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Hilbert Series of Right-Angled Affine Artin Monoid $M(\tilde{A}_n^\infty)$

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Abstract

It is already proved that the growth rate of all the spherical Artin monoids is less than 4. In this paper, we find the Hilbert series of the associated right-angled affine Artin monoid $M(\tilde{A}_n^\infty)$ and also we discuss the recurrence relations and the growth of the monoid $M(\tilde{A}_n^\infty)$.

Keywords: Growth rate; Hilbert series; irreducible words.

MSC 2010: 20F36 (primary), 20M05, 20F10 (secondary)

1. Introduction

We start with basic notions of Coxeter groups. These groups are classified into two categories: finite or spherical type Coxeter groups and affine Coxeter groups. In Iqbal (2011), we gave a linear system for the reducible and irreducible words of the braid monoid MB_n, which leads to compute the Hilbert series MB_n. In Berceanu & Iqbal (2015), we proved that the growth rate of all the spherical Artin monoids is less than 4. In this paper we study one of the affine Artin group A_n^∞ and find the Hilbert series (or spherical growth series) of the associated right-angled affine Artin monoid $M(\tilde{A}_n^\infty)$. We also discuss the recurrence relations and the growth of the monoid $M(\tilde{A}_n^\infty)$.

Let S be a set. A Coxeter matrix over is a square matrix $M = (m_{st}), s, t \in S$ indexed by the elements of S such that

- $m_{st} = 1$ for all $s \in S$
- $m_{st} = m_{ts} \in \{2,3,4,\cdots,\infty\}$ for all $s, t \in S, s \neq t$.

A *Coxeter graph of* Γ is a labeled graph defined by the following data:

- S is a set of vertices of Γ.
- Two vertices $s, t \in S, s \neq t$ are joined by an edge if $m_{ts} \geq 3$. This edge is labeled by m_{ts} if $m_{ts} \geq 4$.

  (A Coxeter matrix $M = (m_{st}), s, t \in S$ is usually represented by its Coxeter graph $\Gamma = \Gamma(M)$.)

Definition 1.1 *Let $M = (m_{st}), s, t \in S$ be the Coxeter matrix of the Coxeter graph Γ. Then the group defined by*

$$W = \langle s \in S : s^2 = 1, (st)^{m_{st}} = 1 \ \forall s, t \in S, s \neq t \rangle$$

is called the Coxeter group (of type Γ).

In a simple way we can write

$$W = \langle s \in S : s^2 = 1, stst\cdots(m_{st} factors) = tsts\cdots(m_{st} factors) \ \forall s, t \in S, s \neq t \rangle$$

We call Γ to be of *spherical type* if W is finite.

An *Artin spherical monoid (or group)* is given by a finite union of connected Coxeter graphs from the well known classical list of Coxeter diagrams (Bourbaki, 1968).

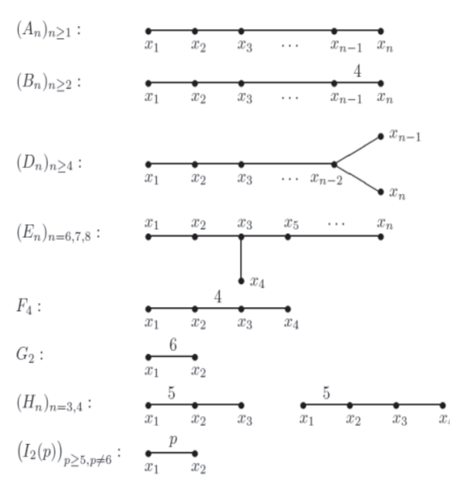

Fig. 1. Coxeter graphs of finite type

By convention m_{ij} is the label of the edge between x_i and $x_j, i \neq j$; if there is no label then $m_{ij} = 3$; if there is no edge between x_i and x_j then $m_{ij} = 2$.

To a given diagram X_n, we associate a monoid $M(X_n)$ with the following presentation (generators correspond to the vertices, and relations correspond to the labels $m_{ij} \geq 3$ of the graphs):

$$W = \langle x_1, x_2, \cdots, x_n : x_1 x_2 x_1 \cdots (m_{ij} factors) = x_2 x_1 x_2 \cdots (m_{ij} factors) \rangle;$$

the corresponding group $G(X_n)$ associated to X_n is defined by the same presentation. Next we will use X_n for $G(X_n)$ for simplicity.

Definition 1.2 If W is a *Coxeter group* (*with the Coxeter matrix* $M = (m_{st}), s, t \in S$ of the *Coxeter graph* Γ) *then* the *Artin group associated to* W *is defined by*

$$\boldsymbol{A} = \langle s \in S : stst \cdots (m_{st} factors) = tsts \cdots (m_{st} factors) \rangle$$

If W *is finite then* $\boldsymbol{A}$ *is called a spherical Artin group*.

Definition 1.3 *In the spherical type Coxeter graphs, if all the labels* $m_{ts} \geq 3$ *are replaced by* ∞ *then the associated groups (monoids) are called right-angled Artin groups (monoids).*

Definition 1.4 (Harpe, 2000) *Let* G *be a finitely generated group and* S *be a finite set of generators of* G. *The word lenth* $l_S(g)$ *of an element* $g \in G$ *is the smallest integer* n *for which there exists* $s_1, \ldots, s_n \in S \cup S^{-1}$ *such that* $g = s_1 \cdots s_n$.

Definition 1.5 (Harpe, 2000) *Let* G *be a finitely generated group and* S *be a finite set of generators of* G. *The growth function of the pair* (G, S) *associates to an integer* $k \geq 0$ *the number* a_k *of elements* $g \in G$ *such that* $l_S(g) = k$ *and the corresponding spherical growth series (also known as the generating function) or the Hilbert series is given by*

$$H_G(t) = \sum_{k=0}^{\infty} a_k t^k.$$

The affine (or infinite) Coxeter groups form another important series of Coxeter groups. These well-known affine Coxeter groups are given as

$$\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{G}_2, \tilde{I}_1.$$

In Berceanu & Iqbal (2015) we proved that the universal upper bound for all the spherical Artin monoids is less than 4.

In this paper we discuss the growth of the associated right-angled affine Artin monoid $M(\tilde{A}_n^{\infty})$ and find the Hilbert series of the monoid. We also study some other results relating the recurrence of polynomials for this monoid.

2. The Hilbert series of right-angled affine Artin monoid $M(\tilde{A}_n^{\infty})$

The first (in the above list) affine Coxeter group $\tilde{A}_n$ is given by the following Coxeter diagram:

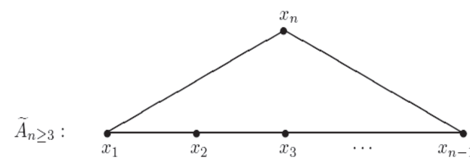

Fig. 2. The Coxeter graph of Affine type $\tilde{A}_n$

If all the labels in the above diagram are replaced by ∞ then we have the associated right-angled Artin group and we denote it by $\tilde{A}_n$. In this paper we consider the associated right-angled Artin monoid $M(\tilde{A}_n^{\infty})$. Thus $M(\tilde{A}_n^{\infty})$ has the following presentation:

$$M(\tilde{A}_n^{\infty}) = \left\langle x_1, x_2, x_3, \ldots, x_n \;\middle|\; \begin{array}{ll} x_i x_j = x_j x_i, & 3 \leq j+2 \leq i \leq n-1 \\ x_n x_k = x_k x_n, & 2 \leq k \leq n-2 \end{array} \right\rangle$$

Let $x_l x_m = x_m x_l$ $(l > m)$. Then the word $x_m x_l$ is said to be an *irreducible word* or *canonical form* of the word $x_l x_m$. If $\alpha = \beta$ be any relation in $M(\tilde{A}_n^{\infty})$, then the change $\gamma\alpha\delta \to \gamma\beta\delta$ gives a rewriting system, called the *reduction*.

If we apply a finite sequence of reductions on a word w and get a word u and, if no further reduction is applicable on u, then u is called the *canonical form* of w.

Let $x_l x_m = x_m x_l$ $(l > m)$ and $x_m x_n = x_n x_m (m > n)$ be any two commutation relations in $M(\tilde{A}_n^{\infty})$. Then the word of the form $x_l x_m x_n$ is said to be an *ambiguity* (for more details see Bergman, 1978).

In a presentation of a monoid we fix a total order of the generators; in all our examples we choose the natural order $x_1 < x_2 < \cdots < x_n$. Such a presentation is complete, if and only if, all the ambiguities are solvable (Bergman, 1978 and Cohn, 2003).

Remark 2.1 *In* $M(\tilde{A}_n^{\infty})$ *all the ambiguities are of type* $x_i x_j x_k, k+4 \leq j+2 \leq i$. Let $w = x_i x_j x_k$, *then*;

If $i \leq n-1$ *then* w *has the canonical form* $x_k x_j x_i$.

If $i = n$ *and* $k > 1$ *then also has the canonical form* $x_k x_j x_i$.

If $i=n$ and $k=1$ then w has the canonical form $x_i x_k x_j$, i.e., we have a relation $x_n x_j x_1 = x_n x_1 x_j$.

Let $\alpha(1,k)$ denote a word in the generators $x_1, x_2, x_3, \dots, x_n$. Then as a consequence of the above remark we have the following

Lemma 2.2 *The presentation*

$$\left\langle x_1,\dots,x_n \;\middle|\; \begin{array}{ll} x_i x_j = x_j x_i, & 3 \le j+2 \le i \le n-1 \\ x_n x_k = x_k x_n, & 2 \le k \le n-2 \\ x_n x_1 \alpha(1,k-2) x_k = x_k x_n x_1 \alpha(1,k-2), & 3 \le k \le n-2 \end{array} \right\rangle$$

is a complete presentation of $M(\tilde{A}_n^{\infty})$.

Now we start to compute the Hilbert series $H_M^{(n)}(t) = \sum_{k\ge0} c_k t^k$ of $M(\tilde{A}_n^{\infty})$, where

$c_k = \#$

{canonical words of lengh k in $M(\tilde{A}_n^{\infty})$}. Let

$c_{k,i} = \#$

{canonical words starting with x_i of lengh k in $M(A_n^{\infty})$}.

Then $H_{M;i}^{(n)}(t) = \sum_{k\ge1} c_{k;i} t^k$ denote the Hilbert series of $M(\tilde{A}_n^{\infty})$ of the canonical words starting with x_i.

Consider a system (Kelley & Peterson, 2001) of linear recurrences

$$u_1(t+1) = a_{11}(t)u_1(t) + \dots + a_{1n}(t)u_n(t) + f_1(t)$$
$$u_2(t+1) = a_{11}(t)u_1(t) + \dots + a_{1n}(t)u_n(t) + f_1(t)$$
$$\vdots$$
$$u_n(t+1) = a_{11}(t)u_1(t) + \dots + a_{1n}(t)u_n(t) + f_1(t).$$

This system can be written as

$$u(t+1) = A(t)u(t) + f(t),$$

where

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix},\; A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix},$$

$$f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

The solution of the homogenous equation (which we need in our work)

$$u(t+1) = A(t)u(t)$$

is given by

$$u(t) = c_1 \lambda_1^t u^1 + \dots + c_k \lambda_k^t u^k,$$

where $\lambda_1, \dots, \lambda_k$ are the eigenvalues of $A(t)$ and u^i is an eigenvector corresponding to λ_i. Hence the largest eigenvalue is the growth rate of the sequence. Therefore, in our case the characteristic equation is very important, which gives us the eigenvalues. Eigenvalues have many applications in different disciplines of science. For example eigenvalues are used for diagonalization of matrices and in the systems of differential equations. In engineering the eigenvalues are used to determine the natural frequencies of vibration in the structures. In physics we use it in oscillations, dynamical systems, rotations and translations of rigid bodies. They are also widely used in economics (e.g., in control theory), statistics (in population growth models), computer graphics, etc.

Corollary 2. 3 *In the monoid $M(\tilde{A}_n^{\infty})$ the following relations are satisfied*

a) $c_0 = 1,\ c_{1;i} = 1,\ c_k = \sum_{i=1}^{n} c_{k;i}\ (k \ge 1)$.

b) $c_{k;i}\ (k \ge 2)$ *are given by the recurrence*

$$\begin{cases} c_{k;1} = \sum_{i=1}^{n} c_{k-1;i}, \\ c_{k;j} = \sum_{i=j-1}^{n} c_{k-1;i}\ (j = 2,\dots,n-1), \\ c_{k;n} = c_{k-1;1} + \sum_{i=n-1}^{n} c_{k-1;i}. \end{cases}$$

The characteristic polynomial of this recurrence is given by:

$$\mathsf{L}_n(\lambda) = \det\begin{bmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 & -1 \\ -1 & 0 & \cdots & 0 & 0 & -1 & \lambda-1 \end{bmatrix}. \quad (1)$$

The characteristic equation $\mathsf{L}_n(\lambda) = 0$ gives the eigenvalues of the matrix of the above system.

Lemma 2. 4 *The polynomials $(\mathsf{L}_n(\lambda))_{n\ge5}$ satisfy the recurrence*

$$\mathsf{L}_n(\lambda) = \lambda \mathsf{L}_{n-1}(\lambda) - \lambda \mathsf{L}_{n-2}(\lambda) - \lambda^{n-3} \quad (n \ge 5)\,(2)$$

with $\mathsf{L}_3(\lambda) = \lambda^3 - 3\lambda^2, \mathsf{L}_4(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2$ as the initial values.

Proof. By adding nth row in $(n-1)$th row and decomposing $\mathsf{L}_n(\lambda)$ as sum of two determinants, say U_n and V_n with the last rows given by $[0,0,\dots,0,\lambda]$ and $[-1,0,\dots,0,-1,-1]$, respectively we have $V_n = 0$. Thus $\mathsf{L}_n(\lambda) = \lambda U_{n-1}(\lambda)$, where

$$U_{n-1}(\lambda)=\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ -1 & 0 & \cdots & 0 & -1 & \lambda-2 \end{vmatrix}.$$

The last determinant is of order $n-1$ and by splitting it from the last row, we have

$$\mathsf{L}_n(\lambda)=\lambda\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ -1 & 0 & \cdots & 0 & -1 & \lambda-1 \end{vmatrix}$$

$$+\lambda\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 \end{vmatrix}.$$

Hence expanding the last determinant by last row and then adding its 2nd row in its last row we have

$$\mathsf{L}_n(\lambda)=\lambda\mathsf{L}_{n-1}(\lambda)-\lambda\begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -2 & \lambda-2 \end{vmatrix}$$

The order of the last determinant now is $n-2$ and by splitting it into two determinants with last rows as $[-1,0,\ldots,0,-1,\lambda-1]$ and $[0,\lambda-1,-1,\ldots,-1]$, respectively, we have

$$\mathsf{L}_n(\lambda)=\lambda\mathsf{L}_{n-1}(\lambda)-\lambda\mathsf{L}_{n-2}(\lambda)-\lambda\begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 \\ 0 & -1 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 \\ 0 & \lambda-1 & -1 & \cdots & -1 & -1 \end{vmatrix}.$$

Let R_k denote the kth row of the last determinant (of order $n-2$). Then by elementary row operations $R_i-R_{n-2};(i=1,\ldots,n-3)$ we have

$$\mathsf{L}_n(\lambda)=\lambda\mathsf{L}_{n-1}(\lambda)-\lambda\mathsf{L}_{n-2}(\lambda)+\lambda^{n-4}\begin{vmatrix} \lambda-1 & -\lambda \\ -1 & 0 \end{vmatrix}.$$

Therefore, we have the result

$$\mathsf{L}_n(\lambda)=\lambda\mathsf{L}_{n-1}(\lambda)-\lambda\mathsf{L}_{n-2}(\lambda)-\lambda^{n-3}.$$

Lemma 2.5 *The Hilbert series $H_M^{(n)}(t)$ of $M(\tilde{A}_n^{\infty})$ is given by the following system of n equations.*

(1) $H_M^{(n)}(t)=1+\sum_{i=1}^{n}H_{M;i}^{(n)}(t)$,

(2) $H_{M;1}^{(n)}(t)=H_{M;2}^{(n)}(t)$,

(3) $H_{M;j}^{(n)}(t)=t+t\sum_{i=j-1}^{n}H_{M;i}^{(n)}(t)\quad(2\le j\le n-1)$,

(4) $H_{M;n}^{(n)}(t)=t+tH_{M;1}^{(n)}(t)+t\sum_{i=n-1}^{n}H_{M;i}^{(n)}(t)$.

Proof. (1) From Corollary 2.3 we have $c_k=\sum_{i=1}^{n}c_{k;i}$ $(k\ge1)$. Therefore

$$\begin{aligned} H_M^{(n)}(t)=\sum_{k\ge1}c_kt^k &= 1+\sum_{k\ge1}\sum_{i=1}^{n}c_{k;i}t^k \\ &= 1+\sum_{i=1}^{n}\sum_{k\ge1}c_{k;i}t^k \\ &= 1+\sum_{i=1}^{n}H_{M;i}^{(n)}(t). \end{aligned}$$

(2) is clear from the recurrence $c_{k;1}=c_{k;2}$.

(3) Again from Corollary 2.3, we have

$c_{k;j}=\sum_{i=j-1}^{n}c_{k-1;i}$ $(j=2,\ldots,n-1)$. Therefore

$$\begin{aligned} H_{M;j}^{(n)}(t) &= \sum_{k\ge1}c_{k;j}t^k \\ &= c_{1;j}t+\sum_{k\ge2}c_{k;j}t^k \\ &= t+\sum_{k\ge2}\sum_{i=j-1}^{n}c_{k-1;i}t^k \\ &= t+t\sum_{i=j-1}^{n}\sum_{k\ge2}c_{k-1;i}t^{k-1} \\ &= t+t\sum_{i=j-1}^{n}H_{M;i}^{(n)}(t). \end{aligned}$$

Proof of (4) is similar to the above proof. The linear system given in the Lemma 2.5 of n equations has the determinant $\det W_n=t^n\mathsf{L}_n(\frac{1}{t})$. The characteristic polynomials $\mathsf{A}_n(\lambda)$ of A_n^{∞} given by

$$\mathsf{A}_n(\lambda)=\det\begin{bmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \lambda-1 \end{bmatrix}$$

satisfies the recurrence:

Lemma 2.6 (Berceanu & Iqbal, 2015) The polynomials $\mathsf{A}_n(\lambda)$ satisfy

$$\mathsf{A}_n(\lambda)=\lambda\mathsf{A}_{n-1}(\lambda)-\lambda\mathsf{A}_{n-2}(\lambda) \quad (n\geq 2) \tag{3}$$

with $\mathsf{A}_0(\lambda)=1$ *and* $\mathsf{A}_1(\lambda)=\lambda-1$.

Lemma 2.7 The *polynomials* $\mathsf{L}_n(\lambda)$ *and* $\mathsf{A}_n(\lambda)$ $(n\geq 0)$ *satisfy the recurrence*

$$\mathsf{L}_n(\lambda)=\lambda\mathsf{A}_{n-1}(\lambda)-\lambda\mathsf{A}_{n-2}-\lambda^{n-2} \quad (n\geq 3). \tag{4}$$

We break the Determinant (2.1) as a sum of two determinants $U_n(\lambda)$ and $V_n(\lambda)$ with the last rows $[0,\ldots,0,0,\lambda]$ and $[-1,0,\ldots,0,-1,-1]$, respectively. By expanding $U_n(\lambda)$ with the last row we have $\mathsf{L}_n(\lambda)=\lambda\mathsf{A}_{n-1}(\lambda)+V_n(\lambda)$, where

$$V_n(\lambda)=\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ -1 & 0 & \cdots & 0 & -1 & -1 \end{vmatrix}.$$

Now subtracting $(n-1)$th column from nth column of $V_n(\lambda)$ and then expanding it by last column we have $V_n(\lambda)=\lambda W_{n-1}(\lambda)$, where

$$W_{n-1}(\lambda)=\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ -1 & 0 & \cdots & 0 & 0 & -1 \end{vmatrix}.$$

Further by decomposing $W_{n-1}(\lambda)$ as a sum of two determinants, say, $\lambda S_{n-1}(\lambda)$ and $\lambda T_{n-1}(\lambda)$ with the last rows $[0,0,\ldots,0,-1]$ and $[-1,0,\ldots,0,0]$, respectively, we have $\lambda S_{n-1}(\lambda)=-\lambda\mathsf{A}_{n-2}(\lambda)$ ..

Thus $\mathsf{L}_n(\lambda)=\lambda\mathsf{A}_{n-1}(\lambda)-\lambda\mathsf{A}_{n-2}(\lambda)+\lambda T_{n-1}(\lambda)$, where

$$T_{n-1}(\lambda)=\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ -1 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix}$$

$$=(-1)^{n-1}Y_{n-2}(\lambda).$$

Let C_k denote the kth column of $(-1)^{n-1}Y_{n-2}(\lambda)$.

Then by elementary column operation $C_{n-2}-C_{n-3}, C_{n-3}-C_{n-4},\ldots,C_2-C_1$, we have finally $\mathsf{L}_n(\lambda)=\lambda\mathsf{A}_{n-1}(\lambda)-\lambda\mathsf{A}_{n-2}(\lambda)-\lambda^{n-2}$.

Lemma 2.8 The Hilbert series $H_{M;m}^{(n)}(t)$ is given by

$$H_{M;m}^{(n)}(t)=\frac{t^{m-1}\ \mathsf{A}_{m-2}(\frac{1}{t})}{t^n\mathsf{L}_n(\frac{1}{t})}, 2\leq m\leq n-1$$

and

$$H_{M;m}^{(n)}(t)=\frac{t^{m-1}\ \mathsf{A}_{m-2}(\frac{1}{t})+t^2}{t^m\mathsf{L}_m(\frac{1}{t})}, m=n.$$

Proof. The system given in the Lemma 2.5 of n equations in n variables $H_{M;i}^{(n)}(t)$, $1\leq i\leq n$ can be written in the form $LY=B$, where $\det L=t^n\mathsf{L}_n(\frac{1}{t})$, $Y=[H_{M;1}^{(n)}(t),H_{M;2}^{(n)}(t),\ldots,H_{M;n}^{(n)}(t)]^t$ and $B=[t,t,\ldots,t]^t$. Using the Cramer's rule we have $H_{M;m}^{(n)}(t)=\frac{D_m}{\det L}$, where D_m is a determinant obtained by replacing m th column of L by column of B. Here have two cases; **Case I**: $2\leq m\leq n-1$.

Let C_k denote the kth column of D_m. Adding C_m in $C_{m+1},C_{m+2},\ldots,C_n$ and simplifying we get a determinant of order m, say L_m. Now adding $(m-1)$th column of L_m in its mth column and

simplifying we have finally $D_m = t^{m-1}\mathsf{A}_{m-2}(\frac{1}{t})$.
Case II: $m = n$.

We compute D_n for even n. The computations for odd n are also same. Adding $(n-1)$ th column in nth column of D_n and expanding from last column, we have

$$H^{(n)}_{M;n}(t) = \frac{-1}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & -t & -t & -t \\ -t & 1-t & \cdots & -t & -t & -t & -t \\ 0 & -t & \cdots & -t & -t & -t & -t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -t & 1-t & -t \\ -t & 0 & \cdots & 0 & 0 & 0 & -t \end{vmatrix}$$

$$= \frac{t}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & -t \\ -t & 1-t & \cdots & -t & -t \\ 0 & -t & \cdots & -t & -t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1-t & -t \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} + \frac{t}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & -t \\ -t & 1-t & \cdots & -t & -t \\ 0 & -t & \cdots & -t & -t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1-t & -t \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= \frac{t}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & -t \\ -t & 1-t & \cdots & -t & -t \\ 0 & -t & \cdots & -t & -t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 1-t \end{vmatrix} + \frac{(-1)^{n-1}t}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} -t & -t & \cdots & -t & -t \\ 1-t & -t & \cdots & -t & -t \\ -t & 1-t & \cdots & -t & -t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1-t & -t \end{vmatrix}$$

$$= \frac{t^{n-1}}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} \frac{1}{t}-1 & -1 & \cdots & -1 & -1 \\ -1 & \frac{1}{t}-1 & \cdots & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \frac{1}{t}-1 \end{vmatrix} + \frac{(-1)^{n-1}t}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} -t & -t & \cdots & -t & -t \\ 1-t & -t & \cdots & -t & -t \\ -t & 1-t & \cdots & -t & -t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1-t & -t \end{vmatrix}$$

Let C_k denote the kth column of the last determinant.

Then by elementary column operations
$C_{n-2}-C_{n-3}, C_{n-3}-C_{n-4}, \ldots, C_2-C_1$ we have

$$H^{(n)}_{M;n}(t) = \frac{t^{n-1}\mathsf{A}_{n-2}(\frac{1}{t})}{t^n\mathsf{L}_n(\frac{1}{t})} + \frac{(-1)^{n-1}t}{t^n\mathsf{L}_n(\frac{1}{t})}\begin{vmatrix} -t & 0 & \cdots & 0 & 0 \\ 1-t & -1 & \cdots & 0 & 0 \\ -t & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & -1 \end{vmatrix}.$$

The determinant involved in the last expression equals $(-1)^{n-1}t$. Therefore we have finally

$$H^{(n)}_{M;n}(t) = \frac{t^{n-1}\mathsf{A}_{n-2}(\frac{1}{t}) + t^2}{t^n\mathsf{L}_n(\frac{1}{t})}.$$

Theorem 2.9 *The Hilbert series of the monoid* $M(\tilde{A}^{\infty}_n)$ *is given by*

$$H^{(n)}_M(t) = \frac{1}{t^n\mathsf{L}_n(\frac{1}{t})}.$$

From Lemma 2.8 and Lemma 2.5 and using the Relations (3) and (4) we have

$$\begin{aligned} H^{(n)}_M(t) &= 1 + \sum_{i=1}^{n} H^{(n)}_{M;i}(t) \\ &= \frac{1}{t^n\mathsf{L}_n(\frac{1}{t})}\left(t^n\mathsf{L}_n\left(\frac{1}{t}\right) + 2t + t^2\mathsf{A}_1\left(\frac{1}{t}\right) + \cdots + t^{n-1}\mathsf{A}_{n-2}\left(\frac{1}{t}\right) + t^2\right) \\ &= \frac{1}{t^n\mathsf{L}_n(\frac{1}{t})}\left(2t + t^{n-1}\mathsf{A}_{n-1}\left(\frac{1}{t}\right) + t^{n-1}\mathsf{A}_{n-2}\left(\frac{1}{t}\right) + t^2\mathsf{A}_1\left(\frac{1}{t}\right) + \cdots \right. \\ &\quad \left. + t^{n-1}\mathsf{A}_{n-2}\left(\frac{1}{t}\right)\right) \\ &= \frac{1}{t^n\mathsf{L}_n(\frac{1}{t})}\left(2t + t^2\mathsf{A}_2\left(\frac{1}{t}\right)\right) \\ &= \frac{1}{t^n\mathsf{L}_n(\frac{1}{t})}\left(2t + t\mathsf{A}_1\left(\frac{1}{t}\right) - t\mathsf{A}_0\left(\frac{1}{t}\right)\right) \\ &= \frac{1}{t^n\mathsf{L}_n(\frac{1}{t})}. \end{aligned}$$

Example 2.10 *Using the above result we see that* $M(\tilde{A}^{\infty}_n)$ *is the free monoid in three variables and its Hilbert series is given by* $H^{(3)}_M(t) = \frac{1}{1-3t}$ *(Similarly* $H^{(4)}_M(t) = \frac{1}{1-4t+2t^2}$*.)*

Now we will separate the zero roots of L_n from the others ($\lfloor x \rfloor$ is the floor function).

Proposition 2.11 *The polynomial* $\mathsf{L}_n(\lambda)$ *has the following form:*

$$\mathsf{L}_n(\lambda) = \lambda^{\left\lfloor \frac{n}{2} \right\rfloor} L_n(\lambda), \qquad (5)$$

where L_n *is a polynomial of degree* $\left\lfloor \frac{n+1}{2} \right\rfloor$. *The sequence* $(L_n)_{n\geq 3}$ *is defined by*

(a) $L_3 = \lambda - 3, L_4 = \lambda^2 - 4\lambda + 2;$

(b) $L_{n+2} = (\lambda-2)L_n - L_{n-2} - (2\lambda+1)\lambda^{\frac{n+1}{2}-3}$.

Proof.(a) We prove these relations by induction. Suppose (5) is true upto $n = 2p$ (for any nonnegative integer p).

Then we have $\mathsf{L}_{2p-3}=\lambda^{p-2}L_{2p-3}$, $\mathsf{L}_{2p-2}=\lambda^{p-1}L_{2p-2}$, $\mathsf{L}_{2p-1}=\lambda^{p-1}L_{2p-1}$ and $\mathsf{L}_{2p}=\lambda^{p}L_{2p}$, respectively.

Hence from Equation (2); we have

$$\begin{aligned}\mathsf{L}_{2p-1} &= \lambda\mathsf{L}_{2p-2}-\lambda\mathsf{L}_{2p-3}-\lambda^{2p-4}\\ &= \lambda^{p-1}(\lambda L_{2p-2}-L_{2p-3}-\lambda^{p-3})\\ &= \lambda^{p-1}L_{2p-1}.\end{aligned}$$

This gives

$$L_{2p-1}=\lambda L_{2p-2}-L_{2p-3}-\lambda^{p-3}. \tag{6}$$

and

$$\begin{aligned}\mathsf{L}_{2p} &= \lambda\mathsf{L}_{2p-1}-\lambda\mathsf{L}_{2p-2}-\lambda^{2p-3}\\ &= \lambda^{p}(L_{2p-1}-L_{2p-2}-\lambda^{p-3})\\ &= \lambda^{p}L_{2p}\end{aligned}$$

gives us

$$L_{2p}=L_{2p-1}-L_{2p-2}-\lambda^{p-3}.$$

Replacing p by $p+1$ in (6) we have $L_{2p+1}=\lambda L_{2p}-L_{2p-1}-\lambda^{p-2}.$

Therefore (5) follows as

$$\begin{aligned}&\mathsf{L}_{2p+1}=\lambda(\mathsf{L}_{2p}-\mathsf{L}_{2p-1}-\lambda^{2p-3})\\ &=\lambda^{p}(\lambda L_{2p}-L_{2p-1}-\lambda^{p-2})=\lambda^{p}L_{2p+1}.\end{aligned}$$

(b) For $n=2p$ we have

$$\begin{aligned}K_{2p+2} &= L_{2p+1}-L_{2p}-\lambda^{p-2}\\ &= (\lambda L_{2p}-L_{2p-1}-\lambda^{p-2})-L_{2p}-\lambda^{p-2}\\ &= (\lambda-1)L_{2p}-L_{2p-1}-2\lambda^{p-2}\\ &= (\lambda-1)K_{2p}-(L_{2p}+L_{2p-2}+\lambda^{p-3})-2\lambda^{p-2}\\ &= (\lambda-2)L_{2p}-L_{2p-2}-(2\lambda+1)\lambda^{p-3}.\end{aligned}$$

Similarly for $n=2p+1$ we have

$$\begin{aligned}L_{2p+1} &= \lambda L_{2p}-L_{2p-1}-\lambda^{p-2}\\ &= \lambda(L_{2p-1}-L_{2p-2}-\lambda^{p-3})-L_{2p-1}-\lambda^{p-2}\\ &= (\lambda-1)L_{2p-1}-\lambda L_{2p-2}-2\lambda^{p-2}\\ &= (\lambda-1)L_{2p-1}-(L_{2p-1}-L_{2p-3}-\lambda^{p-3})-2\lambda^{p-2}\\ &= (\lambda-2)L_{2p-1}-L_{2p-3}-(2\lambda+1)\lambda^{p-3}.\end{aligned}$$

3. Growth of $M(\tilde{A}_n^{\infty})$

The characteristic equation $\mathsf{L}_n(\lambda)=0$ of the recurrence of $M(\tilde{A}_n^{\infty})$ contains the zero roots and the equation $L_n(\lambda)=0$ contains only the nonzero roots. The growth rate is the maximal real root of $L_n(\lambda)=0$. We observe that the growth rate for $M(\tilde{A}_n^{\infty})$ increases (and looks unbounded) as n approaches ∞. We compute few initial growth rates (using any software like Maple 7, Derive 6 etc.) for $M(\tilde{A}_n^{\infty})$. Let r_k denote the growth rate of $M(\tilde{A}_n^{\infty})$, then we have the following few initial values of r_k:

$r_3=3$, $r_4=3.41$, $r_5=3.7$, $r_6=4$,

$r_7=4.24$, $r_8=4.47$, $r_9=4.69$, $r_{10}=4.91$,

$r_{11}=5.11$, $r_{12}=5.32$, $r_{13}=5.53$, $r_{14}=5.73$,

$r_{15}=5.92$, $r_{16}=6.11$, $r_{17}=6.3$, $r_{18}=6.49$,

$r_{19}=6.68$, $r_{20}=6.86$.

The growth of $M(\tilde{A}_n^{\infty})$ is shown in the following graph.

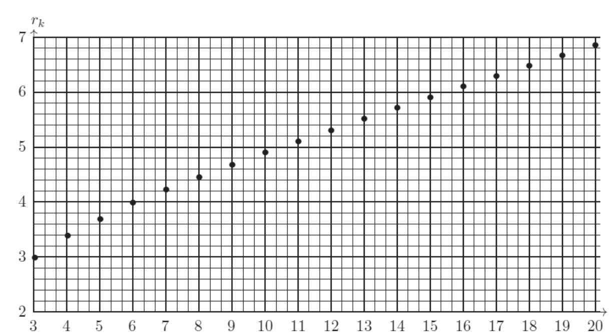

Fig. 3. The graph of the growth rate of for initial vales

We compute the higher values $r_{20}=6.86$, , $r_{60}=13.20$, $r_{80}=16$, $r_{100}=18.65$, $r_{120}=21.2$ (using Mathematica). For the higher values of r_k we have the following graph.

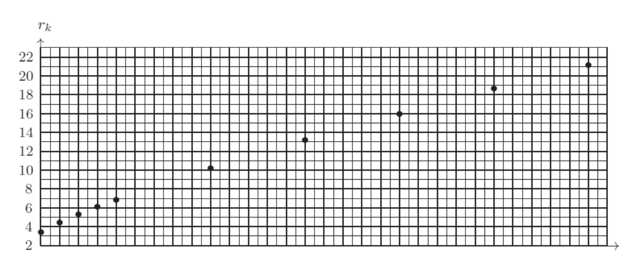

Fig. 4. The graph of the growth rate of $M(\tilde{A}_n^{\infty})$ for higher values

Conjecture: The growth rate of $M(\tilde{A}_n^{\infty})$ is unbounded.

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متسلسلة هيلبرت
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خلاصة

من المعروف أن معدل النمو لجميع أرتين منويد الكروية يقل عن 4. في هذا البحث، نوجد متسلسلة هيلبرت لأرتين منويد المتعامدة التوافقية ونناقش العلاقات المتسلسلة والنمو الخاص بالمنويد .