liao, 2011). However, in the general form it depends on the solution of the direct problem (Rashedi, 2021; Samarskii & Vabishchevich, 2008).

Although the ICPs in the heat equations are well-studied, the particular problem of determining multiple unknown time-dependent coefficients in heat transfer is less investigated (Hussein & Lesnic, 2014; Lesnic *et al.*, 2016). In (Ivanchov & Pabyrivs'ka , 2001) and (Ivanchov & Pabyrivs'ka , 2002), the authors established conditions for the existence and uniqueness of a solution of the inverse problems for a parabolic equation with two unknown time-dependent coefficients. In (Hussein *et al.*, 2014), the authors investigated the numerical approximation of time-dependent thermal conductivity and convection coefficients in a one-dimensional parabolic equation from boundary temperature and heat flux. In (Huntul *et al.*, 2017), the authors studied simultaneous reconstruction of time-dependent coefficients including the thermal conductivity, convection or absorption coefficients in the parabolic heat equation from heat moments. In (Lingde *et al.*, 2017), the authors studied an inverse problem of the simultaneous determination of the right-hand side and the lowest coefficients in parabolic equations and proposed linearized approximations in time using the fully implicit scheme and standard finite difference procedures in space.

In (Shekarpaz & Azari, 2018), a numerical approach based on the forward finite difference and backward finite difference methods was presented for solving the problem given by Equations 1-5. Even though this approach is effective for solving various kinds of partial differential equations, the high computational cost of FD schemes is a difficulty of this method. Moreover, they can often achieve only two or three digits of accuracy (Dehghan & Shamsi, 2006; Shamsi & Dehghan, 2012, 2006). In this paper we use a collocation technique (Canuto *et al.*, 2006; Jahangiri *et al.*, 2016) to provide more accurate and stable numerical solution for the inverse problem 1-5.

The organization of this article is as follows. In Section 2, we review theoretical results concerning the uniqueness of the solution for the inverse problem 1-5 and use new variables to derive the equivalent problem. Section 3, presents the application of Ritz collocation method to the solution of the reformulated problem. In Section 4, some numerical examples are presented to demonstrate the effectiveness of the proposed method. In Section 5, we present some concluding remarks.

2. Uniqueness

In (Kamynin, 2015), the authors established the situations under which the system of Equations 1-5 possesses a unique solution.

Theorem 2.1 *Suppose that all the functions appearing in the Equations 1-5 are measurable and the compatibility conditions of Equation 6 among the boundary and initial conditions hold. Moreover, as-*

sume that there exist the constants

$$C_{1a},\, C_{2a},\, C_{u_0},\, C_g,\, C_{\omega_1},\, C_{\omega_2},\, C_{\mu_1},\, C_{\mu_2} > 0,\, C_a^*,\, C_a^{**},\, C_b,\, C_b^*,\, C_{\omega_1}^*,\, C_{\omega_1}^{**},\, C_{\omega_2}^*,\, C_{\omega_2}^{**},\, C_{\mu_1}^*,\, C_{\mu_2}^* \geq 0,$$

subject to

- $\forall\, (x,t) \in Q,\ C_{1a} \leq a(x,t) \leq C_{2a},\ |a_x(x,t)| \leq C_a^*,\ |a_{xx}(x,t)| \leq C_a^{**},$
- $\forall\, (x,t) \in Q,\ |b(x,t)| \leq C_b,\ |b_x(x,t)| \leq C_b^*,\ |g(x,t)| \leq C_g,$
- $\forall\, x \in [-L,L],\ |\omega_1(x)| \leq C_{\omega_1},\ |\omega_1'(x)| \leq C_{\omega_1}^*,\ |\omega_1''(x)| \leq C_{\omega_1}^{**},\ \omega_1(\mp L) = 0,$ $\omega_1(x) \in W_2^2([-L,L]),$
- $\forall\, x \in [-L,L],\ |\omega_2(x)| \leq C_{\omega_2},\ |\omega_2'(x)| \leq C_{\omega_2}^*,\ |\omega_2''(x)| \leq C_{\omega_2}^{**},\ \omega_2(\mp L) = 0,$ $\omega_2(x) \in W_2^2([-L,L]),$
- $\forall\, x \in [-L,L],\ |u_0(x)| \leq C_{u_0},\ u_0(x) \in W_2^1([-L,L]),$
- $\forall\, t\ \in [0,T],\ |\mu_1(t)| \leq C_{\mu_1},\ |\mu_1'(t)| \leq C_{\mu_1}^*,\ |\mu_2(t)| \leq C_{\mu_2},\ |\mu_2'(t)| \leq C_{\mu_2}^*,$

and denoting $G_{\omega_1}(t) := \int_{-L}^{L} g(x,t)\omega_1(x)dx,\ G_{\omega_2}(t) := \int_{-L}^{L} g(x,t)\omega_2(x)dx$, then there exists C_Δ such that if

$$\forall\, t \in [0,T],\ Det \begin{pmatrix} \mu_1(t) & -G_{\omega_1}(t) \\ \mu_2(t) & -G_{\omega_2}(t) \end{pmatrix} \geq C_\Delta > 0,$$

then, the inverse problem given by Equations 1-5 has a unique solution.

Proof. Please refer to (Kamynin, 2015; Shekarpaz & Azari, 2018).

Next, we employ a method to transform problem 1-5 into a problem of finding an unknown heat source from one additional measurement. Let

$$v(x,t) = r(t)u(x,t), \quad r(t) = e^{\int_0^t c(z)dz}, \tag{7}$$

then, applying transformation 7 in Equations 1-5 results the following system of equations

$$v_t - a(x,t)v_{xx} + b(x,t)v_x = r(t)d(t)g(x,t), \quad (x,t) \in Q, \tag{8}$$

$$v(x,0) = u_0(x), \quad -L < x < L, \tag{9}$$

$$v(-L,t) = v(L,t) = 0, \quad 0 < t < T, \tag{10}$$

$$\int_{-L}^{L} \omega_1(x)v(x,t)dx = \mu_1(t)r(t), \quad t \in [0,T], \tag{11}$$

$$\int_{-L}^{L} \omega_2(x)v(x,t)dx = \mu_2(t)r(t), \quad t \in [0,T]. \tag{12}$$

The unknown function $r(t)$ can be disappeared in Equations 11-12 if either one of the functions $\mu_1(t)$ or $\mu_2(t)$ is nonzero on the interval $[0,T]$. Without loss of generality, we assume that $\forall\, t \in [0,T],\ \mu_1(t) \neq 0$. From Equation 11 we have

$$r(t) = \frac{\int_{-L}^{L} \omega_1(x)v(x,t)dx}{\mu_1(t)}, \tag{13}$$

which by substituting the Equation 13 in Equation 12, the following equation is achieved:

$$\int_{-L}^{L} \omega_2(x)v(x,t)dx = \frac{\mu_2(t)}{\mu_1(t)} \int_{-L}^{L} \omega_1(x)v(x,t)dx, \quad t \in [0,T]. \tag{14}$$

Now by defining
$$H(t) := r(t)d(t), \tag{15}$$
the main problem is reduced to the simplified problem of identifying $\Big(v(x,t), H(t)\Big)$ using the following system of equations

$$v_t - a(x,t)v_{xx} + b(x,t)v_x = H(t)g(x,t), \quad (x,t) \in Q, \tag{16}$$

$$v(x,0) = u_0(x), \quad -L < x < L, \tag{17}$$

$$v(-L,t) = v(L,t) = 0, \quad 0 < t < T, \tag{18}$$

and
$$\mu_2(t)\int_{-L}^{L}\omega_1(x)v(x,t)dx - \mu_1(t)\int_{-L}^{L}\omega_2(x)v(x,t)dx = 0, \quad t \in [0,T]. \tag{19}$$

Theorem 2.2 *Assume that at least one of the functions $\mu_1(t)$ or $\mu_2(t)$ is nonzero over the interval $[0,T]$. Then, the problems given by Equations 1-5 and 16-19 are equivalent.*

Proof. Obviously, if $\Big(u(x,t), c(t), d(t)\Big)$ is a solution of problem 1-5, then from Equations 7 and 15, $\Big(v(x,t), H(t)\Big)$ is a sloution of problem 16-19. Conversely, assuming that $\Big(v(x,t), H(t)\Big)$ is a solution of problem 16-19, the function $r(t)$ is verified from Equation 13 provided that $\mu_1(t) \neq 0$. Then, Equation 15 yields $d(t) = \frac{H(t)}{r(t)}$. Utilizing Equation 7 and differentiating $r(t) = e^{\int_0^t c(z)dz}$ with respect to t we get
$$c(t) = \frac{r'(t)}{r(t)}, \quad u(x,t) = \frac{v(x,t)}{r(t)}. \tag{20}$$
Therefore, we will consider problem 16-19 instead of problem 1-5.

3. Solution method

Suppose that $P_m(z),\ m = 0,1,2,3,...$ denote the well-known Legendre polynomials of order m which are defined on the interval $[-1,1]$ and can be determined via the following recurrence formula:

$$P_0(z) = 1,\ P_1(z) = z,\ P_{m+1}(z) = \frac{2m+1}{m+1}zP_m(z) - \frac{m}{m+1}P_{m-1}(z),\ m = 1,2,3,....$$

Then, we consider $\phi_i(x) := P_i(\frac{x}{L})$ as the shifted Legendre polynomial of degree i in the interval $[-L,L]$ and $\psi_j(t) := P_j(\frac{2t}{T} - 1)$ as the shifted Legendre polynomial of degree j in the interval $[0,T]$. The Ritz approximation $v_{N,N'}(x,t)$ based on polynomial basis functions is sought in the form of the following truncated series

$$v_{N,N'}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij}t(x+L)(x-L)\phi_i(x)\psi_j(t) + u_0(x), \tag{21}$$

and the approximation of $H(t)$ is considered as

$$H_{N''}(t) = \sum_{j=0}^{N''}\alpha_j\psi_j(t). \tag{22}$$

Substituting the approximations $v_{N,N'}(x,t)$ and $H_{N''}(t)$ in Equations 16 and 19 respectively, the following residual functions are constructed

$$Res_1(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij}\Big\{(x^2-L^2)\phi_i(x)(\psi_j(t)+t\psi_j'(t)) - a(x,t)\psi_j(t)\Big(2\phi_i(x)+(x^2-L^2)\phi_i{}''(x) \tag{23}$$

$$+4x\phi_i'(x)\Big)+b(x,t)\Big(2x\phi_i(x)+(x^2-L^2)\phi_i'(x)\Big)\Big\}-\sum_{i=0}^{N''}\alpha_i g(x,t)\psi_i(t)+b(x,t)u_0'(x)-a(x,t)u_0''(x), \tag{24}$$

$$Res_2(t)=\sum_{i=0}^{N}\sum_{j=0}^{N'}c_{ij}t\Big\{\mu_2(t)\psi_j(t)\Delta_i^*-\mu_1(t)\psi_j(t)\Delta_i^{**}\Big\}+\mu_2(t)\mu_1(0)-\mu_1(t)\mu_2(0), \tag{25}$$

where

$$\Delta_i^*=\int_{-L}^{L}\omega_1(x)\phi_i(x)dx, \quad \Delta_i^{**}=\int_{-L}^{L}\omega_2(x)\phi_i(x)dx.$$

Collocating the residual functions $Res_1(x_i,t_j)=0$ and $Res_2(t_k^*)=0$ at the points

$$(x_i,t_j)=\Big(\frac{(2i-2-N)L}{N+2},\frac{jT}{N'+2}\Big),\ t_k^*=\frac{kT}{N''+2} \quad i=\overline{1,N+1},\ j=\overline{1,N'+1},\ k=\overline{1,N''+1}, \tag{26}$$

forms a linear system of algebraic equations

$$AC=g, \tag{27}$$

where C is the vector of unknown constants $c_{ij},\ \alpha_k$. Generally, A is an ill-conditioned matrix, therefore we require using regularization techniques to obtain stable solution. Hence, instead of Equation 27, according to the Tikhonov regularization method we solve the modified system of equations

$$(A^{tr}A+\lambda I)c=A^{tr}g, \tag{28}$$

where I is the identity matrix, A^{tr} denotes the transpose of the matrix A and $\lambda>0$ is the regularization parameter (Hansen, 1992). Therefore, the approximations of functions $v(x,t)$ and $H(t)$ are specified.

It is worthy to note that the approximation given by Equation 21 satisfies the initial and boundary conditions 17-18 exactly, provided that the compatibility conditions of Equation 6 hold. Thus by increasing the parameters N, N' and N'', if the residual functions $Res_1(x,t),\ Res_2(t)\longrightarrow 0$, then the Equations 16 and 19 are satisfied and the approximations $v_{N,N'}(x,t)$ and $H_{N''}(t)$ converge to the exact solutions $v(x,t)$ and $H(t)$, respectively.

In the following, we consider the approximation of the function $r(t)$ as

$$G_{N,N'}(t):=\frac{\int_{-L}^{L}\omega_1(x)v_{N,N'}(x,t)dx}{\mu_1(t)}, \tag{29}$$

and calculate the approximation of the unknown functions $c(t),\ d(t)$ and $u(x,t)$ in two different situations.

Case 1: Suppose that all the initial and boundary conditions 17-19 are given accurately. By substituting the approximations 22 and 29 in Equations 15 and 20, the following approximations are obtained

$$c_{approx}(t)=\frac{\frac{d}{dt}\Big(G_{N,N'}(t)\Big)}{G_{N,N'}(t)},\ d_{approx}(t)=\frac{H_{N''}(t)}{G_{N,N'}(t)},\ u_{approx}(x,t)=\frac{v_{N,N'}(x,t)}{G_{N,N'}(t)}. \tag{30}$$

Case 2: In real applications, due to the presence of inaccuracies in the input data we need to perform the regularization procedure to deal with the derivative of the perturbed data such as $G'(t)$ since it involves perturbed function $\mu_1'(t)$. Therefore, regarding the perturbed boundary data, let $\mu_1^\sigma(t)$ and $G_{N,N'}^\sigma(t)=\frac{\int_{-L}^{L}\omega_1(x)v_{N,N'}(x,t)dx}{\mu_1^\sigma(t)}$ be perturbations such that

$$\max\{\|G_{N,N'}^\sigma(t)-G(t)\|_\infty,\|\mu_1(t)-\mu_1^\sigma(t)\|_\infty\}\le\sigma.$$

Then, we employ the mollification method of (Murio, 1993) by taking into account the Gaussian mollifier $F_\delta(t) = \frac{\exp(-\frac{t^2}{\delta^2})}{\delta\sqrt{\pi}}$ where $\delta > 0$ is the radius of mollification. The mollification of the perturbed data $(G^\sigma_{N,N'}(t))'$ is performed using the convolution

$$\Big\{F_\delta * (G^\sigma_{N,N'})'\Big\}(t) := \int_{-\infty}^{+\infty} F_\delta(r)(G^\sigma_{N,N'})'(t-r)dr. \tag{31}$$

We use

$$\Big\{F_\delta * (G^\sigma_{N,N'})'\Big\}(t) = \Big\{F'_\delta * (G^\sigma_{N,N'})\Big\}(t), \tag{32}$$

such that for a given $\delta > 0$ the function $\Big\{F'_\delta * (G^\sigma_{N,N'})\Big\}(t)$ is calculated numerically using the mid-point integration rule, that is

$$\Big\{F'_\delta * (G^\sigma_{N,N'})\Big\}(t) \simeq \frac{\pi}{m_\delta}\sum_{i=0}^{m_\delta-1} Q(t, -\frac{\pi}{2} + \frac{\pi i}{m_\delta} + \frac{\pi}{2m_\delta}),\ Q(t,r) = F'_\delta(\tan r)G^\sigma_{N,N'}(t-\tan r)\sec^2 r. \tag{33}$$

Then, we consider the following

$$(G^\sigma_{N,N'})'(t) = \Big\{F'_\delta * (G^\sigma_{N,N'})\Big\}(t) \simeq \sum_{i=0}^{N''} \beta_i^{\delta,\sigma}\psi_i(t), \tag{34}$$

and consequently

$$(G^\sigma_{N,N'})(t) \simeq \sum_{i=0}^{N''} \beta_i^{\delta,\sigma}\int_0^t \psi_i(z)dz + G^\sigma_{N,N'}(0), \quad G^\sigma_{N,N'}(0) \approx \frac{\int_{-L}^{L}\omega_1(x)u_0(x)dx}{\mu_1^\sigma(0)}. \tag{35}$$

The strategy given by Equations 32-35 is admissible if for small value $\epsilon > 0$, and the appropriate given values δ and m_δ we find

$$\|\sum_{i=0}^{N''} \beta_i^{\delta,\sigma}\int_0^t \psi_i(z)dz + \frac{\int_{-L}^{L}\omega_1(x)u_0(x)dx}{\mu_1^\sigma(0)} - \frac{\int_{-L}^{L}\omega_1(x)v_{N,N'}(x,t)dx}{\mu_1^\sigma(t)}\|_\infty \le \epsilon. \tag{36}$$

If so, the approximate solution for $c(t)$ is given by

$$c_{approx}(t) = \frac{\mu_1^\sigma(t)\sum_{i=0}^{N''}\beta_i^{\delta,\sigma}\psi_i(t)}{\int_{-L}^{L}\omega_1(x)v_{N,N'}(x,t)dx}, \tag{37}$$

and the approximations of $u(x,t)$ and $d(t)$ are derived as follows

$$d_{approx}(t) = \frac{H_{N''}(t)}{G^\sigma_{N,N'}(t)},\ u_{approx}(x,t) = \frac{v_{N,N'}(x,t)}{G^\sigma_{N,N'}(t)}. \tag{38}$$

4. Numerical experiments

To test the applicability of the proposed technique, we solve two examples. The notations

$$E(u(x,t)) = |u_{exact}(x,t) - u_{approx}(x,t)|, \quad E(d(t)) = |d_{exact}(t) - d_{approx}(t)|$$

and

$$E(c(t)) = |c_{exact}(t) - c_{approx}(t)|,$$

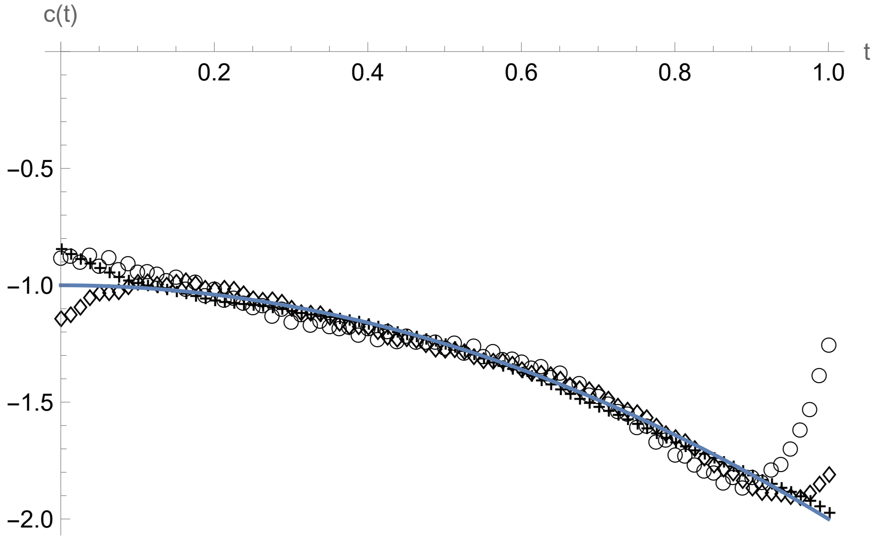

Fig. 1. Representation of the exact (blue line) and approximate solutions for $c(t)$ obtained by applying the proposed method with $N = N' = N'' = 5$ and $\lambda = 10^{-5}$, $\delta = 0.01$, $m_\delta = 600$, $\epsilon = 0.25$ in the presence of the perturbed boundary data subject to different values of σ, i.e. $+++$: corresponding to $\sigma = 1 \times 10^{-2}$, $\diamond\diamond\diamond$: corresponding to $\sigma = 3 \times 10^{-2}$, $\circ\circ\circ$: corresponding to $\sigma = 6 \times 10^{-2}$, discussed in Example 4.0.2.

Table 1. The results of l^2-norm of functions $Res_1(x,t)$ and $Res_2(t)$ and the relative root-mean square error for functions $c(t)$, $d(t)$ and $u(x,t)$ with $M = 50$, discussed in Example 4.0.1.

(N, N', N'')	$\|Res_1(x,t)\|_2$	$\|Res_2(t)\|_2$	$RRMSE(c)$	$RRMSE(d)$	$RRMSE(u)$
$(6, 6, 4)$	8.2×10^{-1}	1.3×10^{-4}	2.6×10^{-2}	1.3×10^{-3}	2.2×10^{-3}
$(8, 8, 5)$	1.9×10^{-1}	3.9×10^{-5}	2.4×10^{-3}	6.4×10^{-4}	6.4×10^{-4}
$(9, 9, 6)$	3.1×10^{-2}	8.83×10^{-7}	5×10^{-4}	8×10^{-5}	1.2×10^{-4}
$(10, 10, 7)$	6×10^{-3}	6.86×10^{-7}	7.4×10^{-5}	1.72×10^{-5}	1.3×10^{-5}

are defined as the absolute error for functions $u(x,t)$, $d(t)$ and $c(t)$ respectively. Moreover, we define the relative root-mean square error for functions $c(t)$, $d(t)$ and $u(x,t)$ as follows

$$RRMSE(c) := \sqrt{\frac{\sum_{i=0}^{M} E^2(c(\frac{iT}{M}))}{\sum_{i=0}^{M} c^2(\frac{iT}{M})}}, \quad RRMSE(d) := \sqrt{\frac{\sum_{i=0}^{M} E^2(d(\frac{iT}{M}))}{\sum_{i=0}^{M} d^2(\frac{iT}{M})}},$$

$$RRMSE(u) := \sqrt{\frac{\sum_{i,j=0}^{M} E^2(u(\frac{2Li}{M} - L, \frac{jT}{M}))}{\sum_{i,j=0}^{M} u^2(\frac{2Li}{M} - L, \frac{jT}{M})}}.$$

Throughout this work, we select the regularization parameters λ by applying the L-Curve criterion (Hansen, 1992) and find the appropriate values for δ and m_δ by trial and error. Numerical implementation is carried out with Wolfram Mathematica software in a personal computer.

Table 2. The results of the infinity norm of errors for the approximations of unknown functions $c(t),\ d(t)$ and $u(x,t)$ in the presence of exact boundary data, discussed in Example 4.0.1.

(N, N', N'')	$\|E(c(t))\|_\infty$	$\|E(d(t))\|_\infty$	$\|E(u(x,t))\|_\infty$	λ
$(6,6,4)$	6.9×10^{-2}	5.8×10^{-3}	8.5×10^{-3}	10^{-11}
$(8,8,5)$	3.61×10^{-3}	3.87×10^{-3}	4×10^{-3}	10^{-12}
$(9,9,6)$	7.1×10^{-4}	3.9×10^{-4}	8.7×10^{-4}	10^{-13}
$(10,10,7)$	5×10^{-5}	5.3×10^{-5}	5.7×10^{-5}	10^{-13}

4.0.1 Example 1

Consider the inverse problem

$$u_t - xtu_{xx} + (x^2+t^2)u_x + c(t)u = d(t)g(x,t), \quad in \quad [-1,1]\times[0,1], \tag{39}$$

where

$$g(x,t) = \sin(\pi x)e^x\Big(1+e^{-t^2}+(t^2+x^2)-tx(1-\pi^2)\Big)+\pi\cos(\pi x)e^x(t-x)^2,$$

with initial condition

$$u_0(x) = e^x\sin(\pi x), \quad -1\le x\le 1, \tag{40}$$

and homogeneous boundary conditions

$$u(-1,t) = u(1,t) = 0, \quad 0\le t\le 1, \tag{41}$$

and overspecifications

$$\int_{-1}^{1}(1-x^2)u(x,t)dx = \frac{2\pi e^{t-1}\Big(5+\pi^2+e^2(-1+3\pi^2)\Big)}{(1+\pi^2)^3}, \tag{42}$$

and

$$\int_{-1}^{1}x^2(x^2-1)u(x,t)dx = \frac{-2\pi e^{t-1}\Big(125-89\pi^2-25\pi^4-3\pi^6+e^2(-25+101\pi^2-59\pi^4+7\pi^6)\Big)}{(1+\pi^2)^5}. \tag{43}$$

The exact solutions of this problem are

$$c(t) = e^{-t^2},\ d(t) = e^t,\ u(x,t) = e^{t+x}\sin(\pi x).$$

We solve the problem by applying the numerical scheme discussed in Section 3 in the presence of exact boundary data and use the approximations given by Equation 30. The results for relative root-mean square error for functions $c(t)$, $d(t)$ and $u(x,t)$ together with l^2-norm of fuctions $Res_1(x,t)$ and $Res_2(t)$ are presented in Table 1. Moreover, in Tables 2-3 we report the infinity norm and l^2-norm of errors for the approximations of unknown functions $c(t)$, $d(t)$ and $u(x,t)$ per different number of basis functions which indicate that the accuracy is improved by increasing the number of basis functions.

Table 3. The results of the l^2-norm of errors for the approximations of unknown functions $c(t),\ d(t)$ and $u(x,t)$ in the presence of exact boundary data, discussed in Example 4.0.1.

(N, N', N'')	$\|E(c(t))\|_2$	$\|E(d(t))\|_2$	$\|E(u(x,t))\|_2$	λ
$(6, 6, 4)$	2×10^{-2}	2.4×10^{-3}	4.9×10^{-3}	10^{-11}
$(8, 8, 5)$	1.81×10^{-3}	1.07×10^{-3}	1.4×10^{-3}	10^{-12}
$(9, 9, 6)$	3.8×10^{-4}	1.3×10^{-4}	2.7×10^{-4}	10^{-13}
$(10, 10, 7)$	5.2×10^{-5}	2.8×10^{-5}	3×10^{-5}	10^{-13}

Table 4. The results of the infinity norm of errors for the approximations of unknown functions $c(t),\ d(t)$ and $u(x,t)$ in the presence of exact boundary data, discussed in Example 4.0.2.

(N, N', N'')	$\|E(c(t))\|_\infty$	$\|E(d(t))\|_\infty$	$\|E(u(x,t))\|_\infty$	λ
$(4, 4, 4)$	0.051	0.056	0.024	10^{-4}
$(6, 6, 5)$	0.0034	0.0027	0.0009	10^{-6}
$(8, 8, 6)$	0.0001	0.00067	0.00021	10^{-9}
$(10, 10, 7)$	2×10^{-6}	1.1×10^{-7}	1.6×10^{-5}	10^{-11}

Table 5. The results of l^2-norm of functions $Res_1(x,t)$ and $Res_2(t)$ and the relative root-mean square error for functions $c(t)$, $d(t)$ and $u(x,t)$ with $M = 50$, discussed in Example 4.0.2.

(N, N', N'')	$\|Res_1(x,t)\|_2$	$\|Res_2(t)\|_2$	$RRMSE(c)$	$RRMSE(d)$	$RRMSE(u)$
$(4, 4, 4)$	1.9×10^{-1}	2.54×10^{-3}	2.4×10^{-2}	7.2×10^{-3}	2×10^{-3}
$(6, 6, 5)$	1.8×10^{-2}	5.2×10^{-5}	1.51×10^{-3}	3.3×10^{-4}	1.4×10^{-4}
$(8, 8, 6)$	1.09×10^{-3}	1.6×10^{-5}	6×10^{-4}	6.7×10^{-5}	1.2×10^{-5}
$(10, 10, 7)$	5.03×10^{-5}	1.58×10^{-6}	2.24×10^{-6}	6.9×10^{-6}	7.7×10^{-7}

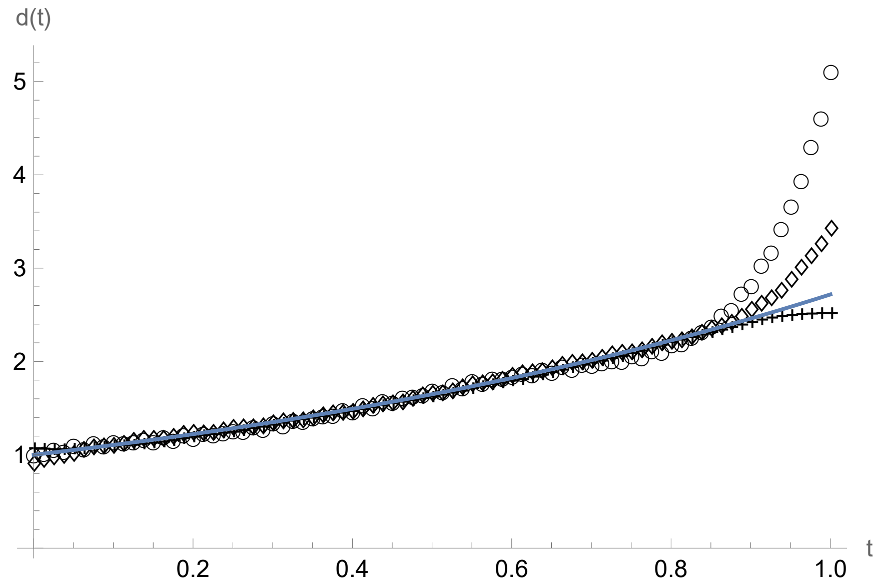

Fig. 2. Representation of the exact (blue line) and approximate solutions for $d(t)$ obtained by applying the proposed method with $N = N' = N'' = 5$ and $\lambda = 10^{-5}$, $\delta = 0.01$, $m_\delta = 600$, $\epsilon = 0.25$ in the presence of the perturbed boundary data subject to different values of σ, i.e. $+++$: corresponding to $\sigma = 1 \times 10^{-2}$, $\diamond\diamond\diamond$: corresponding to $\sigma = 3 \times 10^{-2}$, $\circ\circ\circ$: corresponding to $\sigma = 6 \times 10^{-2}$, discussed in Example 4.0.2.

4.0.2 Example 2

Consider (Shekarpaz & Azari, 2018) the problem given by Equations 1-5 defined over the bounded domain $Q = [-1, 1] \times [0, 1]$ with the following properties:

$$a(x,t) = 1,\ b(x,t) = 1,\ g(x,t) = -2t + (\pi^2 - 2t)\cos(\pi x) + t(2-t)(1+\cos(\pi x)), \tag{44}$$

$$u_0(x) = 1 + \cos(\pi x),\ \omega_1(x) = 1 + x^2,\ \omega_2(x) = 1 - x,\ \mu_1(t) = (\frac{8}{3} - \frac{4}{\pi^2})e^t,\ \mu_2(t) = 2e^t, \tag{45}$$

and the exact solutions

$$c(t) = -1 - t^2,\ d(t) = e^t,\ u(x,t) = e^t(\cos(\pi x) + 1).$$

By using the approximations 30 presented in Section 3 with different values $N,\ N',\ N''$, we produce the results tabulated in Tables 4-5. From the numerical findings it can be seen that the infinity norm of errors as well as the relative root-mean square errors are decreased as the number of basis functions increases gradually which indicate that our method is convergent. Next, we study the numerical stability of the solution with respect to the boundary conditions. Thus, we generate the perturbed boundary data using the following rules (Kirsch, 2011)

$$\mu_1^\sigma(t) = \mu_1(t) + \sigma\sin(\frac{t}{\sigma^2}), \quad \sigma = r \times 10^{-2},\ r \in \mathbf{N}, \tag{46}$$

$$\mu_2^\sigma(t) = \mu_2(t) + \sigma\sin(\frac{t}{\sigma^2}), \quad \sigma = r \times 10^{-2},\ r \in \mathbf{N}. \tag{47}$$

By employing the investigated method with $N = N' = N'' = 5$ and $\sigma \in \{1, 3, 6\} \times 10^{-2}$ and taking the approximations 37 and 38, we obtain the results as shown in Figures 1-2. From the illustrations, it can be seen that the performance of the method is good and the proposed technique finds the stable solution while the amount of noise tends to zero. Indeed, the fair agreement between the exact and approximate solutions holds since the errors imposed to the additional data and propagated with the approximations are of the same order.

5. Conclusion

This article gives a stable numerical solution of an inverse coefficient problem in the one-dimensional heat equation from integral overdetermination conditions. By utilizing new variables, the main problem is converted to a problem of reconstructing an unknown heat source from one additional measurement. We propose a Ritz approximation as the solution of the unknown temperature distribution and consider some truncated series as the approximation of unknown time-dependent function in the heat source. Then, the collocation technique is employed to reduce the inverse problem to the solution of algebraic equations. We take advantage of the mollification method to derive the stable numerical derivatives and solve the ill-conditioned system of equations by using the Tikhonov regularization technique in order to obtain the stable solutions. Following the numerical simulations, it is confirmed that our method proposes a robust approach in dealing with introduced artificial errors in the input boundary data and performs quite well in the presence of exact boundary data since the approximate solutions converge to the exact solutions numerically. Compared to the results presented in (Shekarpaz & Azari, 2018), it can be observed that the algorithm proposed in the present paper yields better results because of providing higher accuracy with lower computational cost. This technique can be extended to solve similar problems in higher dimensions.

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