On weighted noncorona graphs with properties $\mathcal{R}$ and $-\mathcal{SR}$

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Abstract

Let G_w be a simple weighted graph with adjacency matrix $A(G_w)$. The set of all eigenvalues of $A(G_w)$ is called the spectrum of weighted graph G_w denoted by $\sigma(G_w)$. The reciprocal eigenvalue property (or property $\mathcal{R}$) for a connected weighted nonsingular graph G_w is defined as, if $\eta \in \sigma(G_w)$ then $\frac{1}{\eta} \in \sigma(G_w)$. Further, if η and $\frac{1}{\eta}$ have the same multiplicities for each $\eta \in \sigma(G_w)$ then this graph is said to have strong reciprocal eigenvalue property (or property $\mathcal{SR}$). Similarly, a connected weighted nonsingular graph G_w is said to have anti-reciprocal eigenvalue property (or property $-\mathcal{R}$) if $\eta \in \sigma(G_w)$ then $-\frac{1}{\eta} \in \sigma(G_w)$. Furthermore, if η and $-\frac{1}{\eta}$ have the same multiplicities for each $\eta \in \sigma(G_w)$ then strong anti-reciprocal eigenvalue property (or property $-\mathcal{SR}$) holds for the weighted graph G_w. In this article, classes of weighted noncorona graphs satisfying property $\mathcal{R}$ and property $-\mathcal{SR}$ are studied.

Keywords: Adjacency matrix; anti-reciprocal eigenvalue property; corona graphs; strong anti-reciprocal eigenvalue property; weighted graphs

1. Introduction

Spectral graph theory is the branch of mathematics that deals with the properties of graphs in contact with the characteristic polynomial, eigenvectors and eigenvalues of matrices associated with the graphs. Spectral graph theory emerged during 1950s and 1960s. Cvetković summed up virtually all examination to date nearby (Cvetković, 1980). Later on, it was updated by an overview of recent results in the Theory of Graph Spectra (Cvetković *et al.*, 1988). In 2012, discrete geometric analysis was created and developed by Sunada, that dealt with spectral graph theory in terms of discrete Laplacians associated with weighted graphs and discovered applications in different fields, including shape investigation (Sunada, 2012). Nowadays, the spectral graph theory has expanded to vertexvarying graphs often encountered in many real life applications. Also, there are many simple properties of graphs that can be obtained from the eigenvalues of the matrices e.g., the number of edges, the number of connected components (using the adjacency matrix).

Let G be any simple connected graph comprised of the vertex set $V(G)$ and the edge set $E(G)$. Two vertices are called adjacent if there is an edge between them and if one of the vertices of an edge of a graph is a pendant vertex, the edge is said to be pendant. Let G be any graph of order n then the adjacency matrix of the graph G is a matrix of order $n \times n$ defined as, $A(G) = [n_{ij}]$, where n_{ij} is the number of edges between the vertices i and j. A graph G is classified as, singular or nonsingular depending on whether its adjacency matrix is singular or nonsingular. The

characteristic polynomial of a graph G can be written, as
$f(G;t) = \det(tI - A(G))$ and its roots are called the eigenvalues of graph G and the set of all eigenvalues of graph G is called the spectrum of G denoted as $\sigma(G)$.
Let w be a positive weight function defined on edge set of simple connected graph G, which is used to assign weights to the edges and $W(G)$ is the collection of all positive weight functions defined on the edge set of G. A graph G in which the positive weight function w is used to assign weights to the edges of graph is known as weighted graph, denoted by G_w. We use $V(G_w)$ and $E(G_w)$ to denote the vertex set and edge set of weighted graph G_w. Ordinary graphs can be seen as a particular case of weighted graphs in which all the edges are assigned weight 1. An edge between the vertices i and j is denoted by $[i,j]$. Let $A(G_w)$ denotes the adjacency matrix of weighted graph G_w, defined as

$$A(G_w) = [a_{ij}] = \begin{cases} w[i,j], & \text{if } [i,j] \in E(G_w) \\ \\ 0, & \text{otherwise.} \end{cases}$$

The investigation of a graph's structure by associating different matrices to it is a long-standing and fascinating field of study for researchers. The reader can get some initial concepts from (Cvetković, 1980). It would be useful to take a small picture of a large graph that contains information about the graph in a concise way. Studying the spectrum of various matrices, such as the adjacency matrix, the Laplacian matrix, etc. that can be associated with the graph has proven to be one of the most useful ways of doing so.
It is possible to obtain information about a graph by looking at these eigenvalues that might otherwise be difficult to obtain. For instance, a connected graph G is bipartite if and only if $-\eta$ is an eigenvalue of G whenever η is an eigenvalue of G (Godsil & Royle, 2004). In addition η and $-\eta$ have the same multiplicites.

Definition 1.1 *A connected weighted nonsingular graph G_w is said to satisfy the strong reciprocal eigenvalue property (or property $\mathcal{SR}$) if $\frac{1}{\eta} \in \sigma(G)$ whenever $\eta \in \sigma(G)$ and both have the same multiplicities. Weighted Graph G_w has the reciprocal eigenvalue property (property $\mathcal{R}$) when the multiplicity constraint is removed.*

Definition 1.2 *A connected weighted nonsingular graph G_w is said to satisfy the strong anti-reciprocal eigenvalue property (property $-\mathcal{SR}$) if $-\frac{1}{\eta} \in \sigma(G_w)$ whenever $\eta \in \sigma(G_w)$ and both have the same multiplicities. Moreover, if the multiplicity constraint is removed the weighted graph G_w is said to satisfy anti-reciprocal eigenvalue property (property $-\mathcal{R}$).*

Definition 1.3 *A polynomial $f(t) = \sum_{i=0}^{n} a_i t^i$ of degree n is called palindromic polynomial if $a_i = a_{n-i}$ and anti-palindromic polynomial if $a_i = -a_{n-i}$ for $i = 0, 1, \ldots, n$. Property $\mathcal{SR}$ is satisfied by a polynomial $f(t)$ if and only if it is palindromic or anti-palindromic.*

(Frucht & Harary, 1970) defined the corona product of graphs which plays an important role in constructing and characterizing graphs with reciprocal eigenvalue property.

Definition 1.4 *Let L_1 and L_2 be two connected graphs of order n and m, respectively. The corona product $L_1 \circ L_2$ is a graph formed by one copy of graph L_1 and n-copies of L_2 and by connecting each vertex of jth copy of L_2 with the jth vertex of L_1, for $1 \leq j \leq n$.*

We proceed with some previous results. In 1978, graphs with property $\mathcal{SR}$ were investigated for nonsingular trees under the names symmetric property (Godsil & Mckay, 1978) and property C (Cvetković *et al.*, 1978). This property was renamed "property $\mathcal{SR}$" by Barik *et al.* in

2006, and they also introduced property $\mathcal{R}$. They showed that for nonsingular trees, these two properties are the same (Barik *et al.*, 2006).
If specific limits on the weight function are implemented, these properties are similar for weighted trees (Neumann & Pati, 2013), as well as a subclass of connected bipartite graphs with unique perfect matching (Panda & Pati, 2015). In general, however, these properties are not identical (Panda & Pati, 2016).
In 2012, J. D. Lagrange investigated property $-\mathcal{SR}$ first time for the zero-divisor graphs of finite commutative rings with non-zero divisors (Lagrange, 2012).
Authors investigated (Bapat *et al.*, 2016) that if G is a connected bipartite graph having a unique perfect matching M, then weighted graph G_w satisfies property $\mathcal{SR}$, for all $w \in W(G)$ if and only if G is corona.
(Hameed & Ahmad, 2020) analyzed noncorona graphs with zero diagonal entries of the inverse of their adjacency matrix and a single perfect matching, and discovered that they do not meet property $-\mathcal{SR}$ even for a single weight function w.
Property $-\mathcal{SR}$ for the class of connected simple weighted graphs having unique perfect matching M, denoted by G_M, was investigated by (Ahmad *et al.*, 2020). They showed that the weighted graph G_w satisfies property $-\mathcal{SR}$ for all $w \in W(G)$ if and only if G is corona. They also verified property $-\mathcal{SR}$ for some families of noncorona graphs (Ahmad *et al.*, 2021) and authors of (Barik *et al.*, 2021) further generalized these families. They constructed the classes of noncorona graphs by taking a connected corona graph M and by joining each vertex of finite number of copies of corona cycles of different finite length to non-pendant vertices of M, in such a way that no corona cycle is attached to more than one non-pendant vertex.
Until now, the properties $\mathcal{R}$ and $-\mathcal{SR}$ are not studied for weighted noncorona graphs. So, the question arises 'are there any weighted noncorona graphs with these eigenvalue properties?' With the required properties, we constructed families of weighted noncorona graphs. In Section 2, a family of weighted noncorona graphs satisfying property $\mathcal{R}$ and in Section 3 two family of weighted noncorona graphs satisfying property $-\mathcal{SR}$ are constructed. Throughout the paper simple and undirected graphs will be discussed and e_i is the standard unit vector whose i-th entry is equal to 1. Following Lemma gives necessary and sufficient condition for a polynomial to satisfy property $-\mathcal{SR}$.

Lemma 1.1 *(Ahmad et al., 2020) A polynomial* $f(t) = \sum_{i=0}^{2n} a_i t^i$ *satisfies property* $-\mathcal{SR}$ *if and only if*

$$a_{2n-i} = \begin{cases} a_i, & \text{if } i \text{ and } n \text{ have the same parity,} \\ -a_i, & \text{otherwise.} \end{cases} \qquad i = 0, 1, 2, \ldots, 2n.$$

Lemma 1.2 and Lemma 1.3 on determinant and inverse of a block matrix involving the Schur complement are used in the proofs of our main results.

Lemma 1.2 *(Bapat, 2010) If* A *is a block matrix i.e,* $A = \begin{bmatrix} K & L \\ M & N \end{bmatrix}$ *where* K *and* N *are square matrices. Then*

$$\det(A) = \begin{cases} \det(K)\det(N - MK^{-1}L), & \text{if } K \text{ is invertible} \\ \det(N)\det(K - LN^{-1}M), & \text{if } N \text{ is invertible.} \end{cases}$$

Lemma 1.3 *(Bapat, 2010) If A is a block matrix and $A = \begin{bmatrix} K & L \\ M & N \end{bmatrix}$ where K and N are square matrices and N is invertible. Then A is invertible if and only if the Schur complement of N is invertible i.e, $A_N = K - LN^{-1}M$ is invertible, and*

$$A^{-1} = \begin{bmatrix} A_N^{-1} & -A_N^{-1}LN^{-1} \\ -NMA_N^{-1} & N^{-1} + N^{-1}MA_N^{-1}LN^{-1} \end{bmatrix}.$$

The Lemma 1.4 is used in the proof of Theorem 3.1.

Lemma 1.4 *(Barik et al., 2021) Let G be a regular graph of order m and regularity r, and $G_1 = G \circ K_1$. Then*

$$\mathbf{1}^t(tI_{2m} - A(G_1))^{-1}\mathbf{1} = \frac{(2t - r + 2)m}{t^2 - rt - 1}.$$

2. Weighted noncorona graphs satisfying property $\mathcal{R}$

In this Section, we construct a class of weighted noncorona graphs which satisfy property $\mathcal{R}$ but not property $\mathcal{SR}$. In (Panda, 2016) and (Panda & Pati, 2016), authors constructed a class of unweighted noncorona graphs satisfying property $\mathcal{R}$. Now the question arises that 'is it possible to assign weights to some edges so that this class still satisfies property $\mathcal{R}$?' To answer this question, we assign weights to some particular edges of the family of unweighted graphs constructed in (Panda, 2016) and (Panda & Pati, 2016). The new family of weighted noncorna graphs with property $\mathcal{R}$ is as follows.

Consider one copy of P_4, join every vertex of this copy to a new vertex a and name graph as $\acute{G}$

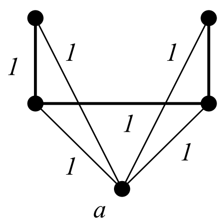

Fig. 1. Graph $\acute{G}$

as shown in Figure 1. Now take k ($k \geq 1$) copies of P_4 named as $P_4^1, P_4^2, \ldots, P_4^k$. With the help of $\acute{G}$ and these k copies of P_4 construct a family $\aleph$ of weighted noncorona graphs in which each weighted graph H_w^k is created by joining every non-pendant vertex in the k copies of P_4 to the vertex a and assigning weights $w_i > 0$ to the joining edges of a and each P_4^i for $i = 1, 2, \ldots, k$ respectively and then add a new vertex b at a. The edges in all k copies of P_4 and $\acute{G}$ are assigned weight 1. A weighted noncorona graph H_w^2 belonging to this family is shown in Figure 2.

The following result proves that weighted noncorona graph $H_w^k \in \aleph$ satisfies property $\mathcal{R}$ but not $\mathcal{SR}$.

Theorem 2.1 *The weighted noncorna graph $H_w^k \in \aleph$ satisfies property $\mathcal{R}$ but not $\mathcal{SR}$.*

Proof:

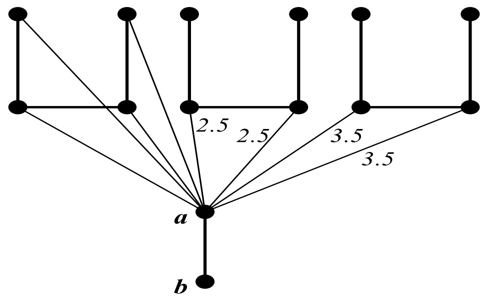

Fig. 2. Weighted noncorona graph H_w^2

The adjacency matrix $A(H_w^k)$ of the graph H_w^k can be written, as

$$A(H_w^k) = \begin{pmatrix} A(\acute{G}) & e_1 & w_1 K_{5,4} & \cdots & w_k K_{5,4} \\ e_1^t & 0 & \mathbf{0}^t & \cdots & \mathbf{0}^t \\ w_1 K_{5,4}^t & \mathbf{0} & A(P_4^1) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_k K_{5,4}^t & \mathbf{0} & O & \cdots & A(P_4^k) \end{pmatrix},$$

where

$$K_{5,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that

$$\mathcal{B} = \begin{pmatrix} tI_4 - A(P_4^1) & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & tI_4 - A(P_4^k) \end{pmatrix}.$$

Then the characteristic polynomial of H_w^k can be written, as

$$f(H_w^k; t) = \det(tI - A(H_w^k))$$

$$= \det \begin{pmatrix} tI_5 - A(\acute{G}) & -e_1 & -w_1 K_{5,4} & \cdots & -w_k K_{5,4} \\ -e_1^t & t & \mathbf{0}^t & \cdots & \mathbf{0}^t \\ -w_1 K_{5,4}^t & \mathbf{0} & tI_4 - A(P_4^1) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_k K_{5,4}^t & \mathbf{0} & O & \cdots & tI_4 - A(P_4^k) \end{pmatrix},$$

using Lemma 1.2

$$= \det(\mathcal{B})\det\left(\begin{bmatrix} tI_5 - A(\acute{G}) & -e_1 \\ -e_1^t & t \end{bmatrix} - \begin{bmatrix} -w_1 K_{5,4} & \cdots & -w_k K_{5,4} \\ \mathbf{0}^t & \cdots & \mathbf{0}^t \end{bmatrix} \mathcal{B}^{-1} \begin{bmatrix} -w_1 K_{5,4}^t & \mathbf{0} \\ \vdots & \vdots \\ -w_k K_{5,4}^t & \mathbf{0} \end{bmatrix} \right),$$

where

$$\mathcal{B}^{-1} = \begin{pmatrix} (tI_4 - A(P_4^1))^{-1} & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & (tI_4 - A(P_4^k))^{-1} \end{pmatrix},$$

and

$$(tI_4 - A(P_4))^{-1} = \frac{1}{t^4 - 3t^2 + 1} \begin{pmatrix} (t^2-1)t & t^2 & t^2-1 & t \\ t^2 & (t^2-1)t & t & t^2-1 \\ t^2-1 & t & t(t^2-2) & 1 \\ t & t^2-1 & 1 & t(t^2-2) \end{pmatrix}.$$

Thus,

$$\begin{aligned} f(H_w^k;t) &= \left(\prod_{i=1}^k f(P_4;t)\right) \det\left(\begin{bmatrix} tI_5 - A(\acute{G}) & -e_1 \\ -e_1^t & t \end{bmatrix} - \begin{bmatrix} \frac{2t}{t^2-t-1} \sum_{i=1}^k w_i^2 K_{5,5} & \mathbf{0} \\ \mathbf{0}^t & 0 \end{bmatrix} \right) \\ &= (t^4 - 3t^2 + 1)^k \det\left(\begin{bmatrix} tI_5 - A(\acute{G}) - \frac{2t}{t^2-t-1} \sum_{i=1}^k w_i^2 K_{5,5} & -e_1 \\ -e_1^t & t \end{bmatrix} \right) \\ &= (t^2 - t - 1)^k (t^2 + t - 1)^k (t^4 - t^3 - 2(\textstyle\sum_{i=1}^k w_i^2 + 3)t^2 - t + 1)(t^2 + t - 1). \end{aligned}$$

Here notice that, $\{1.618033, -0.618033\}$ are the roots of polynomial (t^2-t-1) then $\{0.618033 = \frac{1}{1.618033}, -1.618033 = \frac{1}{-0.618033}\}$ are the roots of polynomial $(t^2 + t - 1)$ and the polynomial $(t^4 - t^3 - 2(\sum_{i=1}^k w_i^2 + 3)t^2 - t + 1)$ is palindromic as a result this polynomial satisfies property $\mathcal{SR}$. However, because $f(H_w^k;t)$ has an additional factor $(t^2 + t - 1)$, we can see that every eigenvalue of H_w^k has its reciprocal as an eigenvalue of H_w^k but multiplicities are different so weighted noncorona graph H_w^k satisfies property $\mathcal{R}$ but not $\mathcal{SR}$.
Following example is an illustration of the weighted noncorna graph belonging to the family $\aleph$, it can be seen from Table 1 that weighted noncorona graph H_w^2 satisfies property $\mathcal{R}$ but not $\mathcal{SR}$.

Example 2.1 *The weighted noncorna graph H_w^2, is shown in Figure 2. The eigenvalues of H_w^2, their reciprocals and their multiplicities are given in the following Table:*

Table 1. Eigenvalues of H_w^2, their reciprocals and their multiplicities

Sr. No.	η	Multiplicity of η	$\frac{1}{\eta}$	Multiplicity of $\frac{1}{\eta}$
1	-2.61803	1	-0.38196	1
2	-1.61803	3	-0.61803	2
3	-0.61803	2	-1.61803	3
4	-0.38196	1	-2.61803	1
5	0.26794	1	3.73205	1
6	0.61803	3	1.61803	2
7	1.61803	2	0.61803	3
8	3.73205	1	0.26794	1

3. Weighted noncorona graphs satisfying property $-\mathcal{SR}$

In this Section, some classes of weighted noncorona graphs are constructed which satisfy property $-\mathcal{SR}$. Consider a connected weighted graph G_w, $w > 0$ of order n and $G_w^1 = G_w \circ K_1$ be its weighted corona graph in which pendant edges are assigned weight 1. Let $F^p = C_p \circ K_1$ be corona cycle where C_p is a cycle of order p, $p \geq 3$. Now, with the help of weighted graph G_w^1 and corona cycles with edges assigned weight 1, we construct families of weighted noncorona

graphs as follows:
Take a copy weighted graph of G_w^1 and k corona cycles $F_1^{p_1}, F_2^{p_2}, \ldots, F_k^{p_k}$ (where p_i's not necessarily same, for $i = 1, 2, \ldots, k$) with edges assigned weight 1. Consider any number of non-pendant vertices $v_1, v_2, \ldots, v_l$, $(1 \leq l \leq n)$ of weighted graph G_w^1. Join each v_j, $(j \leq l)$ to all the vertices of each corona cycle $F_i^{p_i}, i = 1, 2, \ldots, k$. Assign weight w_i to the edges joining a cycle $F_i^{p_i}$, $(i = 1, 2, \ldots, k)$ to all the vertices $v_1, v_2, \ldots, v_l$ and name this weighted graph as $S_{(w_1,w_2,\ldots,w_k)}^{(p_1,p_2,\ldots,p_k;l)}$ as shown in Figure 3. We denote the family containing all weighted noncorona graphs $S_{(w_1,w_2,\ldots,w_k)}^{(p_1,p_2,\ldots,p_k;l)}$ by $\mathfrak{G}$. Now, instead of assigning weight w_i to the edges joining a cycle $F_i^{p_i}$, $(i = 1, 2, \ldots, k)$ to all the vertices $v_1, v_2, \ldots, v_l$, if we assign weight w_j to the edges joining the vertex v_j to each corona cycle for $j = 1, 2, \ldots, l$ we obtain a new weighted graph named as, $S_{(w_1;w_2;\ldots;w_l)}^{(p_1,p_2,\ldots,p_k;l)}$ as shown in Figure 5. We denote the family containing all weighted noncorona graphs $S_{(w_1;w_2;\ldots;w_l)}^{(p_1,p_2,\ldots,p_k;l)}$ by $\mathfrak{H}$.

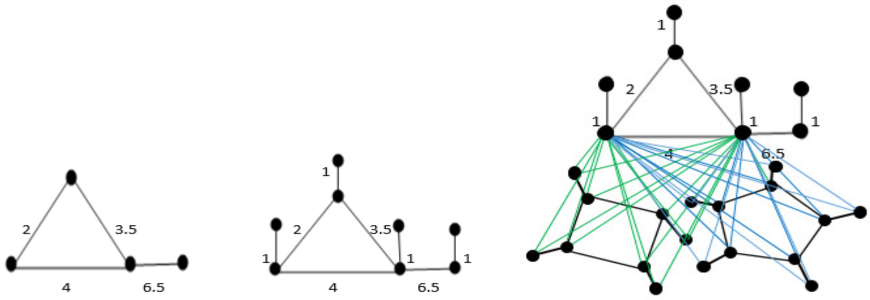

Fig. 3. Weighted graph U, weighted corona graph U_w^1 and $S_{(3.5,6.5)}^{(4,5;2)}$.

Observation 3.1 *For a weighted corona graph G_w^1 of order $2n$, the sum of first $n \times n$ entries of cofactor matrix of $tI - A(G_w^1)$ can be written, as*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} C_{ij} = ct^k g(t),$$

where c is any constant and $g(t)$ is a polynomial of degree $2n - 2k$, $1 \leq k \leq n$, satisfying property $-\mathcal{SR}$. Then note that $f(t) + c\, t^k g(t)$ also satisfies property $-\mathcal{SR}$, where $f(t)$ is the characteristic polynomial of the weighted corona graph G_w^1 of weighted graph G_w and $g(t)$ is the polynomial obtained from the sum of first $n \times n$ entries of the cofactor matrix of $tI - A(G_w^1)$.

We can see this observation with the help of Example 3.1.

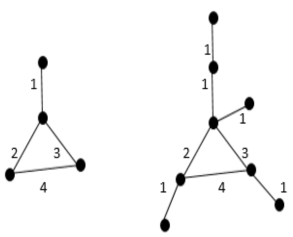

Fig. 4. Weighted graph Z_w and its weighted corona graph Z_w^1

Example 3.1 *Consider a connected weighted graph Z_w of order $n = 4$ and its corona graph as shown in the Figure 4. Then characteristic polynomial of $Z_w^1 = Z_w \circ K_1$ can be determined, as*
$f(Z_w^1;t) = det(tI - A(Z_w^1)) = t^8 - 34t^6 - 48t^5 + 82t^4 + 48t^3 - 34t^2 + 1.$
We can see that it is a polynomial of order $2n = 8$ which satisfies property $-\mathcal{SR}$ as Z_w^1 is weighted corona graph. Now the sum of first 4×4 entries of cofactor matrix of $tI - A(Z_w^1)$ can be written, as

$$\begin{aligned} tg(t) &= 4t^7 + 20t^6 - 10t^5 - 88t^4 + 10t^3 + 20t^2 - 4t \\ &= 2t(2t^6 + 10t^5 - 5t^4 - 44t^3 + 5t^2 + 10t - 2), \end{aligned}$$

which satisfies property $-\mathcal{SR}$ by Lemma 1.1.
Now

$$f(t) + tg(t) = t^8 + 4t^7 - 14t^6 - 58t^5 - 6t^4 + 58t^3 - 14t^2 - 4t + 1,$$

which also satisfies property $-\mathcal{SR}$ by Lemma 1.1.

By Laplace expansion, we can easily obtain the following result.

Lemma 3.1 *Let A be any $2n \times 2n$ matrix, then*

$$\det(A + \begin{bmatrix} J_n & O_n \\ O_n & O_n \end{bmatrix}) = \det(A) + \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{(i+j)}\det(A[i,j]),$$

where J_n is the matrix of ones, O_n is the matrix of zeros and $A[i,j]$ is the sub-matrix of matrix A obtained by deleting ith row and jth column.

The following result proves that weighted noncorona graph $S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)}$ satisfies property $-\mathcal{SR}$.

Theorem 3.1 *The weighted noncorona graph $S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)} \in \mathfrak{G}$ for $1 \le l \le n$ satisfies property $-\mathcal{SR}$.*

Proof:
The adjacency matrix $A(S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)})$ of the weighted noncorona graph $S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)}$ can be written, as

$$A(S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)}) = \begin{pmatrix} A(G_w) & I_n & w_1N_{n,2p_1} & \cdots & w_kN_{n,2p_k} \\ I_n & O & O & \cdots & O \\ w_1N^t_{n,2p_1} & O & A(F_1^{p_1}) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_kN^t_{n,2p_k} & O & O & \cdots & A(F_k^{p_k}) \end{pmatrix},$$

where $N_{n,2p_k} = \begin{bmatrix} J_{l,2p_k} \\ O_{n-l,2p_k} \end{bmatrix}$ for $1 \le l \le n$ is a block matrix in which $J_{l,2p_k}$ is the matrix with all entries 1 of order $l \times 2p_k$ and $O_{n-l,2p_k}$ is the Null matrix of order $(n-l) \times 2p_k$. Let us suppose that

$$\mathcal{D} = \begin{pmatrix} tI_{2p_1} - A(F_1^{p_1}) & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & tI_{2p_k} - A(F_k^{p_k}) \end{pmatrix}.$$

Then the characteristic polynomial of $S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)}$ can be written, as

$$f(S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)};t) = \det(tI - A(S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)}))$$

$$= \det\begin{pmatrix} tI_n - A(G_w) & -I_n & -w_1N_{n,2p_1} & \cdots & -w_kN_{n,2p_k} \\ -I_n & O & O & \cdots & O \\ -w_1N^t_{n,2p_1} & O & tI_{2p_1} - A(F_1^{p_1}) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_kN^t_{n,2p_k} & O & O & \cdots & tI_{2p_k} - A(F_k^{p_k}) \end{pmatrix},$$

using Lemma 1.2

$$= \det(\mathcal{D})det\left(\begin{bmatrix} tI_n - A(G_w) & -I_n \\ -I_n & tI_n \end{bmatrix} - \begin{bmatrix} -w_1N_{n,2p_1} & \cdots & -w_kN_{n,2p_k} \\ O & \cdots & O \end{bmatrix}\right.$$

$$\left.\mathcal{D}^{-1}\begin{bmatrix} -w_1N^t_{n,2p_1} & O \\ \vdots & \vdots \\ -w_kN^t_{n,2p_k} & O \end{bmatrix}\right)$$

$$= (\textstyle\prod_{i=1}^k f(F_i^{p_i};t))\det\left(\begin{bmatrix} tI_n - A(G_w) & -I_n \\ -I_n & tI_n \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^k w_i^2\mathbf{1}^t\mathcal{D}^{-1}\mathbf{1}N_n & O \\ O & O \end{bmatrix}\right).$$

Now, from Lemma 1.4,

$$\mathbf{1}^t\mathcal{D}^{-1}\mathbf{1} = \frac{2t}{t^2-2t-1}\sum_{i=1}^k p_i,$$

Thus,

$$= (\textstyle\prod_{i=1}^k f(F_i^{p_i};t))\det\left(\begin{bmatrix} tI_n - A(G_w) & -I_n \\ -I_n & tI_n \end{bmatrix} - \begin{bmatrix} \frac{2t}{t^2-2t-1}\sum_{i=i}^k p_iw_i^2\, N_n & O \\ O & O \end{bmatrix}\right)$$

$$= (\textstyle\prod_{i=1}^k f(F_i^{p_i};t))\det((tI_{2n} - A(G_w^1)) + \begin{bmatrix} aN_n & O \\ O & O \end{bmatrix}), \text{ where } a = -\frac{2t}{t^2-2t-1}\sum_{i=1}^k p_iw_i^2.$$

Now by using Lemma 3.1

$$= (\textstyle\prod_{i=1}^k f(F_i^{p_i};t))[\det(tI_{2n} - A(G_w^1)) + a\sum_{i=1}^l\sum_{j=1}^l(-1)^{i+j}\det((tI_{2n} - A(G_w^1)[i,j])],$$

and by Observation 3.1

$$f(S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)};t) = \frac{\prod_{i=1}^k f(F_i^{p_i};t)}{t^2-2t-1}(f(t)+ct^kg(t)),$$

where $f(t) = (t^2-2t-1)f(G_w\circ K_1;t)$ satisfies property $-\mathcal{SR}$ and by Observation 3.1, $f(t)+ct^kg(t)$ satisfies property $-\mathcal{SR}$ also for $i=1,2,\ldots k$, $\frac{f(F_i^{p_i};t)}{(t^2-2t-1)}$ satisfies property $-\mathcal{SR}$. Thus, $f(S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1,w_2,\ldots,w_k)};t)$ satisfies property $-\mathcal{SR}$.

Following example is an illustration of the weighted noncorona graph $S^{(4,5;2)}_{(3.5,6.5)}$ for $p_1=4,\ p_2=5,\ w_1=3.5,\ w_2=6.5$ and $l=2$, it can be seen from Table 2 that weighted noncorona graph $S^{(4,5;2)}_{(3.5,6.5)}$ satisfies property $-\mathcal{SR}$.

Example 3.2 *Let M_w be any connected weighted graph of order 4 and $M_w^1 = M_w\circ K_1$ be its weighted corona graph in which pendant edge has weight 1 as shown in Figure 3. Now, construct the weighted noncorona graph $S^{(4,5;2)}_{(3.5,6.5)}$ by using M_w^1 and the corona cycles F_1^4 and*

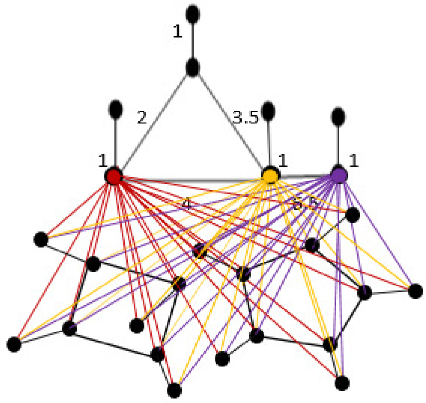

Fig. 5. Weighted noncorona graph $S^{(4,5;3)}_{(0.5;1.5;2.5)}$ in which red edges are assigned weight 0.5, yellow edges are assigned weight 1.5 and purple edges are assigned weight 2.5.

F^5_2, as shown in Figure 3. The weights assigned to the joining edges of corona cycles F^4_1, F^5_2 to 2 selected vertices of M_w are 3.5 and 6.5 represented by green and blue edges respectively. The eigenvalues of $S^{(4,5;2)}_{(3.5,6.5)}$ and with their multiplicities are mentioned in the following table.

Table 2. Eigenvalues of $S^{(4,5;2)}_{(3.5,6.5)}$, their reciprocals and their multiplicities

Sr. No.	η	multiplicity of η	$-\frac{1}{\eta}$	multiplicity of $-\frac{1}{\eta}$
1	-42.194	1	0.0237	1
2	-7.2208	1	0.13849	1
3	-2.4142	1	0.41421	1
4	-2.0953	2	0.47726	2
5	-1	2	1	2
6	-0.99623	1	1.0038	1
7	-0.73764	2	1.3557	2
8	-0.41421	1	2.4142	1
9	-0.2936	1	3.4060	1
10	-0.020767	1	48.154	1
11	0.0237	1	-42.194	1
12	0.13849	1	-7.2208	1
13	0.41421	1	-2.4142	1
14	0.47726	2	-2.0953	2
15	1	2	-1	2
16	1.0038	1	-0.99623	1
17	1.3557	2	-0.73764	2
18	2.4142	1	-0.41421	1
19	3.4060	1	-0.2936	1
20	48.154	1	-0.020767	1

The following theorem can be proved with the same strategy as in Theorem 3.1.

Theorem 3.2 *Weighted noncorona graph $S^{(p_1,p_2,\ldots,p_k;l)}_{(w_1;w_2;\ldots;w_l)}$ satisfies property $-\mathcal{SR}$.*

4. Conclusion

In this article, we constructed three classes of weighted noncorona graphs namely $\aleph$, $\mathfrak{G}$ and $\mathfrak{H}$ which satisfy property $\mathcal{R}$ or $-\mathcal{SR}$. The family of weighted noncorona $\aleph$ satisfies property $\mathcal{R}$ but not $\mathcal{SR}$. The other two families $\mathfrak{G}$ and $\mathfrak{H}$ satisfy property $-\mathcal{SR}$.

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