$\mathcal{Z}$-graphic topology on undirected graph

Hanan Omer Zomam[1,2], Makkia Dammak[3,*]
[1]*Dept. of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia.*
[2] *Dept. of Mathematics, Faculty of Science, Shendi University, Sudan.*
[3] *Dept. of Mathematics, Faculty of Sciences, University of Sfax, Tunisia.*
**Corresponding author: makkia.dammak@gmail.com*

Abstract

In this work, we define $\mathcal{Z}_G$ a topology on the vertex set of a graph G which preserves the connectivity of the graph, called $\mathcal{Z}$-graphic topology. We prove that two isomorphic graphs have homeomorphic and symmetric $\mathcal{Z}$-graphic topologies. We show that $\mathcal{Z}_G$ is an Alexandroff topology and we give a necessary and sufficient condition for a topology to be $\mathcal{Z}$-graphic.

Keywords: Connected components; homeomorphism; graph; symmetric topologies; topology.

1. Introduction

Graph theory is a field applied to many domains. When we discretize a problem by a graph, the properties of the graph help to study the given problem. Having a topology on the graph gives a richer structure to the graph and this have applications in the economy domain, the traffick flow study (Agnarsson *et al.*, 2007; Kandel *et al.*, 2007; Nogly *et al.*, 1996) and many other domains. Also, a graph can be characterized by some topological indices, see (Ali *et al.*, 2016; Cruz *et al.*, 2021; Gutman *et al.*, 2021; Naji *et al.*, 2018) and references therein.

Since the publication of the paper (Jafarian Amiri *et al.*, 2013), other researchers defined some topologies on graphs, as example we can cite (Abdu *et al.*, 2018; Hamza *et al.*, 2013; Kilicman *et al.*, 2018; Sasikala *et al.*, 2019; Shokry, 2015). In (Jafarian Amiri *et al.*, 2013), the authors defined the graphic topology τ_G on a locally finite (i.e. any vertex has a finite order) undirected graph $G = (V, E)$ with no isolated vertices by the subbasis:

$$S_G = \{A_x \mid x \in V\}, \tag{1}$$

where

$$A_x = \{z \in V \mid xz \in E\}. \tag{2}$$

One of the most interesting properties of (V, τ_G) was being an Alexandroff space, that is any intersection of open sets is an open set. This is equivalent to the topology has a unique minimal basis. The Alexandroff spaces were introduced by P. Alexandroff in 1937 in (Alexandroff, 1937) under the name Diskrete Räume spaces. We can find some results about these spaces and their importance and applications in (Herman, 1990; Kronheimer, 1992; Li *et al.*, 2019; McCord, 1966; Stong, 2015; Speer, 2007).

A topological space (V, T) is called graphic space if there exists a graph G such that $T = \tau_G$. In (Jafarian Amiri *et al.*, 2013), the authors posed two open problems: when an Alexandroff space can be graphic? When the graphic topology can be connected?
In (Zomam *et al.*, 2021), a partial answer to the first question was given. In this paper, we define a topology $\mathcal{Z}_G$ on the vertex set of an underacted graph $G = (V, E)$ such that $\mathcal{Z}_G$ is smaller than τ_G, when G is locally finite without isolated vertices, that is $\mathcal{Z}_G \subset \tau_G$. Also, we solve the two open problems of (

Jafarian Amiri *et al.*, 2013) for the $\mathcal{Z}$-graphic topology $\mathcal{Z}_G$.

The outlines of this paper are the following: Section 2 deals with some basic definitions and notations. In section 3, we define $\mathcal{Z}_G$ for an undirected graph $G = (V, E)$ and we prove that it is a topology on V, smaller than τ_G when τ_G exists. We investigate the trace topology of $\mathcal{Z}_G$ on subgraphs of G. In section 4, we prove the equivalence between the connectivity of the graph G and the $\mathcal{Z}$-graphic topology $\mathcal{Z}_G$. And we show that $\mathcal{Z}_G$ is an Alexandroff topology. Finally, in section 5 we prove that being $\mathcal{Z}$-graphic is a topology property and two isomorphic graphs have homeomorphic and symmetric $\mathcal{Z}$-graphic topologies.

2. Preliminaries

In this section, we give some general definitions and properties of a topological space. For more details, we can refer to (Arenas, 1937; Dugundji, 1966; Li *et al.*, 2019; Stong, 2015).
Recall that a topological space (X, T) is a non empty set X with a set T of subsets of X (i.e $T \subset \mathcal{P}(V)$) satisfying:

(i) $\emptyset$ and X are in T.

(ii) If A and B are two subsets of X and $A, B \in T$, then $A \cap B \in T$.

(iii) For any family $\{A_i\}_{i\in I} \subset T$, I a set, we have $\cup_{i\in I} A_i \in T$.

An element A of T will be called an open set of the space (X, T).

Example 1 *Let $X = \{a, b, c\}$, then*

$$T = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}, X\}$$

is a topology for X.

In general, the intersection of open sets is not an open set in a topological space (X, T).

Definition 2.1 *(Alexandroff, 1937) A topological space is called an Alexandroff space if any intersection of open sets is an open set. Also, we say that the topology T is an Alexandroff topology of X.*

The space introduced in Example 1 is an Alexandroff space. In fact, any finite topological space is an Alexandroff space. Later, we will give an example of a non Alexandroff space.

Definition 2.2 *Let (X, T) be a topological space and let $\mathcal{B} \subset T$. $\mathcal{B}$ is called a basis of the topology T if for all $x \in X$, for all O_x an open set containing x, there exists an element $B \in \mathcal{B}$ such that $x \in B \subset O_x$. We say that the topology is generated by the basis $\mathcal{B}$.*

Example 2 *$\mathcal{B} = \{(a, b), -\infty < a < b < +\infty\}$ is a basis for the usual topology T on R.*

Now, if we consider the open sets

$$\Big(-\frac{1}{n}, \frac{1}{n}\Big), \;\; n > 0,$$

we have

$$\bigcap_{n>0} \Big(-\frac{1}{n}, \frac{1}{n}\Big) = \{0\},$$

and so, (R, T) is not an Alexandroff space.
A basis m is called minimal basis for a topology T if for all $\mathcal{B}$ a basis of T, we have $m \subset \mathcal{B}$.

Example 3 *For the topology given in the Example 1, $m = \{\{a\}, \{b\}, \{a, c\}\}$ is a minimal basis.*

Proposition 2.1 *Let (X, T) be an Alexandroff space. Then, T has a minimal basis.*

Proof. Let $x \in X$. The intersection of all open sets containing x is an open set. We set U_x such open set. Consider $\mathcal{U} = \{U_x, \;\; x \in X\}$. We have $\mathcal{U} \subset T$ and, if $x \in X$ and O_x an open set containing x, then $x \in U_x \subset O_x$. Hence, $\mathcal{U}$ is a basis for T.
Now, let B be a basis for the topology T. Since U_x is an open set containing x, there exists $B \in B$ such that $x \in B \subset U_x$ and so $B = U_x$. Hence, $U_x \in B$ and so, $\mathcal{U} \subset B$.

3. $\mathcal{Z}$-graphic topology and some properties

In the sequel, we suppose that all graphs are simple and undirected.
Let $G = (V, E)$ be a graph. In this part, we define a subset $\mathcal{Z}_G$ of the power set $\mathcal{P}(V)$ of V and we prove that $\mathcal{Z}_G$ is a topology on the vertex set V. We call the topology $\mathcal{Z}_G$ the $\mathcal{Z}$-graphic topology of the graph G. We compare the $\mathcal{Z}$-graphic topology and the graphic topology on a graph G. Finally, we study the $\mathcal{Z}$-graphic topologies on subgraphs.

Definition 3.1 *Let $G = (V, E)$ be a graph and $A \subset V$. $A \in \mathcal{Z}_G$ if and if for any vertex $x \in A$, if there exists a path joining x to a vertex y in G then $y \in A$.*

Notation. When two vertices x and y are adjacent, we write $x \sim y$ and when they are joined by a path P, we denote $x \sim_P y$. In particular, $x \sim y$ means $x \sim_{x,y} y$ ($P = x, y$).

Theorem 3.1 *For any graph $G = (V, E)$, $\mathcal{Z}_G$ is a topology on the vertex set V.*

Proof. (i) By definition, $\emptyset$ and V are in $\mathcal{Z}_G$.
(ii) Let A_1 and A_2 two elements in $\mathcal{Z}_G$. Suppose that $x \in A_1 \cap A_2$ and let $y \in V$ such that x joined by a path P to y: $x \sim_P y$.
We get $x \in A_1$ and $x \sim_P y$, so $y \in A_1$ since $A_1 \in \mathcal{Z}_G$.
In a similar way $y \in A_2$ and then $y \in A_1 \cap A_2$. Therefore $A_1 \cap A_2 \in \mathcal{Z}_G$.
(iii) Let $\{A_i\}_{i \in I}$ a countable infinite family of elements in $\mathcal{Z}_G$. Let $x \in \cup_{i \in I} A_i$ and suppose $y \in V$ such that $x \sim_P y$.
Since $x \in \cup_{i \in I} A_i$, there exists $i_0 \in I$ such that $x \in A_{i_0}$. From the fact that $A_{i_0} \in \mathcal{Z}_G$, we get $y \in A_{i_0}$. Therefore, $y \in \cup_{i \in I} A_i$ and then the Theorem 3.1 follows.

Theorem 3.2 *Let $G = (V, E)$ be a graph. If G is locally finite without isolated vertices, then $\mathcal{Z}_G \subset \tau_G$.*

Proof. Let $A \in \mathcal{Z}_G$. Then, $A = \cup_{x \in A} A_x$, where A_x, given by Equation 2. Indeed, If $x \in A$ and $y \in A_x$, then $x \sim_{x,y} y$. Since $A \in \mathcal{Z}_G$, the vertex $y \in A$. That is $A_x \subset A$ and then $\cup_{x \in A} A_x \subset A$.
Conversely, Let $y \in A$. Since G is without isolated vertices, there exists $x \in V$ such that $x \sim y$. So, $y \in A_x$. Also, we have: $A \in \mathcal{Z}_G$, $y \in A$ and $y \sim x$. Therefore, $x \in A$ and $y \in A_x$. Hence $y \in \cup_{x \in A} A_x$ and then $A \subset \cup_{x \in A} A_x$.

Now, since $A = \cup_{x \in A} A_x$, by definition of τ_G we have $A \in \tau_G$.

In the next example, we show that the two topologies $\mathcal{Z}_G$ and τ_G are different.

Example 4

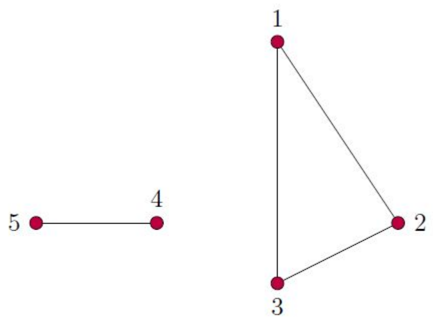

Fig. 1. *Graph with $\mathcal{Z}_G \neq \tau_G$*

In this example, $\mathcal{Z}_G = \{\emptyset, \{4, 5\}, \{1, 2, 3\}, V\}$ and τ_G is the discrete topology.

Recall that a subgraph of a graph $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subset V$and $E' \subset E$. On the set V' we can define the $\mathcal{Z}$-graphic topology $\mathcal{Z}_H$ and we have also the topology induced by $\mathcal{Z}_G$, denoted $\mathcal{Z}_{G,H}$.

Theorem 3.3 *Let $G = (V, E)$ be a graph and $H = (V', E')$ be a subgraph of G. Then, $\mathcal{Z}_H = \mathcal{Z}_{G,H}$.*

Proof. Let $A \in \mathcal{Z}_{G,H}$. Then there exist $O \in \mathcal{Z}_G$ such that $A = O \cap V'$. Suppose that $x \in A$ and $y \in V'$ satisfying $x \sim_P y$ for some path P in H. We get $x \in O$, $y \in G$ and $x \sim_P y$ with P in G. Hence, $y \in O$ and so $y \in O \cap V'$, that is, $y \in A$. So, $A \in \mathcal{Z}_H$.
Conversely, suppose that $A \in \mathcal{Z}_H$ and $A \neq \emptyset$. As in the proof of Theorem 3.2, we prove that $A = \cup_{x \in A}(A_x \cap V')$. Therefore $A = (\cup_{x \in A} A_x) \cap V'$. But $\cup_{x \in A} A_x$ is not necessary in $\mathcal{Z}_G$ as we will see in the Example 2 below. Let us consider C_x the connected component of G containing x. Since $A \in \mathcal{Z}_H$, then $A = \cup_{x \in A}(C_x \cap V')$. Or C_x is an open set of $(V, \mathcal{Z}_G)$ and $A = (\cup_{x \in A} C_x) \cap V'$, it follows that $A \in \mathcal{Z}_{G,H}$.

Example 5 *Consider the following graph G.*

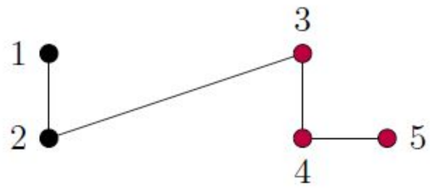

Fig. 2. *$\mathcal{Z}$-graphic topology and subgraph*

Let $H = (V', E')$ with $V' = \{1, 2\}$ and $E' = \{(1, 2)\}$. For $A = V' = \{1, 2\}$, in the graph G, we have $\cup_{x \in A} A_x = \{1, 2, 3\}$ and $\mathcal{Z}_G = \{\emptyset, \{1, 2, 3, 4, 5\}\}$.

4. $\mathcal{Z}$-graphic topology and connectedness

In this section, we will prove the equivalence between the connectivity of a graph G and the connectivity of its Z-graphic topology. Recall that the empty set is called a trivial open set in a topological space V and an open set is called proper if it is not equal to V.

Definition 4.1 *Let V be a topological space. V is called connected if it cannot be written as the union of two proper disjoint open sets. If $\mathcal{T}$ is the topology of V, we say that the topology $\mathcal{T}$ is connected.*

Example 3. Consider $V = \{1, 2, 3\}$, $\tau_1 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, V\}$
and $\tau_2 = \{\emptyset, \{1\}, \{2, 3\}, V\}$. It is clear that τ_1 is connected but the topology τ_2 is not connected.

Definition 4.2 *Let $G = (V, E)$ be a graph. G is called connected if any two vertices can be joined by a path, that is, there exists a path in G from one to the other vertex.*

When a graph is not connected, we can define its connected components.

Definition 4.3 *(Agnarsson et al., 2007; Diestel, 2005) Let $G = (V, E)$ be a graph. Let $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \cdots$ be connected subgraphs of G such that*

(i) $V = \cup_i V_i$*;*

(ii) $E = \cup_i E_i$*;*

(iii) $V_i \cap V_j = \emptyset$*, for all* $i \neq j$*;*

(iv) $E_i \cap E_j = \emptyset$, *for all* $i \neq j$.

Then, each subgraph H_j *is called connected component of the graph* G.

Remark 4.1 *When a graph* G *is connected, it has one connected component and if it is finite, it has a finite connected components.*

We have the following results with an immediate proof for the first theorem, so we omit it.

Theorem 4.1 *Let* $G = (V, E)$ *be a graph. The following properties hold.*

(1) The space $(V, \mathcal{Z}_G)$ *is compact if, and only if,* G *is a finite.*

(2) The topology $\mathcal{Z}_G$ *is discrete if, and only if,* G *is null graph (i.e* $E = \emptyset$*).*

Theorem 4.2 *Let* $G = (V, E)$ *be a graph. The graph* G *is connected if, and only if,* $\mathcal{Z}_G$ *is a connected topology on* V.

Proof. Suppose that the graph G is connected, that is any two points are joined by a path. From the Definition 3.1, the only open sets for $(V, \mathcal{Z}_G)$ are the empty set and the set V itself. And so, the topological space $(V, \mathcal{Z}_G)$ is connected.
Conversely, we suppose that $(V, \mathcal{Z}_G)$ is a connected topological space and we shall prove that the graph G is connected.
We argue by contradiction. Suppose that the graph G is not connected and so it has more than one connected components $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \cdots$.
Denote $W = \cup_{i \geq 2} V_i$. Since H_i is connected, then V_i is in $\mathcal{Z}_G$, for all i. Then, W is a proper open set satisfying $V = V_1 \cup W$ and $V_1 \cap W = \emptyset$. This makes contradiction with the fact that $(V, \mathcal{Z}_G)$ is a connected topological space. Our assumption is false, and so the graph G is connected.

Recall that a topological space is called Alexandroff space if any intersection of open sets is also open. We end this section by proving that the topology $\mathcal{Z}_G$ is an Alexandroff topology, for any graph G.

Theorem 4.3 *Consider a graph* $G = (V, E)$. *Then,* $\mathcal{Z}_G$ *is an Alexandroff topology.*

Proof. Suppose that $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \cdots$ are the connected components of the graph G. From the Definition 3.1, we have A is an open set of $(V, \mathcal{Z}_G)$ if and only if $A = V_i$, for some i or $A = \emptyset$. So, any intersection of open sets is an open set by the characterisation of the connected components given in the Definition 4.3.

5. Isomorphic graphs and $\mathcal{Z}$-graphic topologies

Definition 5.1 *Let* $(X_1, \mathcal{T}_1)$ *and* $(X_2, \mathcal{T}_2)$ *be two topological spaces. A function* $\psi : X_1 \to X_2$ *is called continuous if for all* $A \in \mathcal{T}_2$, $\psi^{-1}(A) \in \mathcal{T}_1$.
When the function ψ *is bijective and,* ψ *and* ψ^{-1} *are continuous, we say that the spaces are homeomorphic and we write* $X_1 \sim_h X_2$.

Definition 5.2 *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two simple graphs. We say that* G_1 *and* G_2 *are isomorphic and we denote* $G_1 \cong G_2$ *if there exists a bijective map* $\phi : V_1 \to V_2$ *such that the function*
$\widetilde{\phi} : E_1 \longrightarrow E_2$
$(x, y) \mapsto (\phi(x), \phi(y))$ *is also bijective.*

Remark 5.1 *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two isomorphic graphs and the isomorphism is* $\phi : V_1 \to V_2$. *It follows that if* $P = x_1 x_2 \cdots x_n$ *is a path joining* x_1 *and* x_n *in* G_1, *then* $P' = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$ *is a path joining* $\phi(x_1)$ *and* $\phi(x_n)$ *in* G_2.
Conversely, if Q *is a path joining* v_1 *and* v_2 *in* G_2, *then we have a path* Q' *joining* $\phi^{-1}(v_1)$ *and* $\phi^{-1}(v_2)$ *in* G_1.

Theorem 5.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two isomorphic graphs. Then the spaces $(V_1, \mathcal{Z}_{G_1})$ and $(V_2, \mathcal{Z}_{G_2})$ are homeomorphic.*

Proof. Let $\phi : V_1 \to V_2$ the bijective map inducing the isomorphism of the two graphs G_1 and G_2. We are going to prove that ϕ and ϕ^{-1} are continuous.
First, let $O \in \mathcal{Z}_{G_2}$ such that $\phi^{-1}(O) \neq \emptyset$. Suppose that $x \in \phi^{-1}(O)$ and $y \in V_1$ such that $x \sim_P y$, that is x and y are joined by a path in G_1. By the Remark 5.1, $\phi(x)$ and $\phi(y)$ are joined by a path in G_2. So, $\phi(y) \in O$ and hence $y \in \phi^{-1}(O)$. Then, $\phi^{-1}(O) \in \mathcal{Z}_{G_1}$.
Conversely, let $O \in \mathcal{Z}_{G_1}$. If $O = \emptyset$, then $\phi(O) = \emptyset \in \mathcal{Z}_{G_2}$.
If $O \neq \emptyset$, suppose that $x \in \phi(O)$ and $x \sim_Q y$ in G_2 (Q is a path in G_2). We have $x = \phi(x_1)$ for some $x_1 \in O$ and $y = \phi(y_1)$ for some $y_1 \in G_1$. From the Remark 5.1, x_1 and y_1 are joined by a path in G_1. Since, O is an open set of V_1, then $y_1 \in O$ and so $y = \phi(y_1) \in \phi(O)$. Therefore $\phi(O) \in \mathcal{Z}_{G_2}$.

In general, the converse of the Theorem 5.1 is not true.
Consider C_4 and K_4, their $\mathcal{Z}$-graphic topologies are homeomorphic but the two graphs are not isomorphic.
in the paper (Hamza *et al.*, 2013), the authors define a symmetry between two topologies. Next, we prove that if two graphs are isomorphic, then their $\mathcal{Z}$-graphic topologies are symmetric.

Definition 5.3 *(Hamza et al., 2013) Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. We say that these two spaces are symmetric and we write $X_1 \sim_s X_2$ (or $\mathcal{T}_1 \sim_s \mathcal{T}_2$) if $|\mathcal{T}_1| = |\mathcal{T}_2|$ and for all $A \in \mathcal{T}_1$ there exists an open set $B \in \mathcal{T}_2$ such that $|A| = |B|$ and conversely for all $B \in \mathcal{T}_2$ there exists an open set $A \in \mathcal{T}_1$ such that $|A| = |B|$.*

Theorem 5.2 *Let $G_i = (V_i, E_i)$, $i = 1, 2$, be two graphs. If $G_1 \cong G_2$ then $\mathcal{Z}_{G_1} \sim_s \mathcal{Z}_{G_2}$.*

Proof. From the proof of the Theorem 4.1, we get a bijective function, still denoted ϕ, $\phi : \mathcal{Z}_{G_1} \to \mathcal{Z}_{G_2}$, defined by $\phi(O) = \{\phi(x);\ \ x \in O\}$. So, $|\mathcal{Z}_{G_1}| = |\mathcal{Z}_{G_2}|$. Since $\phi : V_1 \to V_2$ is bijective, for all $A \in \mathcal{Z}_{G_1}$, the set $B = \phi(A) \in \mathcal{Z}_{G_2}$ and $|A| = |B|$.
Conversely, for all $B \in \mathcal{Z}_{G_2}$, the set $A = \phi^{-1}(B) \in \mathcal{Z}_{G_1}$ and $|A| = |B|$. The Theorem 5.2 follows.

The converse of the Theorem 5.2 is false, since the $\mathcal{Z}$-graphic topologies of C_4 and K_4 are symmetric but the two graphs are not isomorphic.

Definition 5.4 *Let $(V, \mathcal{T})$ be a topological space. $(V, \mathcal{T})$ is said $\mathcal{Z}$-graphic space if there exists a graph $G = (V, E)$ such that $\mathcal{T} = \mathcal{Z}_G$. We say also, $\mathcal{T}$ is a $\mathcal{Z}$-graphic topology.*

Being $\mathcal{Z}$-graphic is a topological property, that is, invariant under homeomorphisms.

Theorem 5.3 *Let $(V, \mathcal{T})$ and $(V', \mathcal{T}')$ be homeomorphic spaces. Suppose that $(V, \mathcal{T})$ is a $\mathcal{Z}$-graphic, then $(V', \mathcal{T}')$ is also a $\mathcal{Z}$-graphic space.*

Proof. Suppose that $\psi : V' \to V$ is a homeomorphism and $G = (V, E)$ is a graph such that $\mathcal{T} = \mathcal{Z}_G$. Consider

$$E' = \{(x', y') \in V' \times V' \mid (\psi(x'), \psi(y')) \in E\}. \tag{3}$$

We claim that $\mathcal{T}' = \mathcal{Z}_{G'}$, where $G' = (V', E')$. Indeed, let $A \in \mathcal{Z}_{G'}$. First, we want to prove that $\psi(A) \in \mathcal{Z}_G$. Let $x \in \psi(A)$ and $y \in V$ such that $x \sim_P y$ for some path P in G. We set $P = x_1, x_2, \cdots, x_n$ with $x_1 = x$ and $x_n = y$. So, since ψ is bijective, we have $x_i = \psi(x_i')$ for $i = 1, \cdots, n$ and also $x_1' \in A$.
Therefore, from the Equation 3, we have a path $P' = x_1', x_2', \cdots, x_n'$ in G' joining x_1' and x_n'. But $x_1' \in A$ and $A \in \mathcal{Z}_{G'}$. From the definition of the $\mathcal{Z}$-graphic topology, we get $x_n' \in A$ and so $y = x_n = \psi(x_n')$ is in $\psi(A)$.
Then, $\psi(A) \in \mathcal{Z}_G$. That is, $\psi(A) \in \mathcal{T}$. Hence $A = \psi^{-1}\Big(\psi(A)\Big) \in \mathcal{T}'$.

Conversely, let $A \in \mathcal{T}'$. In order to prove that $A \in \mathcal{Z}_{G'}$, let $x' \in A$ and $y' \in V'$ such that $x' \sim_{P'} y'$ for some path P' in G'. Denote $P' = x'_1, x'_2, \cdots, x'_n$, where $x'_1 = x'$ and $x'_n = y'$.
$P = \psi(x'_1), \psi(x'_2), \cdots, \psi(x'_n)$ is a path in G joining $\psi(x')$ and $\psi(y')$.
Now, since $A \in \mathcal{T}'$ and ψ is a homeomorphism, $\psi(A) \in \mathcal{T}$. Hence, $\psi(A) \in \mathcal{Z}_G$ and so $\psi(y') \in \psi(A)$. Since, ψ is bijective, $y' \in A$. Therefore, $A \in \mathcal{Z}_{G'}$. So the Theorem 5.3 follows.

Now, we give a necessary and sufficient conditions for a topological space to be $\mathcal{Z}$-graphic (The corresponding problem 1 in (Jafarian Amiri *et al.*, 2013)).

Theorem 5.4 *Consider an Alexandroff topological space $(X, \mathcal{T})$ and denote $S(z)$ the smallest open set containing z, for $z \in X$. $(X, \mathcal{T})$ is $\mathcal{Z}$-graphic if, and only if, for all $z_1, z_2 \in X$, $S(z_1) = S(z_2)$ or $S(z_1) \cap S(z_2) = \emptyset$.*

Proof. First, suppose that $(X, \mathcal{T})$ is a $\mathcal{Z}$-graphic space. Let $G = (X, E)$ be a graph such that $\mathcal{T} = \mathcal{Z}_G$. In this case $S(z)$ is the vertex set of the connected component of G containing x. So, for all $z_1, z_2 \in X$, $S(z_1) = S(z_2)$ or $S(z_1) \cap S(z_2) = \emptyset$, from the Definition 4.3.
Next, suppose $(X, \mathcal{T})$ is a topological space such that $S(z_1) = S(z_2)$ or $S(z_1) \cap S(z_2) = \emptyset$, for all $z_1, z_2 \in X$. Denote

$$E = \{(x, y) \in X \times X \mid S(x) = S(y)\}. \tag{4}$$

Consider the graph $G = (X, E)$, we are going to prove that $\mathcal{T} = \mathcal{Z}_G$. let $A \in \mathcal{T}$. Suppose that $x \in A$ and $y \in X$ such that $x \sim_P y$, where P is a path in G. Since $x \in A$ and A an open set, we have $S(x) \subset A$. Since $x \sim_P y$ and from the definition of the edge set (4), we get $S(x) = S(y)$ and hence $y \in S(y) \subset A$. Therefore $A \in \mathcal{Z}_G$.

Conclusion

Let $G = (V, E)$ an undirected graph. The graphic topology τ_G is a topology defined on V. When the graph G is connected, the topological space (V, τ_G) is not necessarily connected. In this paper, we introduce the $\mathcal{Z}$-graphic topology $\mathcal{Z}_G$ on V which satisfies $G = (V, E)$ is a connected graph if and only if $(V, \mathcal{Z}_G)$ is a connected topological space.
Also, we have proved that two isomorphic graphs have homeomorphic and symmetric $\mathcal{Z}$-graphic topologies. As future work, we can think about graphic topology and $\mathcal{Z}$-graphic topology for directed graphs.

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