Four dimensional matrix mappings and applications

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Abstract

In this paper, we characterize the classes $(\mathcal{L},\mathcal{L}_k)$, $(\mathcal{L}_k,\mathcal{L})$ and $(\mathcal{L}_\infty,\mathcal{L}_k)$, $1\le k<\infty$, of all four dimensional infinite matrices, where $\mathcal{L}_k$ and $\mathcal{L}_\infty$ are the spaces of all absolutely k-summable and bounded double sequences, respectively. Using them, we establish some relations between $\left|\overline{N},p_n,q_n\right|$ and $\left|\overline{N},p'_n,q'_n\right|_k$ summability methods which extend some results of Bosanquet (1950), Sarıgöl (1993), Sarıgöl & Bor (1995), and Sunouchi (1949) to double summability methods, and give a relation between single and double summability methods.

Keywords: Banach space, double matrix mapping, double summability, four dimensional matrix, inclusion theorem

1. Introduction

Let us consider an infinite single series Σx_v of complex (or real) numbers with partial sums s_n, and let (σ_n^α) denote the n-th Cesàro means of order α with $\alpha>-1$ of the sequence (s_n). The series Σx_v is said to be summable $|C,\alpha|_k$, $k\ge 1$, in Flett's notation (Flett, 1957), if $\left(n^{1/k^*}\Delta\sigma_n^\alpha\right)\in\ell_k$, where ℓ_k is the space of the set of absolutely k-summable single sequences and $1/k^*+1/k=1$. Let (p_n) be a sequence of positive numbers satisfying

$$P_n=\sum_{v=0}^{n}p_v\to\infty\ as\ n\to\infty,\ P_{-1}=p_{-1}=0. \tag{1}$$

The sequence-to-sequence transformation $u_n=\sum_{v=0}^{n}p_vs_v/P_n$ defines the sequence (u_n) of the weighted mean or simply $\left(\overline{N},p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (Hardy, 1949). The series Σx_v is said to be summable $\left|\overline{N},p_n\right|_k$, $k\ge 1$, if $\left\{\left(p_n^{-1}P_n\right)^{1/k^*}\Delta u_n\right\}\in\ell_k$, where $\Delta u_n=p_n\left(P_nP_{n-1}\right)^{-1}\sum_{v=1}^{n}P_{v-1}x_v$ (Bor, 2016), which, for $p_n=1$, includes the method $|C,1|_k$.

Throughout the paper, (p_n), (q_n), (p'_n) and (q'_n) will denote the sequences of positive numbers satisfying equation 1 and

$$\mu_{mn}(k)=\begin{cases}\frac{1}{P_{m-1}}\left(\frac{p_m}{P_m}\right)^{1/k}, & n=0,m\ge 1\\ \frac{1}{Q_{n-1}}\left(\frac{q_n}{Q_n}\right)^{1/k}, & m=0,n\ge 1\\ \frac{1}{P_{m-1}Q_{n-1}}\left(\frac{p_mq_n}{P_mQ_n}\right)^{1/k}, & m\ge 1,n\ge 1.\end{cases} \tag{2}$$

A summability method Y is stronger than another method X if each series summable by X implies its summability by Y (not necessarily to the same sum). Hereof, there are many papers in the literature done by various authors, e.g. (see, (Bor, 2016), (Bor & Thorpe, 1987), (Borwein & Cass, 1968), (Bosanquet, 1950), (Das *et al.*, 1967), (Flett, 1957), (Hardy, 1949), (Güleç, 2019), (Mazhar, 1972), (Mishra *et al.*, 2018), (Mohapatra, 1967), (Rhoades, 1998), (Rhoades, 1999), (Rhoades, 2003), (Sarıgöl, 1991), (Sarıgöl, 1992), (Sarıgöl, 1993), (Sarıgöl & Bor, 1995), (Sarıgöl, 2021), (Sarıgöl & Mursaleen, 2021), (Sunouchi, 1949), (Thorpe, 1972), (Zraiqat, 2019)). Among them, in the special case $k = 1$ the following known result is due to Sunouchi (Sunouchi, 1949).

Theorem 1.1. In order that every $\left|\overline{N}, p_n\right|$ summable series should be $\left|\overline{N}, p'_n\right|$ summable, it is sufficient that

$$\frac{p'_n P_n}{P'_n p_n} = O\left(1\right). \tag{3}$$

Reviewing this paper, Bosanquet observed that equation 3 is also necessary for the conclusion and so completed Theorem 1.1 in necessary and sufficient form (see (Bosanquet, 1950)).

In (Sarıgöl, 1993), Theorem 1.1 has been extended to the case $1 \leq k < \infty$ as follows.

Theorem 1.2. Let $1 \leq k < \infty$. Then, in order that every $\left|\overline{N}, p_n\right|$ summable series should be $\left|\overline{N}, p'_n\right|_k$ summable, it is necessary and sufficient that

$$\frac{p'_n}{P'_n}\left(\frac{P_n}{p_n}\right)^k = O\left(1\right).$$

Also, it has been showed in (Sarıgöl & Bor, 1995) that the converse of the implication is not true.

Theorem 1.3. Let $1 < k < \infty$. Then, for every sequences (p_n) and (p'_n), there exists a series which is summable $\left|\overline{N}, p_n\right|_k$ but is not summable by $\left|\overline{N}, p'_n\right|$.

First, we recall related notations. Let $\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} x_{rs}$ be an infinite double series of real or complex numbers with partial sums s_{mn}, *i.e.*,

$$s_{mn} = \sum_{r=0}^{m}\sum_{s=0}^{n} x_{rs}. \tag{4}$$

For the sake of brevity, we denote the summations $\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}$ and $\sum_{r=0}^{m}\sum_{s=0}^{n}$ by $\sum_{r,s=0}^{\infty}$ and $\sum_{r,s=0}^{m,n}$, respectively. By T_{mn}, we denote the double Riesz mean transformation $\left(\overline{N}, p_m, q_n\right)$ of the double sequence (s_{mn}), *i.e.*,

$$T_{mn} = \frac{1}{P_m Q_n}\sum_{r,s=0}^{m,n} p_r q_s s_{rs}. \tag{5}$$

The series $\sum_{r,s=0}^{\infty} x_{rs}$ is said to be summable $\left|\overline{N}, p_m, q_n\right|_k$, $k \geq 1$, if (see (Sarıgöl, 2021))

$$\sum_{m,n=0}^{\infty}\left(\frac{P_m Q_n}{p_m q_n}\right)^{k-1}\left|\overline{\Delta} T_{mn}\right|^k < \infty \tag{6}$$

where $\overline{\Delta} T_{00} = s_{00} = x_{00}$, and, for $m, n \geq 1$,

$$\begin{aligned} \overline{\Delta} T_{m0} &= T_{m0} - T_{m-1,0},\ \overline{\Delta} T_{0n} = T_{0n} - T_{0,n-1}, \\ \overline{\Delta} T_{mn} &= T_{mn} - T_{m-1,n} - T_{m,n-1} + T_{m-1,n-1}. \end{aligned}$$

We note that, in the special case $p_n = q_n = 1$, the summability $\left|\overline{N}, p_m, q_n\right|_k$ reduces to the absolute double Cesàro summability $\left|C, 1, 1\right|_k$, given by Rhoades (1998).

There is a close relationship between the method $\left|\overline{N}, p_m, q_n\right|_k$ and the space $\mathcal{L}_k$, $1 \leq k < \infty$, defined by the set of all double sequences $x = (x_{rs})$ of complex numbers such that $\sum_{r,s=0}^{\infty}\left|x_{rs}\right|^k < \infty$, which reduces to $\mathcal{L}$ for $k = 1$, studied by Zeltser (2001). Also, $\mathcal{L}_k$ is the Banach space (Başar & Sever, 2009) according to its natural norm

$$\|x\|_{\mathcal{L}_k} = \left(\sum_{r,s=0}^{\infty}\left|x_{rs}\right|^k\right)^{1/k}, 1 \leq k < \infty.$$

Further, the space $\mathcal{L}_\infty$ consists of all bounded double sequences and it is a Banach space with the norm $\|x\|_{\mathcal{L}_\infty} = \sup_{r,s} |x_{rs}|$.

Let $x = (x_{rs})$ be a double sequence of complex numbers. If for every $\varepsilon > 0$ there exists a natural integer $n_0(\varepsilon)$ and real number l such that $|x_{rs} - l| < \varepsilon$ for all $r, s \geq n_0(\varepsilon)$, then, the double sequence x is said to be convergent in the Pringsheim sense. Also, a double series $\sum_{r,s=0}^{\infty} x_{rs}$ is convergent if and only if the double sequence (s_{mn}) in equation 4 is convergent.

Let U and V be two double sequence spaces, and $A = (a_{mnrs})$ be a four dimensional infinite matrix of complex (or, real) numbers. Then, A defines a matrix transformation from U into V, written $A \in (U, V)$, if for every sequence $x = (x_{rs}) \in U,$ the A-transform $A(x) = (A_{mn}(x))$ of x exists and belongs to V, where

$$A_{mn}(x) = \sum_{r,s=0}^{\infty} a_{mnrs} x_{rs}$$

provided the double series on right side converges for $m, n \geq 0$.

The transpose $A^t = (a_{rsmn})$ of the matrix $A = (a_{mnrs})$ is defined by

$$A^t_{rs}(x) = \sum_{m,n=0}^{\infty} a_{mnrs} x_{mn} \text{ for } m, n \geq 0.$$

The β-dual U^β of the space U is the set of all double sequences (b_{rs}) such that $\sum_{r,s=0}^{\infty} b_{rs} x_{rs}$ converges for all $x \in U$.

In this paper we characterize the classes $(\mathcal{L}, \mathcal{L}_k)$, $(\mathcal{L}_k, \mathcal{L})$ and $(\mathcal{L}_\infty, \mathcal{L}_k)$, $k \geq 1$, of all four dimensional infinite matrices, and extend Theorem 1.1, Theorem 1.2 and Theorem 1.3 to double summability methods, and also establish a relation between single and double summability methods.

2. Needed Lemmas

We require the following lemmas for the proofs of our theorems.

Lemma 2.1 (Zaanen 1953, p.134) A linear mapping T from a Banach space U into another Banach space V is continuous if and only if it is bounded, i.e., there exists a constant L such that $\|T(x)\|_V \leq L \|x\|_U$ for all $x \in U$.

Lemma 2.2 (Sarıgöl, 1991) Let $k > 0$. Then, there exists two strictly positive constans M_1 and M_2, depending only on k, such that

$$\frac{M_1}{P^k_{r-1}} \leq \sum_{m=r}^{\infty} \mu^k_{m0}(k) \leq \frac{M_2}{P^k_{r-1}} \tag{7}$$

for all $r \geq 1$, where M_1 and M_2 are independent of (p_n) .

Lemma 2.3 (Sarıgöl, 2021) Let $k > 0$. Then, there exists two strictly positive constans N_1 and N_2, depending only on k, such that

$$\frac{N_1}{P^k_{r-1} Q^k_{s-1}} \leq \sum_{m,n=r,s}^{\infty} \mu^k_{mn}(k) \leq \frac{N_2}{P^k_{r-1} Q^k_{s-1}} \tag{8}$$

for all $r, s \geq 1$, where N_1 and N_2 are independent of (p_n) and (q_n) .

3. Main Result

Our results are as follows.

Theorem 3.1 Let $k \geq 1$ and $A = (a_{mnrs})$ be a four dimensional infinite matrix of complex numbers. Then, in order that $A \in (\mathcal{L}, \mathcal{L}_k)$ it is necessary and sufficient that

$$\sum_{m,n=0}^{\infty} |a_{mnrs}|^k = O(1). \tag{9}$$

Proof. Assume equation 9 holds. Then, we should show that $A(x)=(A_{mn}(x)) \in \mathcal{L}_k$ for every $x=(x_{rs}) \in \mathcal{L}$. Now, using equation 9, it follows from Minkowski's inequality that

$$\begin{aligned} \|A(x)\|_{\mathcal{L}_k} &= \left(\sum_{m,n=0}^{\infty} |A_{mn}(x)|^k\right)^{1/k} \leq \left(\sum_{m,n=0}^{\infty}\left(\sum_{r,s=0}^{\infty}|a_{mnrs}x_{rs}|\right)^k\right)^{1/k} \\ &= \sum_{r,s=0}^{\infty}|x_{rs}|\left(\sum_{m,n=0}^{\infty}|a_{mnrs}|^k\right)^{1/k} = O(1)\,\|x\|_{\mathcal{L}} < \infty. \end{aligned}$$

which gives the desired conclusion.

Conversely, let $A \in (\mathcal{L}, \mathcal{L}_k)$. Then, for $k \geq 1$, since $\mathcal{L}_k$ is a Banach space (see (Başar & Sever, 2009)), by Lemma 2.1, there exists a constant K such that $\|A(x)\|_{\mathcal{L}_k} \leq K\,\|x\|_{\mathcal{L}}$, *i.e.*,

$$\left(\sum_{m,n=0}^{\infty}\left|\sum_{r,s=0}^{\infty} a_{mnrs}x_{rs}\right|^k\right)^{1/k} \leq K\,\|x\|_{\mathcal{L}} \tag{10}$$

for all $x \in \mathcal{L}$. So, by applying the double sequence $x \in \mathcal{L}$ to equation 10, where $x_{ij}=1$ for $i=r, j=s$, zero otherwise, we obtain

$$\sum_{m,n=0}^{\infty}|a_{mnrs}|^k \leq K, \text{ for } r,s \geq 0, \tag{11}$$

which gives equation 9.

This step concludes the proof.

Theorem 3.2 Let $1<k<\infty$ and $A=(a_{mnij})$ be an four dimensional infinite matrix of complex numbers. Define $W_k(A)$ and $w_k(A)$ by

$$W_k(A)=\sum_{r,s=0}^{\infty}\left(\sum_{m,n=0}^{\infty}|a_{mnrs}|\right)^k, \tag{12}$$

$$w_k(A)=\sup_{MXN}\sum_{r,s=0}^{\infty}\left|\sum_{(m,n)\in MXN} a_{mnrs}\right|^k \tag{13}$$

where M and N are finite subsets of natural numbers. Then, the following statements are equivalent:

$$(i)\ \ W_{k^*}(A)<\infty \qquad (ii)\ \ A\in(\mathcal{L}_k,\mathcal{L})$$

$$(iii)\ \ A^t\in(\mathcal{L}_\infty,\mathcal{L}_{k^*}) \qquad (iv)\ \ w_{k^*}(A)<\infty.$$

where k^* is the conjugate of k, *i.e.*, $1/k+1/k^*=1$.

Proof. To prove the Theorem, it is enough to show that $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)\Rightarrow(i)$.

$(i)\Rightarrow(ii)$. Assume (i) holds. Then, for all $x\in\mathcal{L}_k$, it follows from Hölder's inequality that

$$\begin{aligned} \|A(x)\|_{\mathcal{L}} &= \sum_{m,n=0}^{\infty}\left|\sum_{r,s=0}^{\infty}a_{mnrs}x_{rs}\right| \leq \sum_{r,s=0}^{\infty}\sum_{m,n=0}^{\infty}|a_{mnrs}x_{rs}| \\ &\leq \left\{\sum_{r,s=0}^{\infty}\left(\sum_{m,n=0}^{\infty}|a_{mnrs}|\right)^{k^*}\right\}^{1/k^*}\|x\|_{\mathcal{L}_k} \qquad (14) \\ &\leq (W_{k^*}(A))^{1/k^*}\,\|x\|_{\mathcal{L}_k}<\infty, \end{aligned}$$

which gives (ii) .

$(ii) \Rightarrow (iii)$. Suppose $A \in (\mathcal{L}_k, \mathcal{L})$. Then, since $\mathcal{L}_k$ is a Banach space, where $k \geq 1$, by Lemma 2.1, there exists a constant L such that

$$\|A(x)\|_{\mathcal{L}} = \sum_{m,n=0}^{\infty} \left| \sum_{r,s=0}^{\infty} a_{mnrs} x_{rs} \right| \leq L \|x\|_{\mathcal{L}_k} \tag{15}$$

for all $x \in \mathcal{L}_k$. Also, it is observed by putting $x_{rs}\text{sgn}a_{mnrs}$ instead of x_{rs} that

$$\sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mnrs} x_{rs}| \leq L \|x\|_{\mathcal{L}_k} . \tag{16}$$

Now, let $u \in \mathcal{L}_\infty$ be given. Then, by equation 15,

$$\begin{aligned} \left| \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} u_{mn} a_{mnrs} x_{rs} \right| &\leq \|u\|_{\mathcal{L}\infty} \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mnrs} x_{rs}| \\ &\leq L \|u\|_{\mathcal{L}\infty} \|x\|_{\mathcal{L}_k} . \end{aligned} \tag{17}$$

In equation 17, taking $x_{rs} = 1$ for $(r,s) = (i,j)$, and zero otherwise, it is easily seen that

$$\left| \sum_{m,n=0}^{\infty} a_{mnrs} u_{mn} \right| \leq \sum_{m,n=0}^{\infty} |a_{mnrs} u_{mn}| \leq L \|u\|_{\mathcal{L}_\infty} ,$$

which gives that $A^t(u)$ is defined for all $r, s \geq 0$, where the double sequence $A^t(u) = \left(A^t_{rs}(u)\right)$ is given by

$$A^t_{rs}(u) = \sum_{m,n=0}^{\infty} a_{mnrs} u_{mn} : m, n \geq 0 \tag{18}$$

Again, it follows by considering equation 17 that

$$\left| \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} A^t_{rs}(u) x_{rs} \right| \leq L \|u\|_{\mathcal{L}\infty} \|x\|_{\mathcal{L}_k} \tag{19}$$

which implies that the series in the left side hand of equation 19 converges. Therefore, since the dual of space $\mathcal{L}_k$ is the space $\mathcal{L}_{k^*}$(see (Başar & Sever, 2009)), we obtain $A^t(u) \in \mathcal{L}_{k^*}$, *i.e.*, $A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k^*})$.

$(iii) \Rightarrow (iv)$. If $A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k^*})$, then, by Lemma 2.1, there exists a constant K such that $\left\|A^t(x)\right\|_{\mathcal{L}_{k^*}} \leq K \|x\|_{\mathcal{L}_\infty}$ for all $x \in \mathcal{L}_\infty$, *i.e.*,

$$\left(\sum_{r,s=0}^{\infty} \left| \sum_{m,n=0}^{\infty} a_{mnrs} x_{mn} \right|^{k^*} \right)^{1/k^*} \leq K \|x\|_{\mathcal{L}_\infty} . \tag{20}$$

Let M and N be any finite subsets of all nature numbers. Take a sequence $x = (x_{mn})$ as $x_{mn} = 1$ for $(r,s) \in MXN$, and zero otherwise. Then, equation 20 is reduced to.

$$\left(\sum_{r,s=0}^{\infty} \left| \sum_{(m,n)\in MXN} a_{mnrs} \right|^{k^*} \right)^{1/k^*} \leq K$$

which proves $w_{k^*}(A) < \infty$.

$(iii) \Rightarrow (iv)$. Suppose (iii) is satisfied and a_{mnrs} are real numbers. Then, for every finite subsets M and N of nature numbers,

$$\sum_{r,s=0}^{\infty}\left|\sum_{(m,n)\in MXN} a_{mnrs}\right|^{k^*} \leq w_{k^*}(A).$$

Let $H^+ = \{(m,n) \in MXN : a_{mnrs} \geq 0\}$ and $H^- = \{(m,n) \in MXN : a_{mnrs} < 0\}$. Then, by considering the inequality $|a+b|^{k^*} \leq 2^{k^*}\left(|a|^{k^*} + |b|^{k^*}\right)$, where a and b are complex numbers, we have

$$\begin{aligned}
W_{k^*}(A) &= \sum_{r,s=0}^{\infty}\left(\sum_{m,n=0}^{\infty}|a_{mnrs}|\right)^{k^*} \\
&= \sum_{r,s=0}^{\infty}\left\{\sum_{(m,n)\in H^+}^{\infty} a_{mnrs} + \sum_{(m,n)\in H^-}^{\infty} -a_{mnrs}\right\}^{k^*} \\
&\leq 2^{k^*}\sum_{r,s=0}^{\infty}\left\{\left(\sum_{(m,n)\in H^+}^{\infty} a_{mnrs}\right)^{k^*} + \left(\sum_{(m,n)\in H^-}^{\infty} -a_{mnrs}\right)^{k^*}\right\} \\
&\leq 2^{k^*+1}w_k(A).
\end{aligned}$$

If a_{mnrs} is complex number for $m,n,r,s \geq 0$, it is easily seen that $W_{k^*}(A) \leq 2^{2k^*+3}w_k(A) < \infty$, which implies (iv) .

Thus the proof of the Theorem is completed.

Theorem 3.3 Let $k \geq 1$. Then, in order that every $\left|\overline{N}, p_m, q_n\right|$ summable double series should be summable $\left|\overline{N}, p'_m, q'_n\right|_k$, it is necessary and sufficient that

$$(i)\quad \frac{p'_m}{P'_m}\left(\frac{P_m}{p_m}\right)^k = O(1) \quad \text{and} \quad (ii)\quad \frac{q'_n}{Q'_n}\left(\frac{Q_n}{q_n}\right)^k = O(1). \tag{21}$$

Proof. Suppose that euation 21i and equation 21ii are satisfied. Let (T_{mn}) and (T'_{mn}) be the double sequences of $\left(\overline{N}, p_n, q_n\right)$ and $\left(\overline{N}, p'_n, q'_n\right)$ means of the series $\sum_{r,s=0}^{\infty} x_{rs}$, respectively, i.e.,

$$T_{mn} = \frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n} p_rq_s\sum_{v,\mu=0}^{r,s} x_{v\mu}, \tag{22}$$

$$T'_{mn} = \frac{1}{P'_mQ'_n}\sum_{r,s=0}^{m,n} p'_rq'_s\sum_{v,\mu=0}^{r,s} x_{v\mu}. \tag{23}$$

Then, since $P_{-1} = Q_{-1} = 0$, it can be written that

$$\begin{aligned}
T_{mn} &= \frac{1}{P_mQ_n}\sum_{v,\mu=0}^{m,n} p_vq_\mu\sum_{r,s=0}^{v,\mu} x_{r,s} \\
&= \frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n} x_{r,s}\sum_{v,\mu=r,s}^{m,n} p_vq_\mu \\
&= \frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n} x_{r,s}\left(P_m - P_{r-1}\right)\left(Q_n - Q_{s-1}\right) \\
&= \sum_{r,s=0}^{m,n} x_{rs}\left(1 - \frac{P_{r-1}}{P_m}\right)\left(1 - \frac{Q_{s-1}}{Q_n}\right),
\end{aligned}$$

which implies

$$\begin{aligned}
y_{00} &= \overline{\Delta} T_{00} = x_{00} \\
y_{m0} &= \overline{\Delta} T_{m0} = \frac{p_m}{P_m P_{m-1}} \sum_{r=1}^{m} P_{r-1} x_{r0} \\
y_{0n} &= \overline{\Delta} T_{0n} = \frac{q_n}{Q_n Q_{n-1}} \sum_{s=1}^{n} Q_{s-1} x_{0s} \\
y_{mn} &= \overline{\Delta} T_{mn} = \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} \sum_{r=1,s}^{m,n} P_{r-1} Q_{s-1} x_{rs}.
\end{aligned} \tag{24}$$

Also, similarly, we get

$$\overline{\Delta} T'_{m,n} = \frac{p'_m q'_n}{P'_m P'_{m-1} Q'_n Q'_{n-1}} \sum_{r,s=1}^{m,n} P'_{r-1} Q'_{s-1} x_{rs}. \tag{25}$$

The double series $\sum_{r,s=0}^{\infty} x_{r,s}$ is summable $\left|\overline{N}, p_m, q_n\right|$ iff $y = (y_{mn}) \in \mathcal{L}$, and also we obtain by solving equation 25 for x_{rs} that, for $m, n \geq 1$,

$$\begin{aligned}
x_{00} &= y_{00} \\
x_{m0} &= \frac{P_m}{p_m} y_{m0} - \frac{P_{m-2}}{p_{m-1}} y_{m-1,0} \\
x_{0n} &= \frac{Q_n}{q_n} y_{0n} - \frac{Q_{n-2}}{q_{n-1}} y_{0,n-1} \\
x_{mn} &= \frac{P_m Q_n}{p_m q_n} y_{mn} - \frac{P_{m-2} Q_n}{p_{m-1} q_n} y_{m-1,n} - \\
&\quad \frac{Q_{n-2} P_m}{q_{n-1} p_m} y_{m,n-1} + \frac{P_{m-2} Q_{n-2}}{p_{m-1} q_{n-1}} y_{m-1,n-1}
\end{aligned} \tag{26}$$

Let

$$y'_{mn} = \left(\frac{P'_m Q'_n}{p'_m q'_n} \right)^{1-1/k} \overline{\Delta} T'_{mn} = \mu'_{mn}(k) \sum_{r,s=1}^{m,n} P'_{r-1} Q'_{s-1} x_{rs} \tag{27}$$

where $\overline{\Delta} T'_{mn}$ is defined by equation 25, and $\mu'_{mn}(k)$ is obtained from $\mu_{mn}(k)$ interchanging p_m and p_m by p'_m and q'_n, respectively . Then, by equation 27, the double series $\sum_{r,s=0}^{\infty} x_{rs}$ is summable $\left|\overline{N}, p'_n, p'_n\right|_k$ iff $y' = (y'_{mn}) \in \mathcal{L}_k$. Further, it follows from equation 26 and equation 27 that, for $m, n \geq 1$,

$$\begin{aligned}
y'_{m0} &= \mu'_{m0}(k) \sum_{r=1}^{m-1} \frac{p_r P'_r - p'_r P_r}{p_r} y_{r0} + \frac{\mu'_{m0}(k) P'_{m-1} P_m}{p_m} y_{m0}, \\
y'_{0n} &= \mu'_{0n}(k) \sum_{s=1}^{n-1} \frac{q_s Q'_s - q'_s Q_s}{q_s} y_{0s} + \frac{\mu'_{0n}(k) Q'_{n-1} Q_n}{q_n} y_{0n},
\end{aligned}$$

$$\begin{aligned}
y'_{mn} &= \mu'_{mn}(k)\sum_{r,s=1}^{m,n} P'_{r-1}Q'_{s-1}\left(\frac{P_rQ_s}{p_rq_s}y_{rs}-\frac{P_{r-2}Q_s}{p_{r-1}q_s}y_{r-1,s}\right.\\
&\left.-\frac{P_rQ_{s-2}}{p_rq_{s-1}}y_{r,s-1}+\frac{P_{r-2}Q_{s-2}}{p_{r-1}q_{s-1}}y_{r-1,s-1}\right)\\
&= \mu'_{mn}(k)\left\{\sum_{r,s=1}^{m,n} P'_{r-1}Q'_{s-1}\frac{P_rQ_s}{p_rq_s}y_{rs}-\sum_{r,s=1}^{m-1,n} P'_rQ'_{s-1}\frac{P_{r-1}Q_s}{p_rq_s}y_{rs}\right.\\
&\left.-\sum_{r,s=1}^{m,n-1} P'_{r-1}Q'_s\frac{P_rQ_{s-1}}{p_rq_s}y_{rs}+\sum_{r,s=1}^{m-1,n-1} P'_rQ'_s\frac{P_{r-1}Q_{s-1}}{p_rq_s}y_{rs}\right\}\\
&= \mu'_{mn}(k)\left\{\frac{P'_{m-1}P_mQ'_{n-1}Q_n}{p_mq_n}y_{mn}+\frac{P'_{m-1}P_m}{p_m}\sum_{s=1}^{n-1}\frac{q_sQ'_{s-1}-q'_sQ_{s-1}}{q_s}y_{ms}\right.\\
&+\frac{Q'_{n-1}Q_n}{q_n}\sum_{r=1}^{m-1}\frac{p_rP'_{r-1}-p'_rP_{r-1}}{p_r}y_{rn}+\\
&\left.\sum_{r,s=1}^{m-1,n-1}\frac{\left(q_sQ'_{s-1}-q'_sQ_{s-1}\right)\left(p_rP'_{r-1}-p'_rP_{r-1}\right)}{q_sp_r}\right\}y_{rs}.
\end{aligned}$$

Therefore we can state

$$y'_{mn}=\sum_{r,s=0}^{m,n} a_{mnrs}y_{rs}=A_{mn}(y),$$

that is, $y\prime=(y'_{mn})$ is the $A-$transform sequence of the sequence $y=(y_{rs})$, where the matrix $A=(a_{mnrs})$ is defined by

$$a_{mnrs}=\begin{cases}
\frac{\mu'_{0n}(k)Q'_{n-1}Q_n}{q_n}, & s=n,\ m=r=0\\
\frac{\mu'_{0n}(k)(q_sQ'_s-q'_sQ_s)}{q_s}, & 1\le s<n,\ m=r=0\\
\frac{\mu'_{m0}(k)P'_{m-1}P_m}{p_m}, & r=m,\ n=s=0\\
\frac{\mu'_{m0}(k)(p_rP'_r-p'_rP_r)}{p_r}, & 1\le r<m,\ n=s=0\\
\frac{\mu'_{mn}(k)P'_{m-1}P_m\left(q_sQ'_{s-1}-q'_sQ_{s-1}\right)}{p_m}, & 1\le s<n\\
\frac{\mu'_{mn}(k)Q'_{n-1}Q_n\left(p_rP'_{r-1}-p'_rP_{r-1}\right)}{q_np_r}, & 1\le r<m\\
\frac{\mu'_{mn}(k)\left(q_sQ'_{s-1}-q'_sQ_{s-1}\right)\left(p_rP'_{r-1}-p'_rP_{r-1}\right)}{q_sp_r}, & 1\le s<n, 1\le r<m\\
\frac{\mu'_{mn}(k)P'_{m-1}P_mQ'_{n-1}Q_n}{p_mq_n}, & r=m,\ s=n\\
0, \text{ otherwise} &
\end{cases}$$

This gives that $\left|\overline{N},p_m,q_n\right|\Rightarrow\left|\overline{N},p'_m,q'_n\right|_k$ iff $(y'_{mn})\in\mathcal{L}_k$ for every $(y_{mn})\in\mathcal{L}, i.e., \mathcal{A}\in(\mathcal{L},\mathcal{L}_k)$. Now, by Theorem 3.1, we should show that equation 21i and equation 21ii are equivalent to the equation 9. To do this, let us write

$$\begin{aligned}
\sum_{m,n=r,s}^{\infty}|a_{mnrs}|^k &= \sum_{m=r}^{\infty}\left(|a_{msrs}|^k+\sum_{n=s+1}^{\infty}|a_{mnrs}|^k\right)\\
&= |a_{rsrs}|^k+\sum_{m=r+1}^{\infty}|a_{msrs}|^k+\sum_{n=s+1}^{\infty}|a_{rnrs}|^k+\sum_{m,n=r+1,s+1}^{\infty}|a_{mnrs}|^k\\
&= L_1+L_2+L_3+L_4,\ \text{say.}
\end{aligned}$$

Then, equation 9 holds iff $L_1 = O(1),\ L_2 = O(1),\ L_3 = O(1)$ and $L_4 = O(1)$. Now, it is written that

$$
\begin{aligned}
L_1' &= |a_{0s0s}| = \left(\frac{q_s'}{Q_s'}\right)^{1/k} \frac{Q_s}{q_s} \\
L_1'' &= |a_{r0r0}| = \left(\frac{p_r'}{P_r'}\right)^{1/k} \frac{P_r}{p_r} \\
L_1''' &= |a_{rsrs}| = \left(\frac{p_r' q_s'}{P_r' Q_s'}\right)^{1/k} \frac{P_r Q_s}{p_r q_s}.
\end{aligned}
$$

Hence, if $L_1' = O(1)$ and $L_1'' = O(1)$, then, since $p_r \leq P_r$ and $q_s \leq Q_s$ for all r, s, then, $p_r' P_r / P_r' p_r = O(1)$ and $q_s' Q_s / Q_s' q_s = O(1)$, and so we have $L_1''' = O(1)$. This shows that $L_1 = O(1)$ if and only if $L_1' = O(1)$ and $L_1'' = O(1)$, or, equivalently, equation 21i and equation 21ii hold. Also, using equation 21i and equation 21ii, it follows from Lemma 2.2 and Lemma 2.3 that

$$
\begin{aligned}
L_2 &= \sum_{m=r+1}^{\infty} |a_{msrs}|^k \leq \sum_{m=r+1}^{\infty} \left(|a_{m0r0}|^k + |a_{msrs}|^k\right) \\
&= \left\{ \left| \left(P_r' - p_r' \frac{P_r}{p_r}\right) \right|^k + \left| \left(\frac{q_s'}{Q_s'}\right)^{1/k} \frac{Q_s}{q_s} \left(P_{r-1}' - \frac{p_r' P_{r-1}}{p_r}\right) \right|^k \right\} \frac{1}{P_r'^k} \\
&= \left| \left(1 - \frac{p_r' P_r}{P_r' p_r}\right) \right|^k + \frac{q_s'}{Q_s'} \left(\frac{Q_s}{q_s}\right)^k \left| \left(1 - \frac{p_r' P_r}{P_r' p_r}\right) \right|^k = O(1),
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= \sum_{n=s+1}^{\infty} |a_{rnrs}|^k \leq \sum_{n=s+1}^{\infty} \left(|a_{0n0s}|^k + |a_{rnrs}|^k\right) \\
&= \left\{ \left| Q_s' - q_s' \frac{Q_s}{q_s} \right|^k + \left| \left(\frac{p_r'}{P_r'}\right)^{1/k} \frac{P_r}{p_r} \left(Q_{s-1}' - \frac{q_s' Q_{s-1}}{q_s}\right) \right|^k \right\} \frac{1}{Q_s'^k} \\
&= \left| 1 - \frac{q_s' Q_s}{Q_s' q_s} \right|^k + \frac{p_r'}{P_r'} \left(\frac{P_r}{p_r}\right)^k \left| \left(1 - \frac{q_s' Q_s}{Q_s' q_s}\right) \right|^k = O(1),
\end{aligned}
$$

$$
\begin{aligned}
L_4 &= \sum_{m,n=r+1,s+1}^{\infty} |a_{mnrs}|^k \\
&= \sum_{m,n=r+1,s+1}^{\infty} \left| \mu_{mn}'(k) \left(Q_{s-1}' - \frac{q_s' Q_{s-1}}{q_s}\right) \left(P_{r-1}' - \frac{p_r' P_{r-1}}{p_r}\right) \right|^k \\
&= \left| \left(Q_{s-1}' - \frac{q_s' Q_{s-1}}{q_s}\right) \left(P_{r-1}' - \frac{p_r' P_{r-1}}{p_r}\right) \right|^k \sum_{m,n=r+1,s+1}^{\infty} \mu_{mn}'^k(k) \\
&= \left| \left(Q_{s-1}' - \frac{q_s' Q_{s-1}}{q_s}\right) \left(P_{r-1}' - \frac{p_r' P_{r-1}}{p_r}\right) \right|^k \frac{1}{P_r'^k Q_s'^k} \\
&= O(1) \left(\frac{q_s' Q_s}{Q_s' q_s} \frac{p_r' P_r}{P_r' p_r}\right)^k = O(1).
\end{aligned}
$$

This completes the proof.

Theorem 1.2 and Theorem 3.3 lead to the following result which gives a important relation between single and double absolute Riesz summability methods.

Corollary 3.4 Let $k \geq 1$. Then, in order that every $\left|\overline{N}, p_m, q_n\right|$ summable double series should be summable $\left|\overline{N}, p_m', q_n'\right|_k$ it is necessary and sufficient that every $\left|\overline{N}, p_m\right|$ and $\left|\overline{N}, q_n\right|$ summable simple series are summable $\left|\overline{N}, p_m'\right|_k$ and $\left|\overline{N}, q_n'\right|_k$, respectively.

For $k=1$, Theorem 3.3 also extends the result of Bosanquet (1950) and Sunouchi (1949) to double summability as follows.

Corollary 3.5 In order that every $\left|\overline{N},p_m,q_n\right|$ summable double series should be summable $\left|\overline{N},p'_m,q'_n\right|_k$ it is necessary and sufficient that

$$(i)\ \ \frac{p'_m P_m}{P'_m p_m}=O(1) \qquad \text{and} \qquad (ii)\ \ \frac{q'_n Q_n}{Q'_n q_n}=O(1).$$

For $p_n=q_n=1$, $\left|\overline{N},p_n,p_n\right|_k$ reduces to $|C,1,1|_k$ and hence one can obtain some new results as:

Corollary 3.6 Let $k\geq 1$.Then, in order that every $\left|\overline{N},p_m,q_n\right|$ summable double series should be summable $|C,1,1|_k$ it is necessary and sufficient that

$$(i)\ \ \frac{1}{m}\left(\frac{P_m}{p_m}\right)^k=O(1) \qquad \text{and} \qquad (ii)\ \ \frac{1}{n}\left(\frac{Q_n}{q_n}\right)^k=O(1).$$

Corollary 3.7 Let $k\geq 1$.Then, in order that every $|C,1,1|$ summable double series should be summable $\left|\overline{N},p_m,q_n\right|_k$ it is necessary and sufficient that

$$(i)\ \ m^k\frac{p_m}{P_m}=O(1) \qquad \text{and} \qquad (ii)\ \ n^k\frac{q_n}{Q_n}=O(1).$$

However the following result shows that converse implication of Theorem 3.3 is not true.

Theorem 3.8 Let $k>1$. Then, for every sequences $(p_m)\,,(q_n)\,,(p'_m)$ and (q'_n) , there exists a series which is summable $\left|\overline{N},p_m,q_n\right|_k$ but not summable $\left|\overline{N},p'_m,q'_n\right|$.

Proof. Let us consider (T_{mn}) and (T'_{mn}) defined by equation 22 and equation 23. Write

$$Y_{mn}=\mu_{mn}(k)\overline{\Delta}T_{mn} \text{ for } m,n\geq 0 \tag{28}$$

where $\overline{\Delta}T=\left(\overline{\Delta}T_{mn}\right)$ is defined by equation 24. Then the double series $\sum_{r,s=0}^{\infty}x_{r,s}$ is summable $\left|\overline{N},p_m,q_n\right|_k$ and $\left|\overline{N},p'_m,q'_n\right|$ if and only if $Y=(Y_{mn})\in\mathcal{L}_k$ and $\overline{\Delta}T'=\left(\overline{\Delta}T'_{m,n}\right)\in\mathcal{L}$, respectively, where $\overline{\Delta}T'_{m,n}$ is given by equation 25 . Further, by equation 2 and equation 28, for $m,n\geq 1$,

$$\begin{aligned}
\overline{\Delta}T'_{m,0} &= \mu'_{m0}(1)\sum_{r=1}^{m-1}\frac{\left(P'_{r-1}P_r-P'_rP_{r-1}\right)Y_{r0}}{p_r\mu_{r0}(k)}+\frac{P'_{m-1}P_m\mu'_{m0}(1)Y_{m0}}{p_m\mu_{m0}(k)}\\
\overline{\Delta}T'_{0,n} &= \mu'_{0n}(1)\sum_{s=1}^{n-1}\frac{\left(Q'_{s-1}Q_s-Q'_sQ_{s-1}\right)Y_{0s}}{q_s\mu_{0s}(k)}+\frac{Q'_{n-1}Q_n\mu'_{0n}(1)Y_{0n}}{q_n\mu_{0n}(k)}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Delta}T'_{m,n} &= \mu'_{mn}(1)\left\{\frac{P'_{m-1}P_mQ'_{n-1}Q_n}{p_mq_n\mu_{mn}(k)}Y_{mn}+\frac{P'_{m-1}P_m}{p_m}\sum_{s=1}^{n-1}\frac{\left(Q'_{s-1}Q_s-Q'_sQ_{s-1}\right)Y_{ms}}{q_s\mu_{ms}(k)}\right.\\
&\quad+\frac{Q'_{n-1}Q_n}{q_n}\sum_{r=1}^{m-1}\frac{\left(P'_{r-1}P_r-P'_rP_{r-1}\right)Y_{rn}}{p_r\mu_{rn}(k)}\\
&\quad\left.+\sum_{r,s=1}^{m-1,n-1}\frac{\left\{P'_rP_{r-1}\left(Q'_sQ_{s-1}-Q'_{s-1}Q_s\right)-P'_{r-1}P_r\left(Q'_sQ_{s-1}-Q'_{s-1}Q_s\right)\right\}Y_{rs}}{p_rq_s\mu_{rs}(k)}\right\}
\end{aligned}$$

Therefore it can be written that

$$\overline{\Delta}T'_{m,n}=\sum_{r,s=0}^{m,n}a_{mnrs}Y_{rs},=A_{mn}(Y)$$

where the matrix $A=(a_{mnrs})$ is given by

$$a_{mnrs}=\begin{cases}\frac{\mu'_{m0}(1)P'_{m-1}P_m}{p_m\mu_{m0}(k)}, & r=m,\ n=s=0\\ \frac{\mu'_{m0}(1)\left(P'_{r-1}P_r-P'_rP_{r-1}\right)}{p_r\mu_{r0}(k)}, & 1\le r<m,\ n=s=0\\ \frac{\mu'_{0n}(1)Q'_{n-1}Q_n}{q_n\mu_{0n}(k)}, & s=n,\ \ m=r=0\\ \frac{\mu'_{0n}(1)\left(Q'_{s-1}Q_s-Q'_sQ_{s-1}\right)}{q_s\mu_{0s}(k)}, & 1\le s<n, m=r=0\\ \frac{\mu'_{mn}(1)P'_{m-1}P_m\left(Q'_{s-1}Q_s-Q'_sQ_{s-1}\right)}{p_mq_s\mu_{ms}(k)}, & 1\le s<n, m\ge 1\\ \frac{\mu'_{mn}(1)Q'_{n-1}Q_n\left(P'_{r-1}P_r-P'_rP_{r-1}\right)Y_{rn}}{q_np_r\mu_{rn}(k)}, & 1\le r<m, n\ge 1\\ \frac{\mu'_{mn}(1)\left\{P'_rP_{r-1}\left(Q'_sQ_{s-1}-Q'_{s-1}Q_s\right)-P'_{r-1}P_r\left(Q'_sQ_{s-1}-Q'_{s-1}Q_s\right)\right\}}{p_rq_s\mu_{rs}(k)}, & 1\le s<n, 1\le r<m\\ \frac{\mu'_{mn}(1)P'_{m-1}P_mQ'_{n-1}Q_n}{p_mq_n\mu_{mn}(k)}, & s=n,\ r=m,\\ 0, & \text{otherwise}\end{cases}$$

This gives that $\left|\overline{N},p_m,q_n\right|_k \Rightarrow \left|\overline{N},p'_m,q'_n\right|$ if and only if $A\in(\mathcal{L}_k,\mathcal{L})$. But, it follows from the definition of the matrix that

$$\begin{aligned} W_{k^*}(A) &= \sum_{r,s=0}^{\infty}\left(\sum_{m,n=0}^{\infty}|a_{mnrs}|\right)^{k^*} \ge \sum_{r=0}^{\infty}|a_{r0r0}|^{k^*} \\ &= \sum_{r=0}^{\infty}\left|\left(\frac{p'_rP_r}{P'_rp_r}\right)\left(\frac{P_r}{p_r}\right)^{1/k}P_{r-1}\right|^{k^*} \ge \sum_{r=0}^{\infty}P_{r-1}^{k^*}=\infty. \end{aligned}$$

Therefore, the proof is completed by Theorem 3.2.

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