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A residual method using Bézier curves for singular nonlinear equations of Lane-Emden type

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Abstract

In this article, we introduce a new method to solve a singular non-linear equation of the Lane-Emden type by approximating the solution with Bernstein polynomials. This method is based on the minimization of a residual function using Taylor's series expansion. We also apply this method to problems that are solved by other methods and the obtained results show that our method is efficient, applicable and has great potential than others.

Keywords : Bézier curves; Bernstein polynomials; continuous linear approximation; Lane- Emden equations; singular non-linear differential equations.

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1. Introduction

We consider the following singular non-linear Lane-Emden equation

$$\begin{gathered} y'' + \frac{a}{x}y' + f(x,y) = g(x), \quad 0 < x < b, \\ y(0) = \alpha, \quad y'(0) = \beta, \end{gathered} \tag{1}$$

where the prescribed function $g(x)$, which represents the source term, is sufficiently smooth; $f \in C^{n-1}([a,b] \times D_1)$, D_1 is a closed interval in $\mathbb{R}$ and a, b, α, and β are finite constants.

Lane-Emden type equations are frequently encountered in various physical phenomena such as reactant concentration in a chemical reactor, boundary layer theory, control and optimization theory, flow networks in biology, some areas of astrophysics, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. But the solutions of (1) neither exists, nor are so practical except for some spe cial cases. So, improving both analytical and numerical schemes for finding the solution becomes very important. So, far many methods have been developed to provide accurate numerical solutions as in Bengochea (2014), Marzban *et al*. (2008), Parand *et al*. (2010), Turkyilmazoglu (2013), Wang *et al*. (2014), Wazwaz (2001), Wazwaz (2014), Kajani *et al*. (2012) and Yüzbaşı (2011).

Most of them reduce initial value problems to a non-linear system of equations, which require another approximation technique in their solutions and they also need to guarantee the convergence. But our method reduces the initial value problem to linear systems of equation, so that the results give us the unknown Bernstein coefficients of the approximate solution on the subintervals. This creates significant advantage for our method. So, we get an iterative method by dividing a given interval $[a, b]$ into subintervals. Also, we can control the bound of error with respect to step size, instead of degree of approximation curve. On each subinterval, we write an initial value problem, whose initial conditions are obtained from the approximate solution on the previous subinterval by constructing Bezier curves with unknown control points. Our goal is to determine the control points in order to minimize the residual. Finally, we get a piecewise approximate solution in the space $C^1[a, b]$.

This technique provides us a lower triangular system with nonzero diagonals. So, most of for Let the non-linear problems can be solved linearly using this technique. The Bernstein polynomials give a preference to the approximate solutions of differential equations as a consequence of these nice properties. Some examples of these type of approximations can be found in Evrenosoglu & Somali (2008), Ghomanjani *et al*. (2012), Wu (2012) and Zheng *et al*. (2004).

The organization of this paper is as follows.

In Section 2 we suggest new technique for the second order non-linear initial value problems using the Bernstein polynomials. Section 3 presents application of the proposed method to the Lane-Emden type equations. The error analysis of this method for the Bernstein polynomials is given in Section 4. In Section 5, three problems are presented by the numerical results to verify the theoretical results in Section 4. Finally, in section of conclusion, we summarize the results of the study and present our suggestions regarding the future works.

2. Residual method for second order nonlinear initial value problems

Consider the initial value problem

$$y'' = F(x, y, y'), \quad y(a) = \alpha, \quad y'(a) = \beta, \tag{2}$$

where $F \in C^{n-1}([a,b] \times D_1 \times D_2)$, D_1 and D_2 are the closed intervals in $\mathbb{R}$ and a, b, α and β are finite constants. Let us divide the interval $[a,b]$ into subintervals $[a_{i-1}, a_i]$ with equal length, where $a_i = a + ih$, $i = 0, 1, ..., N$, $h = (b-a)/N$, N is a positive integer. So, we can define the initial value problem (2) piecewisely as

$$y_i''(x) = F(x, y_i(x), y_i'(x)), \quad x \in S_i = [a_{i-1}, a_i], \tag{3}$$

for $1 \le i \le N$ and

$$\begin{aligned} y_1(a_0) &= \alpha, \quad y_1'(a_0) = \beta, \\ y_i'(a_{i-1}) &= y_{i-1}'(a_{i-1}), \quad y_i'(a_{i-1}) = y_{i-1}'(a_{i-1}), \end{aligned} \tag{4}$$

for $2 \le i \le N$.

Let

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n\left(\frac{x - a_{i-1}}{h}\right) \tag{5}$$

be a n^{th} degree Bezier curve over S_i, where

$$B_j^n\left(\frac{x - a_{i-1}}{h}\right) = \binom{n}{j}\frac{1}{h^n}(x - a_{i-1})^j (a_i - x)^{n-j}$$

are the Bernstein polynomials over the interval S_i and c_j^i are the unknown control points to be determined. So, we have $(n+1)$ unknown control points for $u_i(x)$ over the interval S_i.

Equations (4), initial conditions, should be applied to the approximate solution, that is

$$u_1(a_0) = \alpha, \qquad u_1'(a_0) = \beta, \tag{6}$$

$$u_i(a_{i-1}) = u_{i-1}(a_{i-1}), \quad u_i'(a_{i-1}) = u_{i-1}'(a_{i-1}), \tag{7}$$

for $2 \le i \le N$. Thus, we provided that the continuity of the first derivative of the approximate solution

$$u(x) = u_i(x) \qquad x \in S_i, \tag{8}$$

which means that $u(x) \in C^1[a,b]$.

By the derivation property of the Bezier curves at the end points and Equations (6) and (7), we have

$$\begin{aligned} c_0^1 &= \alpha, \quad c_1^1 = \beta\frac{h}{n} + \alpha, \\ c_0^i &= c_n^{i-1}, \quad c_1^i = 2c_n^{i-1} - c_{n-1}^{i-1} \end{aligned} \tag{9}$$

for $i = 2, \ldots, N$. After that we have $(n-1)$ unknown control points for each subinterval S_i.

Substituting Equation (5) into the differential equation (3) for $i = 1, \ldots, N$, we have the piecewise residual function

$$R(x) = R_i(x) \qquad x \in S_i,$$

where

$$R_i(x) = u_i''(x) - F(x, u_i(x), u_i'(x)), \quad x \in S_i.$$

Our aim is to determine such unknown control points c_j^i so that the sufficiently differentiable residual function $R_i(x)$ will be minimum in the interval S_i. To this end, we force the first $(n-1)$ terms in the Taylor expansion of $R_i(x)$ to be zero at $x = a_{i-1}$, i.e.,

$$R_i^{(k)}(a_{i-1}) = 0 \quad \text{for} \quad k = 0, \ldots, n-2. \tag{10}$$

Since

$$\begin{aligned} R_i(a_{i-1}) = u_i''(a_{i-1}) \\ - F(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})), \end{aligned}$$

using the derivative property of the Bezier curves at the end points given in Equations (9), we have the following linear equation

$$\begin{aligned} &\frac{n(n-1)}{h^2}(c_2^i - 2c_1^i + c_0^i) \\ &\quad - F\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0. \end{aligned} \tag{11}$$

Likewise, the rest of Equation (10)

$$\begin{aligned} R_i^{(k)}(a_{i-1}) &= u_i^{(k+2)}(a_{i-1}) \\ &\quad - F^{(k)}\left(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})\right) \\ &= 0, \end{aligned}$$

for $k = 1, \ldots, n-2$ becomes

$$\begin{aligned} &\frac{n(n-1)\ldots(n-k+1)}{h^{k+2}}\Delta^{k+2}c_0^i \\ &\quad - F^{(k)}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0, \end{aligned} \tag{12}$$

where

$$F^{(k)}(x, y, z) = \frac{\partial^k}{\partial x^k}F(x, y, z)$$

and

$$\Delta^{k+2}c_0^i=\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^{k+2-i}\ c_j^i.$$

So, we solve $c_2^i,\ldots,c_n^i$ from the above lower triangular system of linear equations (11) and (12), hence, we obtain the approximate solution $u_i(x)$ such that $R_i(x)$ will be minimum in S_i for $i=1,\ldots,N$.

3. Application of residual method to Lane- Emden equation

First, we define the initial value problem on each interval $S_i=[a_{i-1},a_i], 1\le i\le N$ as in Section 2

$$y_i''(x)=-\frac{2}{x}y_i'(x)-f(x,y_i(x))+g(x),\quad x\in S_i, \tag{13}$$

for $1\le i\le N$ and

$$y_1(a_0)=\alpha,\quad y_1'(a_0)=\beta,$$

$$y_i(a_{i-1})=y_{i-1}(a_{i-1}),\quad y_i'(a_{i-1})=y_{i-1}'(a_{i-1}), \tag{14}$$

for $2\le i\le N$ where $a_i=a_0+ih, a_0=0$.

We have to change our technique only for the first interval since the residue function

$$R_1(x)=u_1''(x)+\frac{2}{x}u_1'(x)+f(x,u_1)-g(x),\quad x\in S_1$$

and its derivatives are not be defined at $a_0=0$. So, we choose a point $\eta\in(0,a_1]$ to force the residue function and its derivatives to 0, i.e.,

$$R_1^{(k)}(\eta)=0,\quad \text{for}\quad k=0,\ldots,n-2. \tag{15}$$

In this case, we have some non-linear equations to find the unknown control points for the first interval, since we cannot use the property of derivatives of the Bezier curves at the end points. But this situation does not affect the order of convergence (this be proved in Section 4). Here, we use the Newton's method to solve the non-linear equations approximately. For the initial values of the Newton's method, we use the initial values of the Equation (1).

After finding the unknown control points on the interval S_1, we continue our procedure as described in Section 2. We force the residue function with the derivatives to be 0 at x = a_{i-1} for S_i, $2\le i\le N$, i.e.,

$$R_i^{(k)}(a_{i-1})=0, \tag{16}$$

for $k=0,\ldots,n-2$ to find the unknown control points where

$$R_i(x)=u_i''(x)+\frac{2}{x}u_i'(x)+f(x,u_i)-g(x). \tag{17}$$

In this case, we have (n-1) linear equations to find the unknown control points as well but for only the first interval S_1, we try to get zero from the residue function and its derivatives at a different point from a_0 because Equation (17) has the singularity only at a_0.

4. Error analysis

Lemma 1. The residual functions $R_i(x)$ are of order h^{n-1} for $i=1,\ldots,N$.

Proof.

Using (10) and Taylor expansion of $R_i(x)$ at $x=a_{i-1}$ we have

$$|R_i(x)|\le\frac{h^{n-1}}{(n-1)!}\max_{x\in[a,b]}|R_i^{(n-1)}(x)|.$$

Lemma 2. Let $\tilde{u}_i(x)$ be the auxiliary approximate solution of piecewise initial value problem (3) with initial conditions (4). Then, we have

$$y_i^{(l)}(a_{i-1})=\tilde{u}_i^{(l)}(a_{i-1}),\quad l=0,1,\ldots,n \tag{18}$$

$$|y_i(x)-\tilde{u}_i(x)|\le Kh^{n+1},\quad \forall x\in S_i \tag{19}$$

$$|y_i'(x)-\tilde{u}_i'(x)|\le(n+1)Kh^n,\quad \forall x\in S_i, \tag{20}$$

where $y_i(x)$ is the corresponding exact solution and $K=\frac{1}{(n+1)!}\max_{x\in[a,b]}|y_i^{(n+1)}(x)|$.

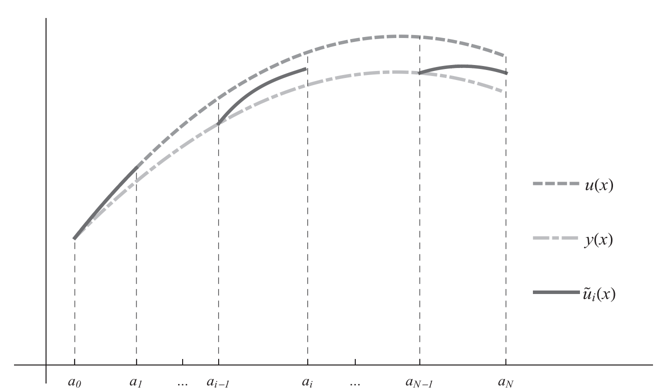

Fig. 1. An illustration of auxiliary approximate solutions $\tilde{u}_i(x)$ approximate solution $u(x)$ and exact solution $y_i(x)$ of non-linear initial value problem (2) on $[a_0,a_N]$.

Proof. Since both $\tilde{u}_i(x)$ and $y_i(x)$ satisfy the initial condition (4), it is obtained that

$$\tilde{u}_i(a_{i-1})=y_i(a_{i-1})\text{ and }\tilde{u}_i'(a_{i-1})=y_i'(a_{i-1}). \tag{21}$$

Define the residual function and its derivatives as

$$\tilde{R}_i(a_{i-1}) = \tilde{u}_i''(a_{i-1}) - F(a_{i-1}, y_{i-1}(a_{i-1}), y'_{i-1}(a_{i-1}))$$

$$\tilde{R}_i^{(k)}(a_{i-1}) = \tilde{u}_i^{(k+2)}(a_{i-1}) - F^{(k)}(a_{i-1}, y_{i-1}(a_{i-1}), y'_{i-1}(a_{i-1})),$$

where $k = 0, 1, \ldots, n-2$ and

$i = 1, 2, \ldots, N,$ which satisfy $\tilde{R}_i^{(k)}(a_{i-1}) = 0$ for $k = 0, 1, \ldots, n-2$ as in (10).

$$\begin{aligned} 0 =&\tilde{R}_i^{(k)}(a_{i-1}) \\ =&\tilde{u}_i^{(k+2)}(a_{i-1}) \\ &- F^{(k)}(a_{i-1}, \tilde{u}_i(a_{i-1}), \tilde{u}'_{i-1}(a_{i-1})) \\ =&\tilde{u}_i^{(k+2)}(a_{i-1}) \\ &- F^{(k)}(a_{i-1}, y_{i-1}(a_{i-1}), y'_{i-1}(a_{i-1})) \\ =&\tilde{u}_i^{(k+2)}(a_{i-1}) - y_i^{(k+2)}(a_{i-1}). \end{aligned}$$

$$\tilde{u}_i^{(k+2)}(a_{i-1}) = y_i^{(k+2)}(a_{i-1}) \tag{22}$$

for $k = 0, 1, \ldots, n-2$. Therefore, Equations (21) and (22) yields,

$$\tilde{u}_i^{(l)}(a_{i-1}) = y_i^{(l)}(a_{i-1}) \qquad \text{for } l = 0, 1, \ldots, n.$$

Writing the Taylor expansion of the difference $\big(y_i(x) - \tilde{u}_i(x)\big)$ about $x = a_{i-1}$

$$\begin{aligned} y_i(x) - \tilde{u}_i(x) = &\, y_i(a_{i-1}) - \tilde{u}_i(a_{i-1}) \\ &+ (x - a_{i-1})(y_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\ &+ \frac{(x - a_{i-1})^2}{2}(y_i''(a_{i-1}) - \tilde{u}_i''(a_{i-1})) \\ &+ \cdots \\ &+ \frac{(x - a_{i-1})^n}{n!}(y_i^{(n)}(a_{i-1}) - \tilde{u}_i^{(n)}(a_{i-1})) \\ &+ \frac{(x - a_{i-1})^{n+1}}{(n+1)!}(y_i^{(n+1)}(\xi_{i-1})) \end{aligned}$$

where $a_{i-1} < \xi_{i-1} < x$. And using Equation (18) we get

$$y_i(x) - \tilde{u}_i(x) = \frac{(x - a_{i-1})^{n+1}}{(n+1)!}\Big(y_i^{(n+1)}(\xi_{i-1})\Big),$$

which gives Equation (19).

Similarly expanding the Taylor series of the difference $\big(y_i'(x) - \tilde{u}_i'(x)\big)$ at $x = a_{i-1}$ and using Equation (18), we obtain the required bound in Equation (20).

The following lemma can be proved by using the Taylor expansion of the expressions about $x = a_{i-1}$

Lemma 3. Let $\tilde{u}_i(x)$ be the auxiliary approximate solution of the piecewise initial value problem (3) with the initial conditions (4) and $u_i(x)$ be the approximate solution of (3) with the initial conditions (6) or (7). Then,

$$\begin{aligned} |u_i(x) - \tilde{u}_i(x)| \le &\, |u_{i-1}(a_{i-1}) - \tilde{u}_{i-1}(a_{i-1})| \\ &\left(1 + \left(\frac{h^2}{2!} + \ldots + \frac{h^k}{k!} + \ldots + \frac{h^n}{n!}\right) C_1\right) \\ &+ |u'_{i-1}(a_{i-1}) - \tilde{u}'_{i-1}(a_{i-1})| \\ &\left(h + \left(\frac{h^2}{2!} + \ldots + \frac{h^k}{k!} + \ldots + \frac{h^n}{n!}\right) C_2\right) \end{aligned} \tag{23}$$

$$\begin{aligned} |u_i'(x) - \tilde{u}_i'(x)| \le &\, |u_{i-1}(a_{i-1}) - \tilde{u}_{i-1}(a_{i-1})| \\ &\left(h + \frac{h^2}{2!} + \ldots + \frac{h^{n-1}}{(n-1)!}\right) C_1 \\ &+ |u'_{i-1}(a_{i-1}) - \tilde{u}'_{i-1}(a_{i-1})| \\ &\left(1 + \left(h + \frac{h^2}{2!} + \ldots + \frac{h^{n-1}}{(n-1)!}\right) C_2\right) \end{aligned} \tag{24}$$

$$\begin{aligned} |y_i'(a_i) - u_i'(a_i)| \le &\, |y_{i-1}(a_{i-1}) - u_{i-1}(a_{i-1})| \\ &\left(h + \frac{h^2}{2!} + \ldots + \frac{h^{n-1}}{(n-1)!}\right) C_1 \\ &+ |y'_{i-1}(a_{i-1}) - u'_{i-1}(a_{i-1})| \\ &\left(1 + \left(h + \frac{h^2}{2!} + \ldots + \frac{h^{n-1}}{(n-1)!}\right) C_2\right) \\ &+ (n+1)Kh^n \end{aligned} \tag{25}$$

where

$$C_1 = \max_{k=0,1,\ldots,n-2}\left\{\max_{(x,v,z)\in\Omega}|F_v^{(k)}(x,v,z)|\right\} \tag{26}$$

$$C_2 = \max_{k=0,1,\ldots,n-2}\left\{\max_{(x,v,z)\in\Omega}|F_z^{(k)}(x,v,z)|\right\} \tag{27}$$

for $\Omega = \{(x, v, z) | x \in [a, b],\ v \in D_1,\ z \in D_2\}$.

Theorem 4. Let $y(x)$ be the exact solution of the second order non-linear initial value problem (2) and $u(x)$ be the corresponding n^{th} degree approximate function (8). Then, we have the inequality

$$|y(x)-u(x)| \leq Mh^{n-1}, \qquad x \in [a,b], \tag{28}$$

where $M = K(b-a)^2\dfrac{(n+1)}{2}$ and $K = \dfrac{1}{(n+1)!}\max_{x\in[a,b]}|y^{(n+1)(x)}|$.

Proof. Let $\tilde{u}_i(x)$ be the auxiliary approximate solution of the piecewise initial value problem (3) with the initial conditions (4) and $u_i(x)$ be the approximate solution of (3) with the initial conditions (6) or (7), then

$$|y_i(x)-u_i(x)| \leq |y_i(x)-\tilde{u}_i(x)|+|\tilde{u}_i(x)-u_i(x)|,$$

$\forall x \in S_i$ for $i = 1,\ldots,N$. Using Equations (19) and (23), we obtain

$$\begin{aligned}|y_{i+1}(x) - u_{i+1}(x)| &\leq |u_i(a_i) - \tilde{u}_i(a_i)| \\ &\left(1+\left(\frac{h^2}{2!}+\ldots+\frac{h^k}{k!}+\ldots+\frac{h^n}{n!}\right)C_1\right) \\ &+|u_i'(a_i) - \tilde{u}_i'(a_i)| \\ &\left(h+\left(\frac{h^2}{2!}+\ldots+\frac{h^k}{k!}+\ldots+\frac{h^n}{n!}\right)C_2\right) \\ &+Kh^{n+1}\end{aligned} \tag{29}$$

Note that $\tilde{u}_1(x)$ and $u_1(x)$ are exactly same since they are the approximate solutions of the same initial value problem. So, from Equations (19) and (20), we get

$$|y_1(x) - u_1(x)| \leq Kh^{n+1}, \quad \forall x \in S_1 \tag{30}$$

$$|y_1'(a_1) - u_1'(a_1)| \leq (n+1)Kh^n. \tag{31}$$

Using the recurrence relations in Equations (29) and (25) and inequalities in Equations (30) and (31), we get by mathematical induction that

$$\begin{aligned}|y_N(x)-u_N(x)| &\leq \left(\frac{1}{2}(n+1)N^2 - \frac{1}{2}(n-1)N\right)Kh^{n+1} + O(h^{n+2}) \\ &\leq \frac{1}{2}(n+1)K(b-a)^2h^{n-1} + O(h^n)\end{aligned}$$

since $N = (b-a)/h$. Hence, we have

$$|y(x) - u(x)| \leq Mh^{n-1}.$$

Lemma 5. Let $\tilde{u}_1(x)$ be the auxihary approximate solution of piecewise initial value problem (13) with initial conditions (14). Then, we have

$$|y_1(x)-u_1(x)| \leq KLh^{n+1}+O(h^{n+2}), \forall x \in S_1 \tag{32}$$

$$|y_1'(x)-u_1'(x)| \leq K\bar{L}h^n + O(h^{n+1}), \forall x \in S_1, \tag{33}$$

where $y_1(x)$ is the corresponding exact solution and

$$\begin{aligned}K &= \frac{1}{(n+1)!}\max_{x\in(a,b]}|y^{(n+1)}(x)|, \\ L &= 2^{n+1} - n - 2 \\ \bar{L} &= (n+1)(2^n - 1).\end{aligned}$$

Proof. Using Taylor expansion of $|u_1^{(k)}(x) - y_1^{(k)}(x)|$, $k = 0,1$ about $x = a_0$ and $|u_1^{(k)}(x) - y_1^{(k)}(x)|$, $k = 2,3,\ldots,n$ about $x = \eta \in (a_0, a_1]$, we have

$$\begin{aligned}|u_1^{(k)}(x) - y_1^{(k)}(x)| &\leq |u_1^{(k)}(a_0) - y_1^{(k)}(a_0)| \\ &+ |x-a_0||u_1^{(k+1)}(a_0) - y_1^{(k+1)}(a_0)| \\ &+ \ldots \\ &+ \frac{|x-a_0|^{n-k}}{(n-k)!}|u_1^{(n)}(a_0) - y_1^{(n)}(a_0)| \\ &+ \frac{|x-a_0|^{n-k+1}}{(n-k+1)!}|y_1^{(n+1)}(\xi_k)|,\end{aligned} \tag{34}$$

where ξ_k, $k = 0,1$ and between a_0 and x;

$$\begin{aligned}|u_1^{(k)}(x)-y_1^{(k)}(x)| &\leq |u_1^{(k)}(\eta) - y_1^{(k)}(\eta)| \\ &+ |x-\eta||u_1^{(k+1)}(\eta) - y_1^{(k+1)}(\eta)| \\ &+ \ldots \\ &+ \frac{|x-\eta|^{n-k}}{(n-k)!}|u_1^{(n)}(\eta) - y_1^{(n)}(\eta)| \\ &+ \frac{|x-\eta|^{n-k+1}}{(n-k+1)!}|y_1^{(n+1)}(\xi_k)|,\end{aligned} \tag{35}$$

where ξ_k, $k = 2,3,\ldots,n$ are between η and x. Equation (15) for $k = 0,1,\ldots,n-2$ yields

$$\begin{aligned}|u_1^{(k)}(\eta) - y_1^{(k)}(\eta)| \leq &C_1|u_1(\eta) - y_1(\eta)| \\ &+ C_2|u_1'(\eta) - y_1'(\eta)|\end{aligned} \tag{36}$$

where C_1 and C_2 are defined in Lemma 3. Using inequalities in Equations (34), (35) and (36) repeatedly, we get the results in Equations (32) and (33).

The following corollary can be proved with mathematical induction using the recurrence relations in Equations (29), (25) together with the inequalities in Equations (32) and (33).

Corollary 6. Let $y(x)$ be the exact solution of the Lane-Emden type equation (1) and $u(x)$ be the corresponding n^{th} degree approximate function. Then, we have

$$|y(x) - u(x)| \leq Mh^{n-1}, \qquad x \in [0, b],$$

where M is a constant, which does not depend on h.

5. Numerical Examples

In this section, we provide some numerical ex where we have are between and and amples to illustrate the applicability of the proposed method. The tables and figures demonstrate the power of the current study.

Example 1. Consider the Lane-Emden equation in Wang *et al*. (2014)

$$\begin{aligned} y'' + \frac{2}{x}y' + y^5 = 0, \quad &\text{for} \quad 0 \leq x \leq 1 \\ y(0) = 1, \qquad &y'(0) = 0 \end{aligned} \tag{37}$$

with the exact solution

$$y(x) = (1 + x^2/3)^{-1/2}$$

Algorithm of the method for the above problem with $n = 3$ and $N = 50$ as follows

Step 1: Set $N = 50, h = 1/N$.

Step 2: Compute $a_i = ih$ for $i = 1, 2, \ldots, N$.

Step 3: Construct

$$u_i(x) = \sum_{j=0}^{3} c_j^i B_j(\frac{x - a_{i-1}}{h}),$$

$$R_i(x) = u_i''(x) + \frac{2}{x}u_i'(x) + u_i(x)^5,$$

$i = 1, 2, \ldots, 50.$

Step 4: Set $c_0^1 = 1$, $c_1^1 = 1$.

Step 5: Solve c_2^1 and c_3^1 from following non-linear equations

$$\begin{aligned} R_1(a_1) = &\frac{6}{h^2}(c_3^1 - 2c_2^1 + c_1^1) \\ &+ \frac{2}{a_1}\frac{3}{h}(c_3^1 - c_2^1) + (c_3^1)^5 = 0, \end{aligned}$$

$$\begin{aligned} R_1'(a_1) = &\frac{6}{h^3}(c_3^1 - 3c_2^1 + 3c_1^1 - c_0^1) \\ &- \frac{2}{a_1^2}\frac{3}{h}(c_3^1 - c_2^1) + \frac{2}{a_1}\frac{6}{h^2}(c_3^1 - 2c_2^1 + c_1^1) \\ &+ \frac{3}{h}5(c_3^1)^4(c_3^1 - c_2^1) = 0 \end{aligned}$$

by Newton's method.

Step 6: Set *i* = 2.

Step 7: Compute $c_0^i = c_3^{i-1}$ and $c_1^i = 2c_3^{i-1} - c_2^{i-1}$.

Step 8: Solve c_2^i and c_3^i from the following linear equations

$$\begin{aligned} R_i(a_{i-1}) = &\frac{6}{h^2}(c_2^i - 2c_1^i + c_0^i) \\ &+ \frac{2}{a_{i-1}}\frac{3}{h}(c_1^i - c_0^i) + (c_0^1)^5 = 0, \end{aligned}$$

$$\begin{aligned} R_i'(a_{i-1}) = &\frac{6}{h^3}(c_3^i - 3c_2^i + 3c_1^i - c_0^i) \\ &- \frac{2}{a_{i-1}^2}\frac{3}{h}(c_1^i - c_0^i) \\ &+ \frac{2}{a_{i-1}}\frac{6}{h^2}(c_2^i - 2c_1^i + c_0^i) \\ &+ \frac{3}{h}5(c_0^i)^4(c_1^i - c_0^i) = 0. \end{aligned}$$

Step 9: Repeat steps 7 and 8 for

$i = 3, 4, \ldots, N.$

Step 10: Write the piecewise approximate solution as

$$u(x) = \begin{cases} u_1(x), & x \in [a_0, a_1] \\ u_2(x), & x \in [a_1, a_2] \\ \vdots & \\ u_N(x), & x \in [a_{N-1}, a_N] \end{cases}.$$

Table 1 gives a comparison ofnumerical results with the results of the reproducing kernel method (RKM) given in Wang *et al*. (2014) and we observe that the accuracy obtained is high enough. In Table 2, we give the observed orders obtained using the following formulae

$$ord(N/2N) = \frac{\log(\frac{\max|y(x) - u(x,N)|}{\max|y(x) - u(x;2N)|})}{\log(2)}, \tag{38}$$

where *u(x;N)* denotes the approximate solution of *y(x)* obtained by *N* subintervals. It can be seen that observed orders are well to confirm the theoretical results. Figure 2 shows the error functions for N = 50, 100, 200.

Table 1. Comparison of the errors of the present method with RKM errors.

x	Exact Values	RKM ($N=50$) (in Wang *et al.* (2014))	Presented Method $n=3$ $N=50$	$n=4$ $N=50$	$n=5$ $N=50$	$n=5$ $N=200$
0.	1.	0.	0.	0.	0.	0.
0.1	0.998337	5.115E-7	2.243E-7	1.762E-10	3.194E-11	1.893E-13
0.2	0.993399	7.322E-7	6.306E-7	1.633E-9	8.934E-11	6.780E-13
0.3	0.985329	7.218E-7	1.144E-6	5.405E-9	1.596E-10	8.862E-13
0.4	0.974355	2.963E-7	1.711E-6	1.199E-8	2.250E-10	1.556E-12
0.5	0.960769	1.077E-6	2.248E-6	2.127E-8	2.660E-10	1.960E-12
0.6	0.944911	8.175E-7	2.675E-6	3.259E-8	2.682E-10	1.169E-12
0.7	0.927146	1.459E-6	2.927E-6	4.496E-8	2.253E-10	3.251E-13
0.8	0.907841	1.701E-6	2.960E-6	5.721E-8	1.390E-10	5.829E-13
0.9	0.887357	2.491E-6	2.757E-6	6.828E-8	1.770E-11	6.768E-13
1.	0.866025	3.596E-6	2.324E-6	7.732E-8	1.265E-10	9.200E-13

Table 2. Observed orders of Example 1 with n = 3, 4, 5 and = 25, 50, 100, 200.

N	$n=3$	$n=4$	$n=5$
25/50	2.281	3.028	4.339
50/100	2.119	3.017	4.176
100/200	2.049	3.010	3.278

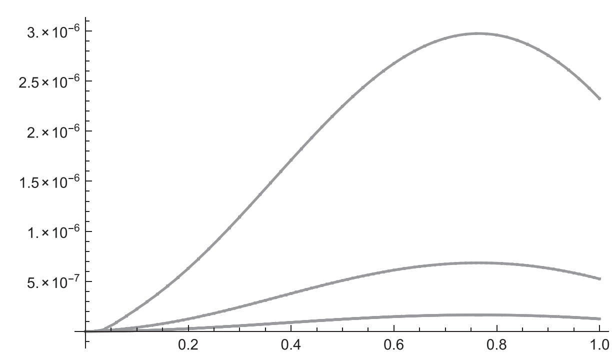

Fig. 2. Graphs of Error functions of Example 1 obtained using 3rd degree approximate function with number of intervals $N = 50,\ 100,\ 200$.

Example 2. Consider the Lane-Emden equation in Parand *et al*. (2010) and Wang *et al*. (2014)

$$y'' + \frac{2}{x}y' + 4(2e^y + e^{y/2}) = 0, \quad \text{for } x \geq 0 \qquad (39)$$
$$y(0) = 0, \qquad y'(0) = 0$$

with the exact solution $y(x) = -2\ln(1 + x^2)$.

Table 5 gives a comparison of numerical results with the results of Hermite functions collocation method (HFC) and RKM given in Parand *et al*. (2010) and Wang *et al*. (2014), respectively. In Table 3, observed orders obtained using (38) are given.

Example 3. Consider the Lane-Emden equation in Parand *et al*. (2010)

$$y'' + \frac{2}{x}y' + \sin y = 0, \quad \text{for } x \geq 0$$
$$y(0) = 1, \qquad y'(0) = 0$$

with the asymptotic solution given in Wazwaz

$$\begin{aligned} y(x) \;& \simeq 1 - \frac{1}{6}k_1x^2 + \frac{1}{120}k_1k_2x^4 \\ & + k_1\left(\frac{1}{3024}k_1^2 - \frac{1}{5040}k_2^2\right)x^6 \\ & + k_1k_2\left(-\frac{113}{3265920}k_1^2 + \frac{1}{362880}k_2^2\right)x^8 \\ & + k_1\left(\frac{1781}{898128000}k_1^2k_2^2 - \frac{1}{399168000}k_2^4 \right. \\ & \left. - \frac{19}{23950080}k_1^4\right)x^{10}, \end{aligned}$$

where $k_1 = \sin(1)$ and $k_2 = \cos(1)$.

Table 6 gives a comparison of some numerical results of the presented method with an asymptotic solution given in Wazwaz (2001) and the results of HFC given in Parand *et al*. (2010). In Table 4, we give orders using

$$order(N/2N) = \frac{\log\left(\frac{\max|u(x;N)-u(x;2N)|}{\max|u(x;2N)-u(x;4N)|}\right)}{\log(2)}$$

where $u(x;N)$, $u(x;2N)$, $u(x;4N)$ are approximate solutions of $y(x)$ obtained by respectively. From Tables 2, 3 and 4, it is concluded that observed orders are equal to as proved in Corollary 6.

6. Conclusion

In this paper, we have improved a linear residual method. The obtained results give us a piecewise approximate solution of the Lane- Emden type equations. An advantage of this method is that we will have more accurate solutions for the fixed degree of polynomials by solving the linear systems of equations. We have seen that the obtained numerical results are compatible with the theoretical aspects. This method can be extended to some singularly perturbed boundary value problems, the Emden-Fowler equations, strongly non-linear boundary value problems and the chaotic initial value systems such as Lorenz systems, Genesio-Tesi systems, Rossler systems etc.

Table 3. Observed orders of Example2 with n=3;4;5 and N=25;50;100;200.

N	$n=3$	$n=4$	$n=5$
25/50	1.900	3.000	2.264
50/100	2.093	3.033	3.903
100/200	2.061	3.028	4.022

Table 4. Observed orders of Example3 with n=3;4;5 and N=25;50;100.

N	$n=3$	$n=4$	$n=5$
25/50	2.345	2.975	4.423
50/100	2.163	2.998	4.010

Table 5. Comparison of the errors of the present method with RKM and HFC errors.

x	Exact Values	RKM ($N=50$) (in Wang *et al.* (2014))	HFC (in Parand *et al.* (2010))	Presented Method $N=50$ $n=5$
0	0.	0	0	0.
0.01	-0.00019999	6.666E-13	2.931E-6	1.886E-13
0.1	-0.0199007	3.829E-8	3.939E-6	2.224E-7
0.5	-0.446287	1.026E-7	3.018E-6	4.755E-6
1	-1.38629	4.361E-6	9.314E-7	2.632E-6
2	-3.21888	4.175E-6	4.999E-7	5.099E-6
3	-4.60517	1.860E-7	8.104E-7	2.257E-6
4	-5.66643	3.312E-6	7.692E-7	1.967E-7

Table 6. Comparison of the errors of the present method with asymptotic solution and HFC errors.

x	Wazwaz (2001)	Error in Parand *et al.* (2010)	Error in presented method
0	1.	0	0.
0.1	0.998598	7.207E-6	8.427E-9
0.2	0.994396	9.997E-6	8.446E-9
0.5	0.965178	1.038E-5	8.538E-9
1	0.863681	7.027E-6	2.267E-8
1.5	0.705042	1.049E-5	3.309E-6
2	0.506372	9.672E-5	9.156E-5

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طريقة البواقي باستخدام منحنيات بزير للمعادلات غير الخطية الأحادية من نوع لاين–أمدن Lane–Emden

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خلاصة

في هذا البحث نقدم طريقة جديدة لإيجاد حل معادلة أحادية غير خطية من نوع لاين-امدن Lane-Emden لإيجاد الحل التقريبي باستخدام متعددات الحدود من نوع برنشتين Bernstein. تعتمد هذه الطريقة على تصغير دالة البواقي باستخدام مفكوك تيلور Taylor. قمنا أيضاً بتطبيق هذه الطريقة على مسائل تم حلها باستخدام طرق أخرى وتوضح النتائج التي تم الحصول عليها أن الطريقة الجديدة كفوء وقابلة للاستخدام ولها مستقبل واعد مقارنةً بالطرق الأخرى.