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An optimal fourth-order second derivative free iterative method for nonlinear scientific equations

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Abstract

In the present paper, we develop an efficient second derivative free two-step optimal fourth-order iterative method for nonlinear equations. We explore the convergence criteria of the proposed method and also exhibit its validity and efficiency by considering some test problems. We present both numerical as well as graphical comparisons. Further, the dynamical behavior of the proposed method is explored.

Keywords: Approximation scheme; iterative methods; nonlinear equations; order of convergence; polynomiography.

1. Introduction

In everyday life, most of the phenomena lead to some kind of nonlinear equations. In general, these nonlinear equations cannot be dealt analytically or for exact solutions. So, naturally, the scientists focused on numerical methods for solving such type of equations, (Dogan, 2013; Uddin & Imdad, 2015; Bayat *et al.*, 2015; Karakaya *et al.*, 2016; Demiray & Bulut, 2017; Al-jawary & Nabi, 2020; Ozer, 2021; Eze, 2022). Particularly, finding the numerical solution of the nonlinear equation

$$f(x) = 0, \tag{1}$$

has been a sweltering problem in the fields of science and engineering, e.g.(Babolian & Baizar, 2002; Abbasbandy, 2003; Bhalekar & Daftardar-Gejji, 2011; Chun, 2018; He, 2016; He *et al.*, 2020; He *et al.*, 2021; He & El-Dib, 2020) and references therein. In the construction and performance of a numerical method for nonlinear equations, its convergence order and the number of evaluations per iteration are significant.

Definition 1. Let $\{x_n\}$ be the sequence of approximations that converges to the root α of $f(x) = 0$ i.e. $\lim_{n\to\infty} x_n = \alpha$. If there exist positive real numbers p and k such that $\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = k$, then p is called the order of convergence of the method.

Definition 2. If p is the convergence order of an iterative method and m denotes the number of function evaluations per iteration, then the efficiency index (E.I) of the method is $p^{\frac{1}{m}}$.

One of the most significant and well-known techniques, for solving nonlinear equations, is the Newton's method which converges quadratically, Ostrowski, (1973):

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right), \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \ldots . \tag{2}$$

Noor & Gupta, (2007) modified Householder iterative method and developed the following fourth-order method which requires four evaluations per iteration i.e. its EI = 1.4141.

$$x_{n+1}=y_n-\left(\frac{f(y_n)}{f'(y_n)}\right)-\left(\frac{1}{2}\right)\left[\left(\frac{f(y_n)}{f'(y_n)}\right)\right]^2\left[\left(\frac{f'(x_n)}{f(x_n)}\right)\right]\left[\left(\frac{f'(x_n)+f'(y_n)}{f'(y_n)}\right)\right],\ n=0,1,2,\ldots, \tag{3}$$

where, $y_n=x_n-\left(\frac{f(x_n)}{f'(x_n)}\right), \qquad f'(x_n)\neq 0, \qquad n=0,1,2,\ldots$.

An immense literature is available regarding third-order and fourth-order iterative methods for solving nonlinear equations, (Chun, 2007; Herceg & Herceg, 2008; Saeed & Khthr, 2011; Thukral, 2013; Singh & Jaiswal, 2014; Jaiswal, 2014; Ali *et al.*, 2018; Huang *et al.*, 2018; Naseem *et al.*, 2020; Sana *et al.* 2021) and references therein.

Keeping in view the importance of convergence order and the number of evaluations per iteration required in an iterative method, Kung & Turab, (1974) gave a conjecture. According to this conjecture, an iterative method is said to be an optimal one if it needs $(n+1)$ evaluations per iteration and posses convergence order 2^n. Some useful optimal fourth-order iterative methods have been constructed by various researchers (Sharma *et al.*, 2020; Ali *et al.*, 2020; Shams *et al.*, 2020; Cordero *et al.*, 2021; Hafiz & Khirallah, 2021).

Cordero *et al.*, (2010) introduced the following optimal fourth-order method:

$$x_{n+1}=x_n-\left(\frac{f(x_n)+f(y_n)}{f'(x_n)}\right)-\left[\frac{f(y_n)}{f'(x_n)}\right]^2\left[\frac{2f(x_n)+f(y_n)}{f'(x_n)}\right], \tag{4}$$

where, $y_n=x_n-\left(\frac{f(x_n)}{f'(x_n)}\right), \qquad f'(x_n)\neq 0, \qquad n=0,1,2,\ldots$.

Obviously, the above method is of order $2^2=4$ and requires $(2+1)=3$ evaluations per iteration, so it is an optimal fourth-order iterative method with EI = 1.5874 .

Sherma & Bahl, (2015) also developed an optimal fourth-order method:

$$x_{n+1}=x_n-\left[-\left(\frac{1}{2}\right)+\left(\frac{9f'(x_n)}{8f'(y_n)}\right)+\left(\frac{3f'(y_n)}{8f'(x_n)}\right)\right]\left(\frac{f(x_n)}{f'(x_n)}\right), \qquad n=0,1,2,\ldots, \tag{5}$$

where, $y_n=x_n-\left(\frac{2}{3}\right)\left(\frac{f(x_n)}{f'(x_n)}\right), \qquad f'(x_n)\neq 0, \qquad n=0,1,2,\ldots$.

A second derivative free optimal fourth-order method was introduced by Shengfeng Li, (2019):

$$x_{n+1}=x_n-\left(\frac{[f(x_n)-f(y_n)]f(x_n)}{[f(x_n)-2f(y_n)]f'(x_n)}\right), \qquad n=0,1,2,\ldots, \tag{6}$$

where, $y_n=x_n-\left(\frac{f(x_n)}{f'(x_n)}\right), \qquad f'(x_n)\neq 0, \qquad n=0,1,2,\ldots$.

In this paper, being inspired from the literature regarding optimal iterative methods for nonlinear equations, we present a rapidly convergent and efficient optimal fourth-order iterative method. In order to demonstrate the validity and effectiveness of the proposed method, we explore the numerical as well as graphical comparisons. We also consider some complex polynomials to study the dynamics of the suggested method. We present four polynomiographs against four different polynomials. These polynomiographs clearly exhibit the corresponding roots along with the region of convergence according to the chosen initial guess.

2. Construction of Iterative Method

Noor *et al.*, (2007) presented the following sixth-order iterative method which involves the second derivative of the function and needs five evaluations per iteration, i.e. its EI = 1.4309.

$$x_{n+1}=y_n-\frac{f(y_n)}{f'(y_n)}-\left(\frac{(f(y_n))^2 f''(y_n)}{2\left(f'(y_n)\right)^3}\right), \tag{7}$$

where, $y_n=x_n-\left(\frac{f(x_n)}{f'(x_n)}\right)$, $\quad f'(x_n)\neq 0, \quad n=0,1,2,\ldots$.

We consider the following interpolation scheme, Rehman *et al.*, (2021):

$$H(t)=a+b(t-y_k)+c(t-y_k)^2+d(t-y_k)^3, \tag{8}$$

where a, b, c and d are unknowns and can be determined by applying the following conditions:

$$H(x_k)=f(x_k),\ H(y_k)=f(y_k),\ H'(x_k)=f'(x_k),\ H'(y_k)=f'(y_k),\ H''(y_k)=f''(y_k). \tag{9}$$

Using the above conditions, we obtain the following system of equations:

$$f(y_k)=a, \tag{10}$$

$$f(x_k)=a+b(x_k-y_k)+c(x_k-y_k)^2+d(x_k-y_k)^3, \tag{11}$$

$$f'(x_k)=b+2c(x_k-y_k)+3d(x_k-y_k)^2, \tag{12}$$

$$f'(y_k)=b, \tag{13}$$

$$f''(y_k)=2c+6d(x_k-y_k). \tag{14}$$

Solving equations (10)-(14), simultaneously, we obtain

$$f''(y_k)=\frac{6[f(x_k)-f(y_k)]-2(x_k-y_k)[2f'(x_k)+f'(y_k)]}{(x_k-y_k)^2}=P(x_k, y_k). \tag{15}$$

Let us now consider the following interpolation scheme:

$$G(t)=a+b(t-y_k)+c(t-y_k)^2, \tag{16}$$

where a, b and c are the unknowns, which can be determined by applying the following conditions:

$$G(x_k)=f(x_k),\ G(y_k)=f(y_k),\ G'(x_k)=f'(x_k),\ G'(y_k)=f'(y_k). \tag{17}$$

Using the above conditions, we obtain the following system of equations:

$$f(y_k) = a, \tag{18}$$

$$f(x_k) = a + b(x_k - y_k) + c(x_k - y_k)^2, \tag{19}$$

$$f'(x_k) = b + 2c(x_k - y_k), \tag{20}$$

$$f'(y_k) = b. \tag{21}$$

Solving equations (18)-(21), simultaneously, we get

$$f'(y_k) = \frac{2[f(x_k) - f(y_k)]}{(x_k - y_k)} - f'(x_k) = Q(x_k, y_k). \tag{22}$$

From equations (15) and (22), we get

$$f''(y_k) = \frac{6[f(x_k) - f(y_k)] - 2(x_k - y_k)[2f'(x_k) + Q(x_k, y_k)]}{(x_k - y_k)^2} = R(x_k, y_k). \tag{23}$$

Thus, using equations (7), (22) and (23), we are in a position to formulate the following optimal fourth-order second derivative free iterative method for solving nonlinear equation (1).

2.1 Algorithm For a given x_0, compute the approximate solution $x_{n+1}$by the following iterative scheme:

$$x_{n+1} = y_n - \frac{f(y_n)}{Q(x_n, y_n)} - \left(\frac{f^2(y_n)R(x_n, y_n)}{2Q^3(x_n, y_n)}\right), \tag{24}$$

where, $y_n = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right)$, $\quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \ldots$.

3. Convergence Analysis

The convergence criteria for the newly proposed iterative method i.e. algorithm 2.1 is described in the following theorem.

3.1 Theorem

Assume that the function $f: I \subset \mathbb{R} \to \mathbb{R}$ (where I is an open interval) has a simple root $\alpha \in I$. If $f(x)$ is a sufficiently differentiable function in the neighborhood of α, then the method given in algorithm 2.1 has the convergence order at least 4.

Proof Since $f(x)$ is sufficiently differentiable, therefore, the Taylor's series expansions of $f(x_n)$ and $f'(x_n)$ about α are given by:

$$f(x_n) = f'(\alpha)\left\{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_2 e_n^5 + O\left(e_n^6\right)\right\}, \tag{25}$$

and

$$f'(x_n) = f'(\alpha)\left\{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O\left(e_n^6\right)\right\}, \tag{26}$$

where, $e_n = x_n - \alpha$ *and* $c_j = \left(\frac{1}{j!}\right)\left(\frac{f^{(j)}(\alpha)}{f'(\alpha)}\right)$, $\quad j = 2, 3, \ldots$.

From equations (25) and (26), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right) e_n^3 + \left(-4c_2^3 + 7c_2c_3 - 3c_4\right) e_n^4 + (8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + O\left(e_n^6\right). \tag{27}$$

Using equation (27), we get

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 c_n^2 - 2\left(c_2^2 - c_3\right) e_n^3 + \left(4c_2^3 - 7c_2c_3 + 3c_4\right) e_n^4 \\ &\quad + (-8c_2^4 + 20c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{28}$$

Using equation (28), the Taylor's series of $f(y_n)$ is given by

$$f(y_n) = c_2 e_n^2 - 2\left(c_2^2 - c_3\right) e_n^3 + \left(5c_2^3 - 7c_2c_3 + 3c_4\right) e_n^4 + (-12c_2^4 + 24c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + O\left(e_n^6\right). \tag{29}$$

Using equations (25), (26), (28) and (29), we get

$$Q(x_n, y_n) = 1 + \left(2c_2^2 - c_3\right) e_n^2 + \left(-4c_2^3 + 6c_2c_3 - 2c_4\right) e_n^3 + (8c_2^4 - 16e_2^2c_3 + 8c_2c_4 + 4c_3^2 - 3c_5)e_n^4 + O\left(e_n^5\right). \tag{30}$$

Using equations (25), (26), (28) and (29), we get

$$R(x_n, y_n) = 2c_2 + 4c_3 e_n + (2c_2c_3 + 6c_4) e_n^2 + \left(-4c_2^2c_3 + 4c_2c_4 + 4c_3^2 + 8c_5\right) e_n^3 + O\left(e_n^4\right). \tag{31}$$

Thus, using equations (28)-(31), the error term for algorithm 2.1 becomes:

$$e_{n+1} = -c_2c_3 e_n^4 + \left(2c_2^2c_3 - 2c_2c_4 - 2c_3^2\right) e_n^5 + O\left(e_n^6\right). \tag{32}$$

This completes the proof.

4. Numerical Examples

In order to exhibit the validity and effectiveness of the newly proposed optimal fourth-order iterative method given in algorithm 2.1 (AM), we compare the same, numerically as well as graphically, with the standard Newton's method (NM), Sharma & Bahl, (2015) method (equation 5) (RM), Noor & Gupta, (2007) method (equation 3) (HM), Cardero *et al.*, (2010) method (equation 4) (CM), recently developed method by Shengfeng Li, (2019) (equation 6) (LM) and Noor *et al.*, (2000) method (equation 7) (NR) in the context of standard nonlinear equations. The numerical comparison, by using the software Maple-18, is presented in the following table, in which NFE column represents the number of function evaluations, whereas figure 1 to figure 8 exhibit the graphical comparison, performed with the help of Matlab software. In order to stop the iteration process, we use the condition $|x_{n+1} - x_n| < \varepsilon$ with $\varepsilon = 10^{-15}$. Both comparative studies clearly indicate that the newly developed method performs more efficiently.

Table 1: Numerical Examples

$f(x)$	x_0	*Method*	n	x_k	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	**NFE**
x^3-x-8	0.5	NM	21	2.1663127473977890	2.906126e^{-16}	1.213442e^{-10}	42
		RM	7	2.1663127473977890	2.906126e^{-16}	3.999760e^{-10}	28
		HM	14	2.1663127473977890	2.906126e^{-16}	1.721625e^{-7}	56
		LM	14	2.1663127473977890	2.906126e^{-16}	1.604659e^{-8}	42
		CM	50	-22713.26588077619	2.906126e^{-16}	2.264950e^{4}	150
		NR	8	2.1663127473977890	2.906126e^{-16}	8.622847e^{-9}	40
		AM	6	2.1663127473977890	2.906126e^{-16}	2.998342e^{-10}	18
$x^3 - e^{(sinx)}-1.3$	0.43	NM	13	1.5897513629099752	1.028154e^{-16}	1.597951e^{-9}	26
		RM	7	1.5897513629099752	1.028154e^{-16}	5.554208e^{-12}	21
		HM	6	1.5897513629099752	1.028154e^{-16}	2.063160e^{-10}	24
		LM	21	1.5897513629099752	1.028154e^{-16}	3.616346e^{-6}	23
		CM	41	1.5897513629099752	2.187217e^{-15}	1.130202e^{-4}	123
		NR	6	1.5897513629099752	1.028154e^{-16}	1.769922e^{-14}	30
		AM	5	1.5897513629099752	1.028154e^{-16}	1.728701e^{-12}	15
$x^3+sinx-0.5$	-.70	NM	11	0.4324702259081946	1.486126e^{-18}	1.290499e^{-14}	22
		RM	20	0.4324702259081946	1.486126e^{-18}	1.379866e^{-12}	60
		HM	5	0.4324702259081946	1.486126e^{-18}	1.284167e^{-8}	20
		LM	5	0.4324702259081946	1.486126e^{-18}	2.552682e^{-6}	15
		CM	6	0.4324702259081946	1.486126e^{-18}	1.626361e^{-6}	18
		NR	5	0.4324702259081946	1.486126e^{-18}	7.327305e^{-15}	25
		AM	4	0.4324702259081946	1.486126e^{-18}	9.966395e^{-08}	12
$x^4-2tan^{-1}(x)$-1	-5.5	NM	11	-0.5048496838915417	1.013082e^{-17}	4.613243e^{-11}	22
		RM	43	-0.5048496838915417	1.095395e^{-17}	5.949597e^{-9}	129
		HM	5	-0.5048496838915417	1.013082e^{-17}	2.401892e^{-13}	20
		LM	6	-0.5048496838915417	1.013082e^{-17}	9.637054e^{-15}	18
		CM	6	-0.5048496838915417	1.095395e^{-17}	5.956829e^{-05}	18
		NR	5	-0.5048496838915417	1.013082e^{-17}	2.898906e^{-11}	25
		AM	4	-0.5048496838915417	1.013082e^{-17}	1.850485e^{-11}	12
$x^5+xsin(x-1)$	-.65	NM	17	0.7230912060028413	8.893281e^{-18}	5.426442e^{-11}	34
		RM	9	0.7230912060028413	8.893281e^{-18}	1.796728e^{-14}	36
		HM	7	0.7230912060028413	8.893281e^{-18}	3.322140e^{-07}	28
		LM	7	0.7230912060028413	8.893281e^{-18}	3.440059e^{-07}	21
		CM	50	227793617.43987802	6.133491e^{41}	1.050057e^{08}	150
		NR	7	0.7230912060028413	8.893281e^{-18}	9.934670e^{-6}	35
		AM	5	0.7230912060028452	6.950625e^{-15}	7.280912e^{-06}	15

$f(x)$	x_0	*Method*	n	x_k	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	**NFE**
x^3+x+e^x+5	9.00	NM	14	-1.5426515636094549	2.718112e^{-15}	2.413960e^{-8}	28
		RM	25	-1.5426515636094549	2.121712e^{-16}	3.274434e^{-12}	75
		HM	6	-1.5426515636094549	2.121712e^{-16}	1.070996e^{-6}	24
		LM	7	-1.5426515636094549	5.224052e^{-15}	2.834313e^{-4}	21
		CM	8	-1.5426515636094549	2.121712e^{-16}	8.797138e^{-15}	24
		NR	6	-1.5426515636094549	2.121712e^{-16}	8.510917e^{-9}	30
		AM	5	-1.5426515636094549	2.121712e^{-16}	2.213423e^{-12}	15
x^3+4x^2+1	0.7	NM	21	-4.0606470275541425	2.121712e^{-16}	1.345687e^{-08}	42
		RM	7	-4.0606470275541425	7.712486e^{-16}	1.241081e^{-13}	28
		HM	12	-4.0606470275541425	7.712486e^{-16}	2.098410e^{-10}	48
		LM	32	-4.0606470275541425	7.712486e^{-16}	1.607614e^{-12}	96
		CM	27	-4.0606470275541425	7.712486e^{-16}	6.873173e^{-12}	81
		NR	12	-4.0606470275541425	7.712486e^{-16}	1.353735e^{-10}	60
		AM	6	-4.0606470275541425	7.712486e^{-16}	4.036553e^{-15}	18
$x^{10}-1$	2.50	NM	14	1.0000000000000000	0.000000e^{0}	1.203329e^{-14}	28
		RM	50	390114007548790110	8.164234e^{645}	1.334660e^{64}	150
		HM	6	1.0000000000000000	0.000000e^{0}	2.197544e^{-11}	24
		LM	6	1.0000000000000000	0.000000e^{0}	1.922172e^{-05}	18
		CM	8	1.0000000000000000	0.000000e^{0}	9.125819e^{-09}	24
		NR	6	1.0000000000000000	0.000000e^{0}	5.712945e^{-11}	30
		AM	5	1.0000000000000000	0.000000e^{0}	7.647115e^{-09}	15

In the above Table 1, we consider 8 examples involving different types of functions i.e. trigonometric functions, inverse trigonometric functions and exponential function. It is notable that, in each case, proposed method converges to the actual root in least number of iterations with minimum number of function evaluations.
The following figures i.e. figure 1 to figure 8 exhibit the plots of number of iterations against log of residuals in the context of each example considered in Table 1. Clearly, the proposed method converges in least number of iterations.

4.1 Graphical Comparison

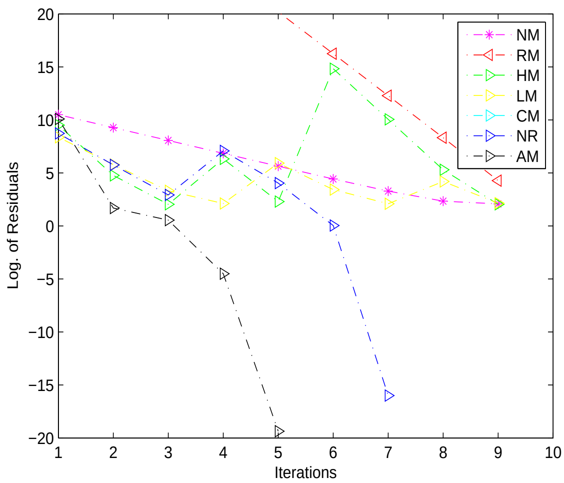

Fig. 1. $(f(x) = x^2 - x - 8)$

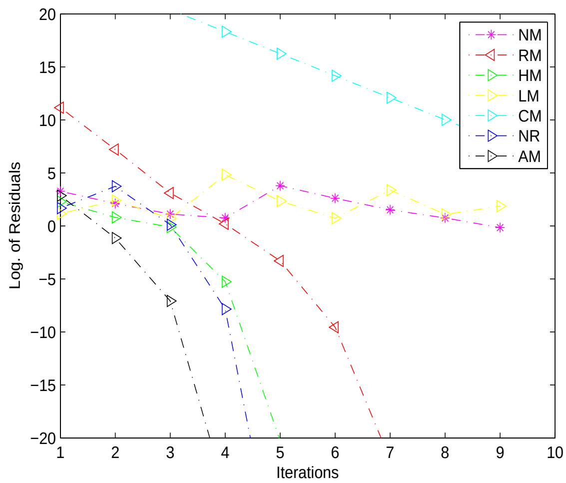

Fig. 2. $(f(x) = x^3 - e^{sinx} - 1.3)$

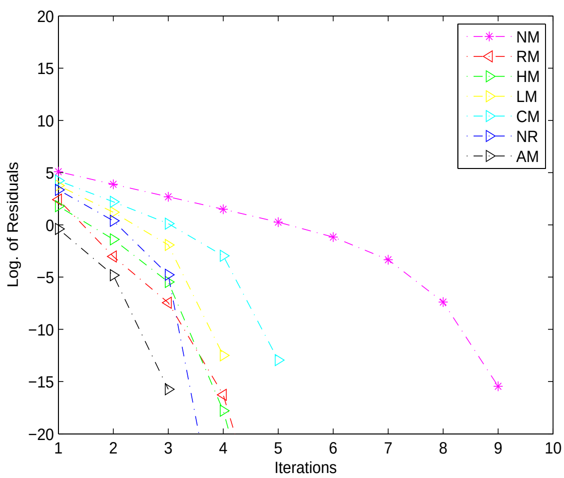

Fig. 3. $(f(x) = x^3 + sinx - 0.5)$

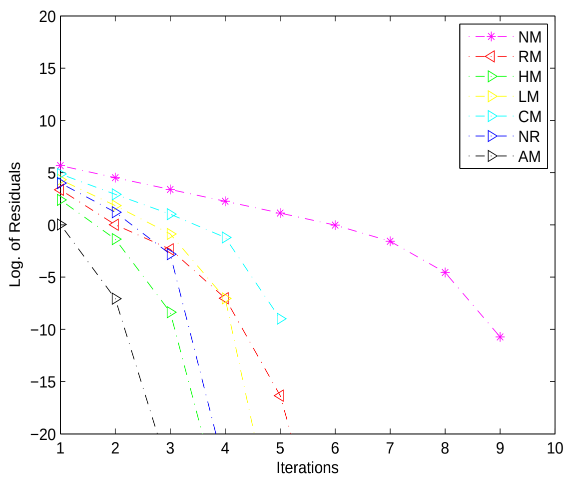

Fig. 4. $(f(x) = x^4 - 2tan^{-1}(x) - 1)$

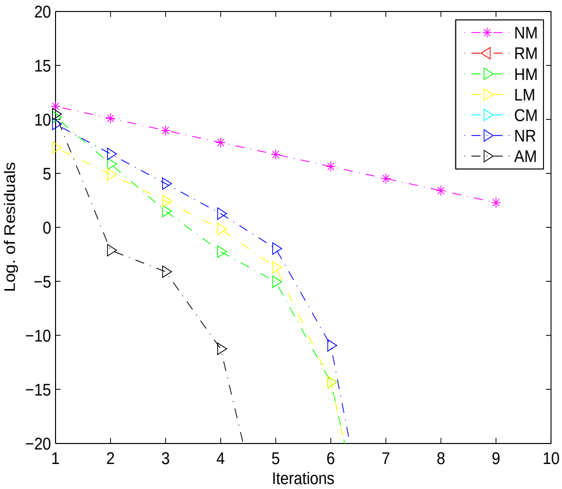

Fig. 5. $(f(x) = x^5 + xsin(x-1))$

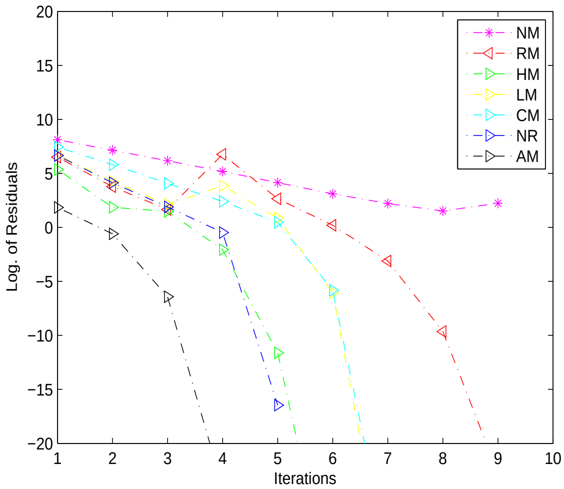

Fig. 6. $(f(x) = x^3 + x + e^x + 5)$

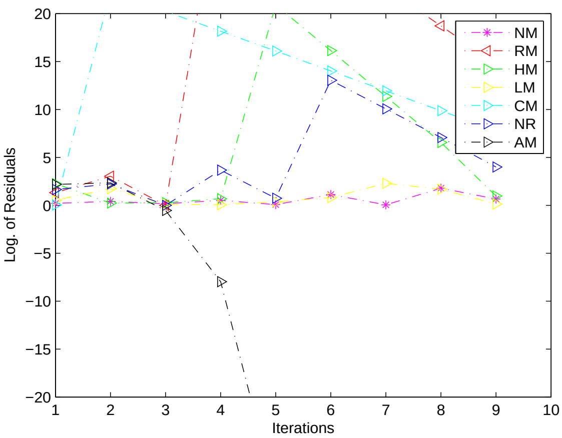

Fig. 7. $(f(x) = x^3 + 4x^2 + 1)$

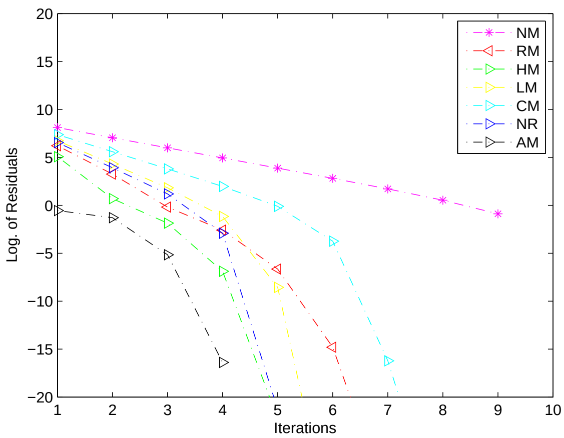

Fig. 8. $(f(x) = x^{10} - 1)$

5. Polynomiography

Kalantary, 2009, introduced the concept of polynomiography, which is an art and science of visualization of the zeros of complex polynomials through fractal and non-fractal images obtained by using the convergence properties of iteration functions. Through polynomiography, nice looking graphics are generated. An individual image is known as polynomiograph. Polynomiography is a modern technique to solve problems with the help of computer technology. It has vast and diverse applications in science, art, design, industry; especially in textile industry. Fundamental theorem of algebra describes that a polynomial of degree $n \geqslant 1$ has n roots. In the study of polynomiography, the degree of a polynomial describes the number of basins of attraction. The colors of polynomiograph indicate the number of iterations required to achieve the approximate root of a certain polynomial with a given accuracy corresponding to a chosen initial guess. For further description and applications of polynomiography, we refer (Kalantary, 2005a; Kalantary, 2005b; Kotarski *et al.*, 2012) The following figures i.e. figure 9 to figure 12 display the polynomiographs and basins of attraction of some standard complex polynomials in the context of newly proposed method. We have used Matlab software for the purpose.

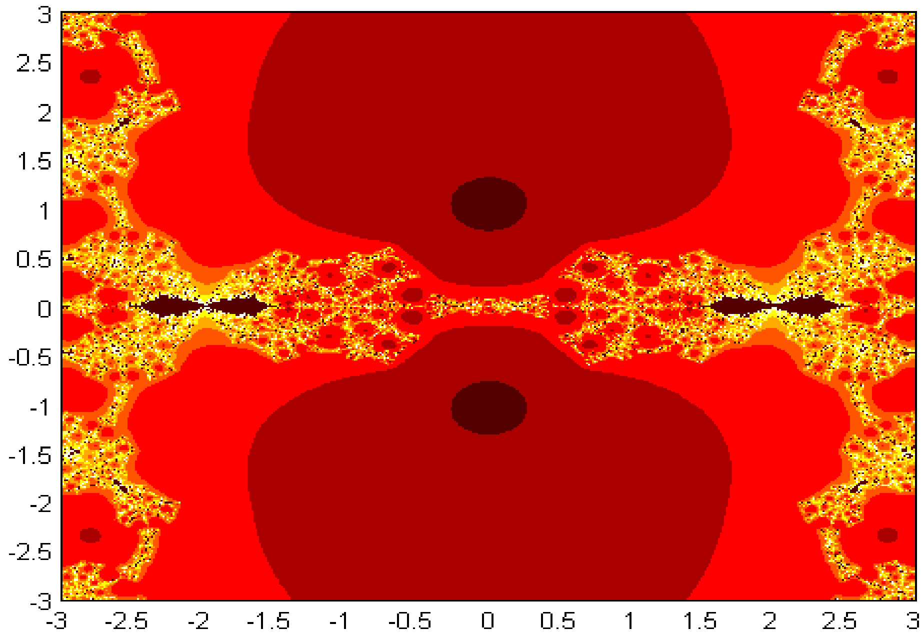

Fig. 9. Polynomiograph of z^2+1

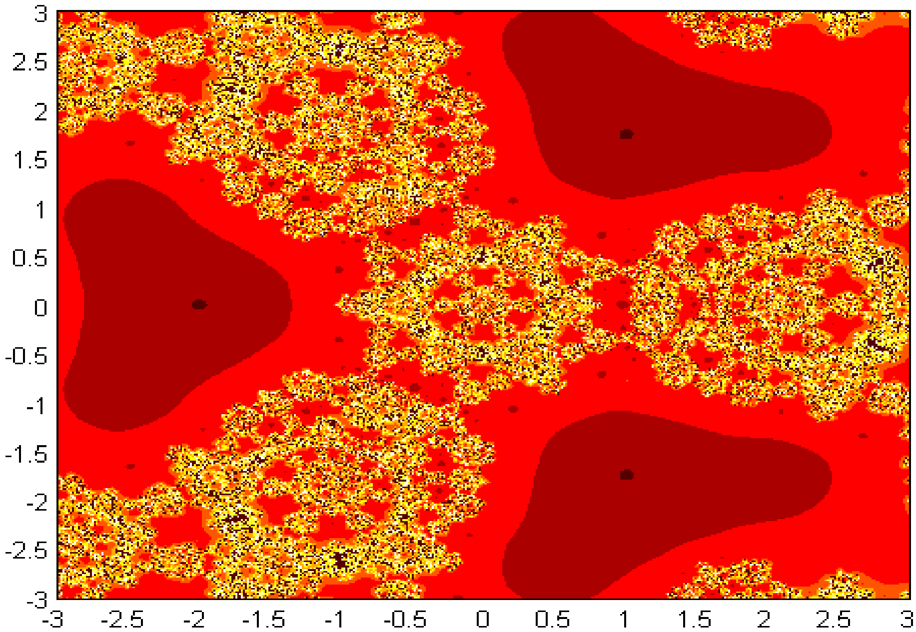

Fig. 10. Polynomiograph of z^3+8

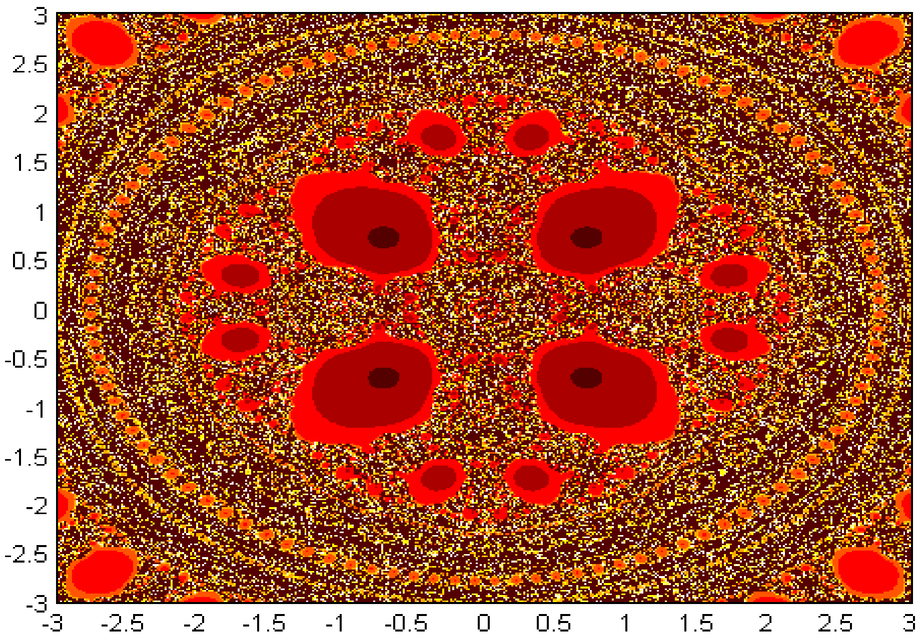

Fig. 11. Polynomiograph of z^4+1

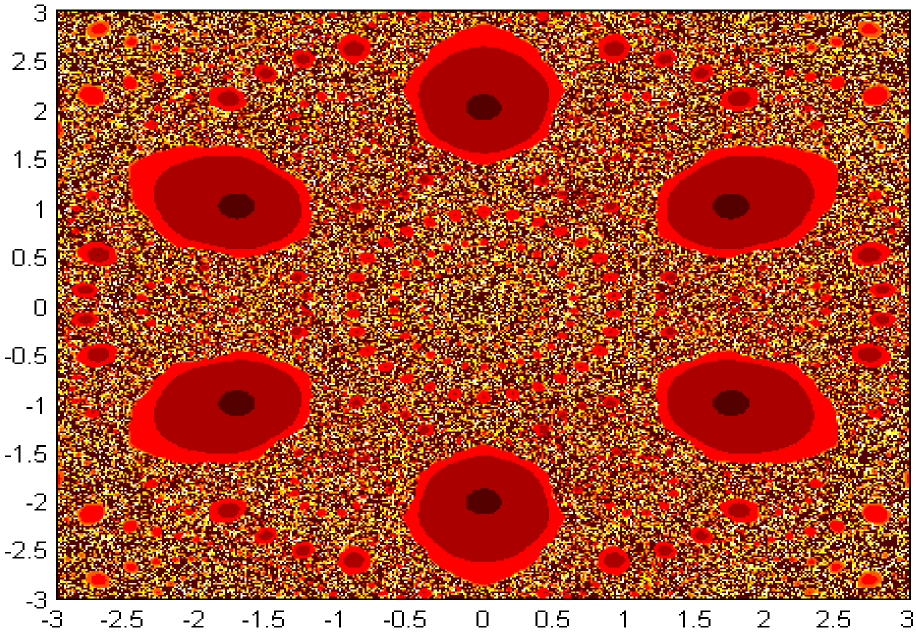

Fig. 12. Polynomiograph of z^6+64

6. Conclusion

A second derivative free optimal fourth-order iterative method has been established in this paper. The numerical and graphical comparisons clearly indicate that the newly constructed method performs efficiently with least computational cost compared to other existing iterative methods. Dynamical behavior of the developed method has also been explored to envisage the visualization of the roots of complex polynomials and is of significant interest.

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