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Characteristic properties of the ruled surface with Darboux frame in $\mathbb{E}^3$

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ABSTRACT

In this study, the ruled surface with Darboux frame is defined. Then, the ruled surfaces characteristic properties which are related to the geodesic curvature, the normal curvature and the geodesic torsion are investigated. The relation between the Darboux frame and the Frenet frame on the ruled surface is presented. Moreover, some theorems about the pitch and the angle of the pitch which are the integral invariants of the ruled surface with darboux frame are given.

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INTRODUCTION

In differential geometry, ruled surface is a special type of surface, which can be defined by choosing a curve and a line along that curve. The ruled surfaces are one of the easiest of all surfaces to parametrize. These surface were found and investigated by Gaspard Monge who established the partial differential equation that satisfies all ruled surface. Hlavaty (1945) and Hoschek (1973) also investigated ruled surfaces which are formed by one parameter set of lines. In addition, H. R. Müller (1978) showed that the pitch of closed ruled surfaces are integral invariants.

From past to today, many properties of the ruled surface and their integral invariants have been examined in Euclidean and non-Euclidean spaces; for example in (Ravani & Ku, 1991; Hacısalihoğlu, 1994; Sarıoğlugil & Tutar 2007; Gray *et al*., 2006; Aydemir & Kasap, 2005). Moreover, an application of ruled surfaces is of the nature that they are used in civil engineering, computer programming, architecture and solid modelling.

Another one of the most important subjects of the differential geometry is the Darboux frame which is a natural moving frame constructed on a surface. It is the version of the Frenet frame as applied to surface geometry. A Darboux frame exists on a surface in Euclidean or non-Euclidean spaces. It is named after the French mathematician Jean Gaston Darboux, in four volume collection of the studies he published between 1887 and 1896. Since that time, there have been many important repercussions of Darboux frame, having been examined for example in (Darboux, 1896; O'Neill, 1996).

In this study, the ruled surface with Darboux frame $\mathbb{E}^3$ in is taken into consideration. Furthermore, we present the characterization of the ruled surface related to the geodesic curvature, the normal curvature and the geodesic torsion by using the Darboux frame. In the following section, the integral invariants of this surface are examined and some special cases of the rulings are demonstrated according to $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$Frenet frame with $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$Darboux frame.

PRELIMINARIES

In this section, we will present some basic concepts related to ruled surface (Ravani & Ku, 1991; Hacısalihoğlu, 1994; Sarıoğlugil & Tutar 2007; Gray *et al.*, 2006) and the Darboux Frame (O'Neill, 1996).

Differential Geometry of Ruled Surface

A ruled surface M in $\mathbb{R}^3$ is generated by a one-parameter family of straight lines. The straight lines are called the rulings. The equation of the ruled surface can be written as,

$$\varphi(s,v) = \alpha(s) + ve(s)$$

where, (α) is curve which is called the base curve of the ruled surface and (e) is called the unit direction vector of the straight line.

The striction point on ruled surface is the foot of the common perpendicular line of the successive rulings on the main ruling. The set of striction points of the ruled surface generates its striction curve. It is given by

$$c(s) = \alpha(s) - \frac{<\alpha_s, e_s>}{<e_s, e_s>} e(s). \tag{1}$$

Theorem 1. *If successive rulings intersect, the ruled surface is called developable. The unit tangent vector of the striction curve of a developable ruled surface is the unit vector with direction e. (Ravani & Ku, 1991)* .

The distribution parameter of the ruled surface is identified by

$$P_e = \frac{det(\alpha_s, e, e_s)}{<e_s, e_s>}. \tag{2}$$

Theorem 2. The ruled surface is developable if and only if $P_e = 0$. (Ravani & Ku, 1991)

The ruled surface is said to be a noncylindrical ruled surface provided that $< e_s, e_s > \neq 0$. In this case, we will take the striction curve as the base curve of the ruled surface. So we can write

$$\varphi(s,v) = c(s) + ve(s). \tag{3}$$

Theorem 3. Let M be a noncylindrical ruled surface and defined by its striction curve. The Gaussian curvature of M is given by its distribution parameter by (Gray *et al.*, 2006)

$$K = -\frac{P_e^2}{(P_e^2 + v^2)^2}. \tag{4}$$

Theorem 4. Let M be a noncylindrical ruled surface and defined by its striction curve. If P_e never vanishes, then K is continuous and $|K|$ assumes its maximum value $\frac{1}{P_e^2}$ at $v = 0$. (Gray *et al.*, 2006)

If the ruled surface satisfies $\varphi(s + 2\pi, v) = \varphi(s, v)$ for all $s \in \mathrm{I}$, then the ruled surface is called closed.

A curve which intersects perpendicularly each one of rulings is called an orthogonal trajectory of the ruled surface. It is calculated by

$$< e, d\varphi >= 0. \tag{5}$$

The pitch of closed ruled surface is defined by

$$L_e = -\oint_\alpha < e, d\alpha >= dv \tag{6}$$

An orthogonal trajectory of the closed ruled surface, starting from the point P_0, on the ruling e, intersects the same ruling at the point P_1, generally different from P_0 Thus $L_e = \overrightarrow{P_0 P_1}$.

The angle of pitch of closed ruled surface is defined by

$$\lambda_e = \oint_\alpha < \vec{D}, \vec{e} > \tag{7}$$

where $\vec{D}$ is Steiner rotation vector of the motion.

Darboux Frame of Surface

Let $\alpha(s)$ be a unit speed curve on an oriented surface M. Since $\alpha(s)$ is a space curve, there exists the moving Frenet frame {**T, N, B**} along the curve. **T** is a unit tangent vector, **N** is a principal normal vector and **B** is a binormal vector. The Frenet equations are given the following,

$$\begin{cases} \mathbf{T}' = \kappa\mathbf{N} \\ \mathbf{N}' = \kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' = -\tau\mathbf{N} \end{cases}$$

,where κ is the curvature and τ is the torsion of the curve. Due to the curve $\alpha(s)$ that lies on the surface there exists the Darboux Frame and it is denoted by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. In Darboux Frame **T** is the unit tangent vector of the curve like the Frenet Frame. **n** is the unit normal vector of the surface and **g** is the unit vector which is defined by $\mathbf{g} = \mathbf{n} \times \mathbf{T}$. Due to the unit tangent vector **T** is common Frenet Frame and Darboux Frame, the vectors **N**, **B**, **g**, **n** lie on the same plane. Therefore the relations between these frames can be given by:

$$\begin{bmatrix} T \\ g \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where θ is the angle between the vectors **g** and **N** . The derivative formulae of the Darboux frame is given as follows:

$$\begin{bmatrix} T' \\ g' \\ n' \end{bmatrix} \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} = \begin{bmatrix} T \\ g \\ n \end{bmatrix} \tag{8}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the relative (also called geodesic) torsion of $\alpha(s)$. In this article, we prefer using "prime" to denote the derivative with respect to the arc length parameter of a curve. The relations between the geodesic curvature, the normal curvature, the geodesic torsion are given the following:

$$\kappa_g = \kappa\cos\theta, \kappa_n = \kappa\sin\theta, \tau_g = \tau + \frac{d\theta}{ds}.$$

In addition, the geodesic curvature κ_g and geodesic torsion τ_g of the curve $\alpha(s)$ can be calculated as follows:

$$\kappa_g = <\frac{d\alpha}{ds}, \frac{d^2\alpha}{ds^2} \times n >, \tau_g = <\frac{d\alpha}{ds}, n \times \frac{dn}{ds} >.$$

In the differential geometry of surface, for a curve $\alpha(s)$ lying on a surface, the following points are well-known and acknowledged (O'Neill, 1996) :

- $\alpha(s)$ is a geodesic curve $\Leftrightarrow$ $\kappa_g = 0$,
- $\alpha(s)$ is an asymptotic line $\Leftrightarrow$ $\kappa_n = 0$,
- $\alpha(s)$ is a principal line $\Leftrightarrow$ $\tau_g = 0$.

CHARACTERISTIC PROPERTIES OF THE RULED SURFACE WITH DARBOUX FRAME IN $\mathbb{E}^3$

A ruled surface can be written as:

$$\varphi(s,v) = \alpha(s) + ve(s).$$

Since $\alpha(s)$ is a space curve, there exists the moving Frenet frame {**T, N, B**} along the curve. The unit normal vector field of the ruled surface is **n**. We can define $\mathbf{g} = \mathbf{n} \times \mathbf{T}$ unit vector, which satisfies $< T, g >=< n, g >=0$. Therefore {**T, g, n**} is a Darboux frame the ruled surface. The derivative formulae of the ruled surface with Darboux frame can be defined by the Equation 8.

A unit direction vector of a straight line e is stretched by the system {**T, g**}. So it can be written as:

$$e = T\cos\phi + g\sin\phi \tag{9}$$

where ϕ is the angle between the vectors **T** and e. It is shown in Figure.1.

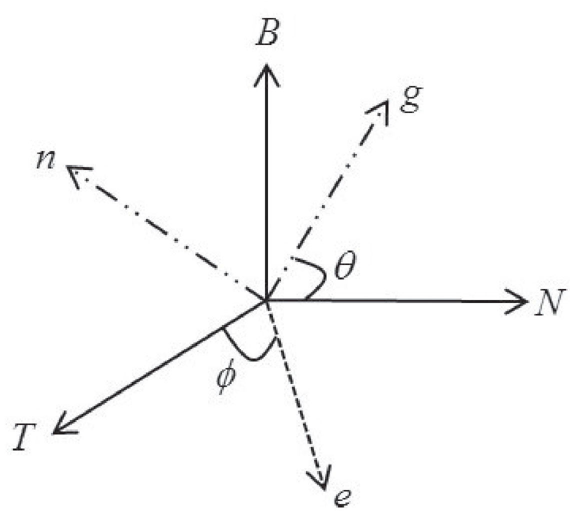

Fig. 1. The relation between Darboux frame, Frenet frame and the unit vector *e*

e_s is also provided by the equation,

$$e_s = -T(\phi' + \kappa_g)\sin\phi + g(\phi' + \kappa_g)\cos\phi + n(\kappa_n \cos\phi + \tau_g \sin\phi). \qquad (10)$$

Holding v constant, we obtain a curve $\beta(s) = \alpha(s) + ve(s)$ on a ruled surface whose vector field is

$$T^* = (1 - v(\phi' + \kappa_g)\sin\phi)T + v(\phi' + \kappa_g)\cos\phi g + v(\kappa_n \cos\phi + \tau_g \sin\phi)n. \qquad (11)$$

Moreover, the relation between the vectors e and T^* is:

$$< T^*, e >= \cos\phi. \qquad (12)$$

In the Equation 2 the distribution parameter of the ruled surface is defined by

$$P_e = \frac{det(\alpha_s, e, e_s)}{< e_s, e_s >}.$$

If we use α_s, e and e_s values respectively which are obtained for {**T, g, n**} Darboux frame, we find the distribution parameter of ruled surface with Darboux frame

$$P_e = \frac{sin\phi(\kappa_n \cos\phi + \tau_g \sin\phi)}{(\phi' + \kappa_g)^2 + (\kappa_n \cos\phi + \tau_g \sin\phi)^2}. \qquad (13)$$

Theorem 5 The ruled surface with Darboux frame is developable if and only if $\kappa_n \cos\phi + \tau_g \sin\phi = 0.$

Proof. The tangent space of the ruled surface with Darboux frame is defined in the Equation 11 by:

$$T^* = (1 - v(\phi' + \kappa_g)\sin\phi)T + v(\phi' + \kappa_g)\cos\phi g + v(\kappa_n \cos\phi + \tau_g \sin\phi)n.$$

Since this tangent space is a constant along a main ruling, the coefficient of the normal vector field of the surface *n* is equal to zero. In this case P_e distribution parameter is zero.

Conversely, supposing that distribution parameter of the ruled surface with Darboux frame is zero, then $sin\phi(\kappa_n \cos\phi + \tau_g \sin\phi)$ is equal to zero from the Equation 13. In this case $\sin\phi = 0$ or $\kappa_n \cos\phi + \tau_g \sin\phi = 0$.

If $\sin\phi$ is zero then by the Equation 9 *e* is equal to *T*. In that case $T^* = T$ is obtained and the tangent plane is constant along a main ruling.

If $\kappa_n \cos\phi + \tau_g \sin\phi$ is zero then from the Equation 11

$$T^* = (1 - v(\phi' + \kappa_g)\sin\phi)T + v(\phi' + \kappa_g)\cos\phi\, g.$$

is obtained so the tangent plane is perpendicular to the normal of the ruled surface with Darboux frame. Therefore the ruled surface with Darboux frame is developable.

In the Equation 5 orthogonal trajectories of ruled surface are defined by

$$< e, d\varphi >= 0.$$

In this equation, if we use α_s , e and e_s values respectively which are obtained for $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ Darboux frame, we find that the orthogonal trajectories of ruled surface with Darboux frame as,

$$\cos\phi ds = -dv . \tag{14}$$

From the Equation 1 striction curve of ruled surface is defined by

$$c(s) = \alpha(s) - \frac{< \alpha_s, e_s >}{< e_s, e_s >} e(s).$$

In this equation, if we use α_s , e and e_s values respectively which are obtained for $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ Darboux frame, we find the striction curve of the ruled surface with Darboux frame is,

$$c(s) = \alpha(s) + \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 + (\kappa_n \cos\phi + \tau_g \sin\phi)^2} e(s). \tag{15}$$

Theorem 6. Let M be a ruled surface with Darboux frame which is given by $\varphi(s,v) = \alpha(s) + ve(s)$. So, the shortest distance between the rulings of the ruled surface with Darboux frame along the orthogonal trajectories is

$$v = \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 + (\kappa_n \cos\phi + \tau_g \sin\phi)^2}$$

along the curve $\varphi_v : I \to M$.

Proof. Supposing that the two rulings pass through the points α_{s_1} and α_{s_2} where $s_1 < s_2$, the distance between these rulings along an orthogonal trajectory is given:

$$J(v) = \int_{s_1}^{s_2} \| T^* \| \, \mathrm{d}s$$

where $T^* = (1 - v(\phi' + \kappa_g)\sin\phi)T + v(\phi' + \kappa_g)\cos\phi g + v(\kappa_n \cos\phi + \tau_g \sin\phi)g$. From there we obtain

$$J(v) = \int_{s_1}^{s_2} \sqrt{1 - 2v\sin\phi(\phi' + \kappa_g) + v^2(\phi' + \kappa_g)^2 + v^2(\kappa_n \cos\phi + \tau_g \sin\phi)^2} \, \mathrm{d}s.$$

To find value of s which minimizes $J(v)$, we have to use

$$J'(v)=\int_{s_1}^{s_2}\frac{-2\sin\phi(\phi'+\kappa_g)+2v(\phi'+\kappa_g)^2+2v(\kappa_n\cos\phi+\tau_g\sin\phi)^2}{\sqrt{1-2v\sin\phi(\phi'+\kappa_g)+v^2(\phi'+\kappa_g)^2+v^2(\kappa_n\cos\phi+\tau_g\sin\phi)^2}}ds=0$$

which satisfies

$$v=\frac{\sin\phi(\phi'+\kappa_g)}{(\phi'+\kappa_g)^2+(\kappa_n\cos\phi+\tau_g\sin\phi)^2}$$

Theorem 7. *Let* M *be a ruled surface with Darboux frame. Moreover, the point* $\varphi(s,v_0)$, $v_0\in\mathbb{R}$, *on the main ruling which passes the point* $\alpha(s)$, *is a striction point if and only if* e_s *is the unit normal vector field of tangent plane in the point* $\varphi(s,v_0)$.

Proof. While suggesting that the point $\varphi(s,v_0)$ on the main ruling which passes through the point $\alpha(s)$ is a striction point, we have to show that $<e_s,e>=<e_s,T^*>=0$. We know that $<e,e>=1$ so if we take differential this equation with respect to *s*, we obtain $<e_s,e>=0$. Also if we calculate the value of $<e_s,T^*>$, we get

$$<e_s,T^*>=-\sin\phi(\phi'+\kappa_g)+v(\phi'+\kappa_g)^2+v(\kappa_n\cos\phi+\tau_g\sin\phi)^2. \qquad (16)$$

From $\varphi(s,v_0)$, we can write the striction point as

$$v_0=\frac{\sin\phi(\phi'+\kappa_g)}{(\phi'+\kappa_g)^2+(\kappa_n\cos\phi+\tau_g\sin\phi)^2}.$$

If we calculate the value v_0 into the Equation 16, then we get $<e_s,T^*>=0$. So, e_s is normal to e and the vector field T^* .

Conversely, suppose that e_s is a unit normal vector field of the tangent plane at the point $\varphi(s,v_0)$. When holding v constant as v_0, the tangent vector field of $\varphi(s,v_0)$ is:

$$T^*=(1-v_0(\phi'+\kappa_g)\sin\phi)T+v_0(\phi'+\kappa_g)\cos\phi g+v_0(\kappa_n\cos\phi+\tau_g\sin\phi)n.$$

Since e_s is a unit normal to the tangent plane at the point $\varphi(s,v_0)$, we can write that $<e_s,T^*>=0$. Thus we get:

$$v_0=\frac{\sin\phi(\phi'+\kappa_g)}{(\phi'+\kappa_g)^2+(\kappa_n\cos\phi+\tau_g\sin\phi)^2}.$$

Hence the point $\varphi(s,v_0)$ is a striction point on the ruled surface with Darboux frame.

Theorem 8. Let M be a ruled surface with Darboux Frame. The absolute value of the Gaussian curvature K of the ruled surface M along a ruling takes the maximum value at the striction point on that ruling.

Proof. Let M is a ruled surface with Darboux frame can be written as:

$$\varphi(s,v) = \alpha(s) + ve(s).$$

The Gaussian curvature in terms of the coefficients of the fundamental forms is

$$K = -\frac{M^2}{EG - F^2}. \tag{17}$$

When we calculate the Gaussian curvature of the ruled surface with Darboux frame by using Equation 17, we get:

$$K(s,v) = \frac{(\kappa_n \cos\phi + \tau_g \sin\phi)^2 sin^2\phi}{((1-2v\sin\phi(\phi' + \kappa_g) + v^2(\phi' + \kappa_g)^2 + v^2(\kappa_n \cos\phi + \tau_g \sin\phi)^2) - cos^2\phi)^2}$$

and to find the maximum absolute value of the Gaussian curvature, differentiating this equation with respect to v gives

$$\frac{\partial K}{\partial v} = \frac{(\kappa_n \cos\phi + \tau_g \sin\phi) sin^2\phi \frac{\partial(((1-2v\sin\phi(\phi' + \kappa_g) + v^2(\phi' + \kappa_g)^2 + v^2(\kappa_n \cos\phi + \tau_g \sin\phi)^2) - cos^2\phi)^2)}{\partial v}}{((1-2v\sin\phi(\phi' + \kappa_g) + v^2(\phi' + \kappa_g)^2 + v^2(\kappa_n \cos\phi + \tau_g \sin\phi)^2) - cos^2\phi)^4}. \tag{18}$$

If we solve the Equation 18 with respect to v, we get

$$v = \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 + (\kappa_n \cos\phi + \tau_g \sin\phi)^2}.$$

That point corresponds to the striction point of the ruled surface with Darboux frame therefore the absolute value of the Gaussian curvature K of the ruled surface M along a ruling takes the maximum value at the striction point on that ruling.

Example 1 Let $\alpha(s) = (\frac{\sqrt{3}}{3}\cos s, 1 - \sin s, -\frac{\sqrt{6}}{3}\cos s)$ be a curve. Since the tangent developable ruled surface of α is as follows,

$$\varphi(s,v) = \alpha(s) + v\alpha'(s)$$

we get the tangent developable ruled surface as

$$\varphi(s,v) = (\frac{\sqrt{3}}{3}\cos s, 1 - \sin s, -\frac{\sqrt{6}}{3}\cos s) + v(-\frac{\sqrt{3}}{3}\sin s, -\cos s, -\frac{\sqrt{6}}{3}\sin s).$$

which is shown in Fig. 2.

The Darboux frame of the tangent developable ruled surface is as follows:

$$\begin{cases} \mathbf{T(s)} = (-\frac{\sqrt{3}}{3}\sin s, -\cos s, -\frac{\sqrt{6}}{3}\sin s) \\ \mathbf{g(s)} = \mathbf{n(s)} \times \mathbf{T(s)} = (0,0,-1) \\ \mathbf{n(s)} = (\frac{\sqrt{6}}{3}, 0, \frac{\sqrt{3}}{3}) \end{cases}$$

The striction curve of the tangent developable ruled surface with Darboux frame from Equation 15 is defined as

$$c(s) = \alpha(s) + \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 + (\kappa_n \cos\phi + \tau_g \sin\phi)^2} e(s).$$

Since $\sin\phi = 0$ then striction curve is obtained as $c(s) = \alpha(s)$.

The distribution parameter of the tangent developable ruled surface with Darboux frame from Equation 13 is defined as

$$P_e = \frac{sin\phi(\kappa_n \cos\phi + \tau_g \sin\phi)}{(\phi' + \kappa_g)^2 + (\kappa_n \cos\phi + \tau_g \sin\phi)^2}.$$

Since $\sin\phi = 0$, the distribution parameter is obtained as $P_e = 0$. So the ruled surface with Darboux frame is developable.

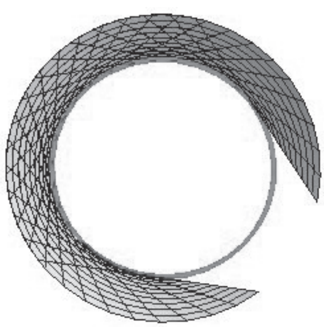

Fig. 2. . The tangent developable ruled surface with Darboux frame

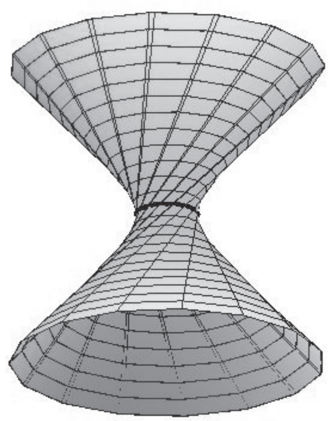

Fig. 3. Hyperboloid of one sheet with Darboux frame

Example 2 Let us demonstrate the hyperboloid of one sheet as a ruled surface.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1$$

Let us find its Darboux frame, striction curve and distribution parameter.

If we choose $\alpha(s)$ and $e(s)$ as follows,

$$\alpha(s)=(a\cos\frac{s}{a},a\sin\frac{s}{a},0),e(s)=-(\frac{a}{\sqrt{a^2+c^2}}\sin\frac{s}{a},\frac{a}{\sqrt{a^2+c^2}}\cos\frac{s}{a},\frac{c}{\sqrt{a^2+c^2}})$$

$$\varphi(s,v)=(a\cos\frac{s}{a},a\sin\frac{s}{a},0)+v(-\frac{a}{\sqrt{a^2+c^2}}\sin\frac{s}{a},\frac{a}{\sqrt{a^2+c^2}}\cos\frac{s}{a},\frac{c}{\sqrt{a^2+c^2}})$$

$$e(s)=\cos\phi T+\sin\phi g$$

we can say that $\varphi(s,v)$ is a ruled surface which is shown in Fig. 3.

Since a unit direction vector is:

$$e(s)=\cos\phi T+\sin\phi g$$

we get Darboux frame of the ruled surface as follows:

$$\begin{cases}\mathbf{T(s)}=(-\sin\frac{s}{a},\cos\frac{s}{a},0)\\ \mathbf{g(s)}=\mathbf{n(s)}\times\mathbf{T(s)}=(0,0,1)\\ \mathbf{n(s)}=(\cos\frac{s}{a},\sin\frac{s}{a},0)\end{cases}$$

Then we write

$$\begin{bmatrix}T'\\ g'\\ n'\end{bmatrix}=\begin{bmatrix}0 & 0 & \frac{-1}{a}\\ 0 & 0 & 0\\ \frac{1}{a} & 0 & 0\end{bmatrix}\begin{bmatrix}T\\ g\\ n\end{bmatrix}.$$

The striction curve of the ruled surface with Darboux frame is defined by the Equation 15 as

$$c(s)=\alpha(s)+\frac{sin\phi(\phi'+\kappa_g)}{(\phi'+\kappa_g)^2+(\kappa_n\cos\phi+\tau_g\sin\phi)^2}e(s).$$

Since $\kappa_g=0$ and $\phi'=0$, the striction curve is obtained as $c(s)=\alpha(s)$

The distribution parameter of the ruled surface with Darboux frame is defined form the Equation 13 is as

$$P_e=\frac{sin\phi(\kappa_n\cos\phi+\tau_g\sin\phi)}{(\phi'+\kappa_g)^2+(\kappa_n\cos\phi+\tau_g\sin\phi)^2}.$$

So the distribution parameter is obtained as $P_e = -c.$

PROPERTIES OF INTEGRAL INVARIANTS OF THE RULED SURFACE WITH DARBOUX FRAME IN $\mathbb{E}^3$

By the Equation 14 orthogonal trajectories of the ruled surface with Darboux frame is defined as

$$\cos\phi ds = -dv .$$

So the pitch of closed ruled surface with Darboux frame are defined as

$$L_e = -\oint_\alpha <e, d\alpha> = \oint_\alpha dv = \oint_\alpha \cos\phi ds. \qquad (19)$$

Let

$$e = T\ \cos\phi + N\ \sin\phi\cos\theta + B\ \sin\phi\sin\theta, \| e \| = 1 \qquad (20)$$

where θ is the angle between the vectors $\mathbf{g}$ and $\mathbf{N}$ and where ϕ is the angle between the vectors $\mathbf{T}$ and e, be a unit vector line in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ Frenet frame on the ruled surface with Darboux frame drawn by a line e .

Theorem 9. Let $\alpha : I \to \mathbb{E}^3$ be a closed space curve and H/H' be a closed space motion which is defined by α . The angle of pitch of the ruled surface with the Darboux frame, which is drawn by a fixed line in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' in fixed space H is,

$$\lambda_e = \lambda_T \cos\phi + \lambda_B \sin\phi\sin\theta \qquad (21)$$

where λ_T and λ_B are the angle of pitches of the ruled surfaces with Darboux frame which are drawn by the vectors $\mathbf{T}$ and $\mathbf{B}$, respectively.

Proof. While $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is moving during the motion, e draws a ruled surface with Darboux frame. Thus, we calculate the angle of pitch of this ruled surface as follows:

$$\lambda_e = -<\overrightarrow{D}, e>$$

$$\lambda_e = -<\lambda_T T + \lambda_B B, T\cos\phi + N\sin\phi\cos\theta + B\sin\phi\sin\theta>$$

or

$$\lambda_e = \lambda_T \cos\phi + \lambda_B \sin\phi\sin\theta.$$

Theorem 10. The pitch of the ruled surface with the Darboux frame drawn by a fixed line in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' in fixed space H is equal to the multiplication of L_T and the tangential component of e which is $\cos\phi$.

Proof. For the pitch of the ruled surface with the Darboux frame, which is drawn by the fixed line e, we get

$$L_e = -\oint_\alpha < d\alpha, e >$$

$$L_e = \oint_\alpha < Tds, T\cos\phi + N\sin\phi\cos\theta + B\sin\phi\sin\theta >$$

or

$$L_e = \cos\phi \oint_\alpha ds$$

or

$$L_e = \cos\phi L_T. \qquad (22)$$

Theorem 11. If the ruled surface with Darboux frame drawn by a line e in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' is developable, then the harmonic curvature, which is calculated as follows:

$$h = \frac{\kappa}{\tau} = \frac{\sin^2\phi}{\sin\phi\cos\phi\sin\theta} = \frac{(L_T^2 - L_e^2)\lambda_B}{(\lambda_e L_T - L_e\lambda_T)L_e} \qquad (23)$$

of the base curve of the ruled surface with Darboux frame, is constant.

Proof. Let e draws a developable ruled surface with Darboux frame. In this case, the distribution parameter of the ruled surface is zero. Hence,

$$\frac{de}{ds} = \kappa N \cos\phi + (-\kappa T + \tau B)\sin\phi\cos\theta + (-\tau N)\sin\phi\sin\theta$$

or

$$\frac{de}{ds} = -\kappa T \sin\phi\cos\theta + N(\cos\phi\kappa - \sin\phi\sin\theta\tau) + \tau B \sin\phi\cos\theta \qquad (24)$$

and

$$\frac{d\alpha}{ds} \times e = T \times e = \sin\phi\cos\theta B - \sin\phi\sin\theta N$$

so

$$det(\frac{d\alpha}{ds}, e, \frac{de}{ds}) = < T \times e, \frac{de}{ds} >$$

or

$$det(\frac{d\alpha}{ds}, e, \frac{de}{ds}) = \sin^2\phi\cos^2\theta\tau - \sin\phi\sin\theta(\cos\phi\kappa - \sin\phi\sin\theta\tau) = 0 \qquad (25)$$

Using the Equation (25), we get the following

$$\sin\phi^2\tau - \sin\phi\cos\phi\sin\theta\kappa = 0 \qquad (26)$$

$$\frac{\kappa}{\tau}=\frac{\sin^2\phi}{\sin\phi\cos\phi\sin\theta} \tag{27}$$

Solving $\cos\phi$ and $\sin\phi\sin\theta$ from the Equation 22 and the Equation 21 , we get the following equations

$$\cos\phi=\frac{L_e}{L_T} \tag{28}$$

$$\lambda_e=\frac{L_e}{L_T}\lambda_T+\sin\phi\sin\theta\lambda_B$$

$$\sin\phi\sin\theta=\frac{(\lambda_e L_T-L_e\lambda_T)}{L_T\lambda_B} \tag{29}$$

Then, substituting the Equation 28 and the Equation 29 into the Equation 27 gives

$$h=\frac{\kappa}{\tau}=\frac{\sin^2\phi}{\sin\phi\cos\phi\sin\theta}=\frac{(L_T^2-L_e^2)\lambda_B}{(\lambda_e L_T-L_e\lambda_T)L_e}$$

Besides, e is a fixed line in $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$. Hence, the components of e in $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ are fixed. So that from the Equation 27, $\frac{\kappa}{\tau}$ is constant. So the harmonic curvature of the ruled surface with Darboux frame is constant.

Theorem 12. The harmonic curvature of the closed curve $\alpha(s)$ of the ruled surface with Darboux frame, during the space motion H/H', is calculated as follows:

$$(\frac{\kappa}{\tau})^2=\frac{P_B}{P_N}-1=(\frac{\sin^2\phi}{\sin\phi\cos\phi\sin\theta})^2-1 \tag{30}$$

where P_B and P_N are the distribution parameters of the ruled surface with Darboux frame which are drawn by **B** and **N**.

Proof. Using the definition of the distribution parameter in the Equation 13, we may calculate the distribution parameter of the closed ruled surface which is drawn by tangent line **T**, as follows:

$$P_T=\frac{det(T,T,\kappa N)}{<\kappa N,\kappa N>}=0$$

Similarly, the distribution parameter of the closed ruled surface which is drawn by a normal line **N**, as follows:

$$P_N=\frac{det(T,N,-\kappa T+\tau B)}{<-\kappa T+\tau B,-\kappa T+\tau B>}=\frac{\tau}{\kappa^2+\tau^2} \tag{31}$$

and the distribution parameter of the closed ruled surface which is drawn by binormal line **B**, is

$$P_B = \frac{det(T, B, -\tau N)}{< -\tau N, -\tau N >} = \frac{1}{\tau} \tag{32}$$

Moreover, the Equation 31 can be written as

$$P_N = \frac{\frac{1}{\tau}}{(\frac{\kappa}{\tau})^2 + 1}$$

Then, if we consider the Equation 32,

$$P_N = \frac{P_B}{(\frac{\kappa}{\tau})^2 + 1}$$

or

$$(\frac{\kappa}{\tau})^2 = \frac{P_B}{P_N} - 1$$

and if we use the Equation 30,

$$(\frac{\kappa}{\tau})^2 = \frac{P_B}{P_N} - 1 = (\frac{\sin^2\phi}{\sin\phi\cos\phi\sin\theta})^2 - 1$$

can be obtained.

Theorem 13. The ruled surface with Darboux frame drawn by a line *e* in a normal plane in {**T, N, B**} during the motion H/H' is developable if and only if (α) is a plane curve.

Proof. If *e* is a line in a normal plane then from Equation 20,

$$\cos\phi = 0 \tag{33}$$

can be obtained. Since the ruled surface with Darboux frame is developable from the Equation 26,

$$\sin^2\phi\tau = 0 \tag{34}$$

can be obtained. When we use the Equation 33 and the Equation 34, τ is zero, so (α) is a plane curve.

Theorem 14. The ruled surface with Darboux frame drawn by a line e in an osculator plane in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' is developable if and only if (α) is a plane curve or $\phi = 0$.

Proof. If e is a line in an osculator plane from, form the Equation 20 we get,

$$\sin\phi \sin\theta = 0$$

Hence $\sin\phi = 0$ or $\sin\theta = 0$. Moreover, the ruled surface with Darboux frame is developable from the Equation 26,

$$\sin^2\phi\tau = 0$$

is obtained.

If $\sin\phi = 0$ when $\sin^2\phi\tau = 0$, then $\phi = 0$.

If $\sin\theta = 0$ when $\sin\phi = 0$, then $\phi = 0$.

If $\sin\phi = 0$ when $\tau = 0$, then (α) is a plane curve.

Theorem 15. The ruled surface with Darboux frame which drawn by a line e in a rectifian plane in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' is developable if and only if

$$\frac{\kappa}{\tau} = \frac{\sin\phi}{\cos\phi}$$

or

$$\phi = 0.$$

Proof. If e is a line in a rectifian plane from the Equation 20, then

$$\sin\phi \cos\theta = 0$$

Hence $\sin\phi = 0$ or $\cos\theta = 0$. Moreover, the ruled surface with Darboux frame is developable from the Equation 26, so

$$\sin\phi^2\tau - \sin\phi\cos\phi\sin\theta\kappa = 0$$

If $\sin\phi = 0$ when $\cos\theta \neq 0$, then $\phi = 0$.

If $\cos\theta = 0$ when $\sin\phi \neq 0$, then $\frac{\kappa}{\tau} = \frac{\sin\phi}{\cos\phi}$.

Both of $\sin\phi$ and $\cos\theta$ are both zero, then $\phi = 0$.

Theorem 16 The ruled surface with Darboux frame drawn by a line e in a rectifian plane in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' is developable then the relation between P_B and P_N is calculated as follows,

$$\frac{P_B}{P_N} = \frac{(\lambda_e L_T - L_e\lambda_T)^2}{L_e^2\lambda_B^2} + 1 = (\frac{\sin\phi\sin\theta}{\cos\phi})^2 + 1. \tag{35}$$

Proof. If e is a line in a rectifian plane from the Equation 20,

$$\sin\phi\cos\theta = 0.$$

Therefore

$$\frac{\kappa}{\tau} = \frac{\sin\phi\sin\theta}{\cos\phi} \tag{36}$$

If we substitute the Equation 36 into the Equation 23,

$$h = \frac{\kappa}{\tau} = \frac{\sin\phi\sin\theta}{\cos\phi} = \frac{\lambda_e L_T - L_e\lambda_T}{L_e\lambda_B}$$

then

$$\frac{P_B}{P_N} = \frac{(\lambda_e L_T - L_e\lambda_T)^2}{L_e^2\lambda_B^2} + 1 = (\frac{\sin\phi\sin\theta}{\cos\phi})^2 + 1$$

Special Cases

- $\cos\theta \neq 0$ when $\sin\phi = 0$, then $\frac{P_B}{P_N} = 1$.
- $\sin\phi \neq 0$ when $\cos\theta = 0$, then $\frac{P_B}{P_N} = (\frac{\sin\phi}{\cos\phi})^2 + 1$.
- If $\sin\phi$ and $\cos\theta$ are both zero, then $\frac{P_B}{P_N} = 1$.

Theorem 17. The ruling of the ruled surface with Darboux frame drawn by a line e in $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ during the motion H/H' in fixed space is always in the rectifian plane of the striction curve.

Proof. Since the base curve of the ruled surface is the striction curve,

$$<\alpha_s, e_s>= 0. \tag{37}$$

Hence if we substitute $\mathbf{T}$ and the Equation 24 into the Equation 37,

$$<T, -T\sin\phi\cos\theta\kappa + N(\cos\phi\kappa - \sin\phi\sin\theta\tau) + B\sin\phi\cos\theta\tau >= 0$$

or

$$-\sin\phi\cos\theta\kappa = 0$$

can be obtained. Since $\kappa \neq 0$, $\sin\phi\cos\theta$ is zero. So $e = T\cos\phi + B\sin\phi\sin\theta$ is always in the rectifian plane of the striction curve.

Theorem 18. Since the ruled surface with Darboux frame drawn by a line e in $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ during the motion H/H' in fixed space is developable, then the surface is tangent developable ruled surface.

Proof. If we use Theorem 17, we get $\sin\phi\cos\theta = 0$. In this case, if we calculate P_e,

$$P_e = \frac{det(T, T\cos\phi + B\sin\phi\sin\theta, N(\cos\phi\kappa - \sin\phi\sin\theta\tau))}{\| N(\cos\phi\kappa - \sin\phi\sin\theta\tau)\|^2}$$

or

$$P_e = \frac{-\sin\phi\sin\theta}{(\cos\phi\kappa - \sin\phi\sin\theta\tau)}$$

can be obtained. Since the ruled surface is developable, P_e will be zero. Moreover, if we use $\sin\phi\cos\theta = 0$ and $\sin\phi\sin\theta = 0$, we get $\sin\phi$ is zero. Hence the ruled is the tangent developable.

Theorem 19. The ruled surface with Darboux frame which has a closed the spherical striction curve is developable if and only if the angles of the pitch are zero.

Proof. Since the total curvature of the closed spherical curve is zero,

$$\lambda_e = \lambda_T = \oint \tau ds = 0$$

can be obtained. Conversely, since the striction curve of the ruled surface is closed spherical,

$$\lambda_e = 0$$

and

$$\lambda_e = \lambda_T \cos\phi + \lambda_B \sin\phi\sin\theta = 0 \Rightarrow \sin\phi\sin\theta\oint \kappa ds = 0$$

are obtained. So $\sin\phi\sin\theta$ or $\oint \kappa ds$ is zero. If we take that $\sin\phi\sin\theta$ is zero and use the Theorem 18, we get $\cos\phi = 1$ or $e = T$. If we take $\oint \kappa ds$ as zero, *kds* must be zero. Hence *ds* and s is a constant.

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الخصائص المميزة للسطوح التي لها إطار داربو في E3

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خلاصة

نعرف في هذه الدراسة السطح المسطر الذي له إطار داربو. ثم نقوم بدراسة الخصائص المميزة المتعلقة بالتقوس الجيوديزي، التقوس الناظم والفتل الناظم. كما نقوم بعرض العلاقة بين إطار داربو وإطار فرينيه للسطح المسطر. إضافة إلى ذلك، نعطي أيضاً بعض المبرهنات حول درجة الميل وزاوية الميل، وهما في اللامتغيرات للسطح المسطر الذي له إطار داربو.