Kuwait J. Sci. **45 (1)** pp 79-88, 2018

Generalized ratio-product-type estimator for variance using auxiliary information in simple random sampling

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Abstract

This paper suggests a new generalized ratio-product-type estimator for population variance of study variable utilizing information obtained from two auxiliary variables. Efficiency of the new estimator has been compared mathematically with the generalized ratio-product-type estimator based on information from auxiliary variable under simple random sampling without replacement. Empirically, the estimator proves more efficient than the usual unbiased estimator and some previously existing biased variance estimators under the derived conditions and for suitable choice of scalars and constants at which bias is also smaller in comparison. It is also worth-mentioning that all the estimators under discussion are the special cases of the new generalized ratio-product-type estimator for population variance.

Keywords: Generalized estimator for population variance;population variance estimator using two auxiliary variable information; ratio-product-type variance estimator; ratio-type variance estimator; transformed sample variances.

1. Introduction

A statistical population can be described by several characteristics, one of which is variance. It is a routine practice that population variance is not known. Remarkable attempts have been made to estimate this characteristic in the best possible way. Objective is to minimize the dispersion of the desired estimator, if obtained from different possible samples. Unbiasedness is one of the ideal properties of an estimator, but it may be sacrificed, if a biased estimator provides less scattered results in repeated sampling. Moreover, need for the use of information available from one or more auxiliary variables has also been emphasized to raise the efficiency of estimator. Numerous efforts have already been made in this regard including ratio, ratio-type and transformed ratio-product-type estimators for variance.

Consider y, the study variable in a finite population of size N from which a sample of size n is drawn using simple random sampling without replacement. Let x and z be the auxiliary variables about which the information in the form of observations or some useful parameters is available. The available variances are transformed by using the parameters of the relevant variable.

Here are some important results and notations to be used later.

N: Population size $\qquad$ n: Sample size

the population means of the variables:

$$\bar{Y}=\sum_{i=1}^{N} y_i/N,\ \bar{X}=\sum_{i=1}^{N} x_i/N,\ \bar{Z}=\sum_{i=1}^{N} z_i/N$$

the population variances:

$$S_y^2=\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)^2/(N-1),$$

$$S_x^2=\sum_{i=1}^{N}\left(x_i-\bar{X}\right)^2/(N-1),$$

$$S_z^2=\sum_{i=1}^{N}\left(z_i-\bar{Z}\right)^2/(N-1)$$

the sample means:

$$\bar{y}=\sum_{i=1}^{n} y_i/n,\ \bar{x}=\sum_{i=1}^{n} x_i/n,\ \bar{z}=\sum_{i=1}^{n} z_i/n,$$

the unbiased sample variances:

$$s_y^2=\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2/(n-1)$$

$$s_x^2=\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2/(n-1),$$

$$s_z^2=\sum_{i=1}^{n}\left(z_i-\bar{z}\right)^2/(n-1) \qquad (1)$$

the population correlation coefficients:

$$\rho_{xy}=S_{xy}/S_xS_y,\ \rho_{zy}=S_{zy}/S_zS_y,$$

$\rho_{zx}=S_{zx}/S_zS_x$ where

$$S_{xy} = \sum_{i=1}^{N}\left(x_i - \bar{X}\right)\left(y_i - \bar{Y}\right)/\left(N-1\right),$$

$$S_{zy} = \sum_{i=1}^{N}\left(z_i - \bar{Z}\right)\left(y_i - \bar{Y}\right)/\left(N-1\right),$$

$$S_{zx} = \sum_{i=1}^{N}\left(z_i - \bar{Z}\right)\left(x_i - \bar{X}\right)/\left(N-1\right)$$

$\lambda' = \frac{1}{n}\left(1 - \frac{n}{N}\right)$ and

$N-1 \cong N$ for sufficiently large population.

Population coefficients of variation

$C_y = S_y/\bar{Y}$, $C_x = S_x/\bar{X}$, $C_z = S_z/\bar{Z}$

Population coefficients of kurtosis

$\beta_2(y) = \mu_{400}/\mu_{200}^2$, $\beta_2(x) = \mu_{040}/\mu_{020}^2$,

$\beta_2(z) = \mu_{004}/\mu_{002}^2$.

Some other functions as described by Jhajj *et al*. (2005)

$h = \mu_{220}/\mu_{200}\mu_{020}$, $k = \mu_{202}/\mu_{200}\mu_{002}$,

$l = \mu_{022}/\mu_{020}\mu_{002}$

where

$$\mu_{rst} = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^r\left(x_i - \bar{X}\right)^s\left(z_i - \bar{Z}\right)^t$$

and *r, s, t* are positive integers.

Some important results as presented in Yadav *et al*. (2013)

$e_0 = \left(S_y\right)^{-2}\left(s_y^2 - S_y^2\right)$, $e_1 = \left(S_x\right)^{-2}\left(s_x^2 - S_x^2\right)$,

$e_2 = \left(S_z\right)^{-2}\left(s_z^2 - S_z^2\right)$

the expected values of e_0, e_1 and e_2 are all zero as:

$E\left(s_y^2\right) = S_y^2$, $E\left(s_x^2\right) = S_x^2$, $E\left(s_z^2\right) = S_z^2$

$E\left(e_0^2\right) = \lambda'\beta_2'(y)$, $E\left(e_1^2\right) = \lambda'\beta_2'(x)$,

$E\left(e_2^2\right) = \lambda'\beta_2'(z)$

$E\left(e_0 e_1\right) = \lambda' h'$, $E\left(e_0 e_2\right) = \lambda' k'$,

$E\left(e_1 e_2\right) = \lambda' l'$

where $\beta_2'(y) = \beta_2(y) - 1$

$\beta_2'(x) = \beta_2(x) - 1$, $\beta_2'(z) = \beta_2(z) - 1$ and

$h' = h - 1$, $k' = k - 1$, $l' = l - 1$

some further notations to be used in deriving bias and mean-squared error are:

$\upsilon' = h'/\beta_2'(x)$, $\mu' = k'/\beta_2'(z)$

$\theta_0 = \frac{S_y^2}{S_y^2 + d}$, $\theta_1 = \frac{b}{b+c}$, $\theta_2 = \frac{m}{m+p}$,

$w_1 = \frac{\theta_1}{\theta_0}$, $w_2 = \frac{\theta_2}{\theta_0}$.

where b,c,d,m and p are constants or usually the known parameters of variables as: b,c the known parameters of x; m,p of z ; and d of y .

Isaki (1983) proposed an estimator of population variance using auxiliary information that has been cited by the successors. Some other references are Ahmed *et al*. (2000); Arcos *et al*. (2005); Gupta & Shabbir (2008); Subramani & Kumarapandiyan (2012a); Subramani & Kumarapandiyan (2012b); Singh & Solanki (2013a); Subramani & Kumarapandiyan (2013); Singh & Solanki (2013b); Yadav *et al*. (2013).

Yadav *et al*. (2013) developed a class of estimators of variance as the combination of ratio-type and product-type estimators based on some parametric information of an auxiliary as well as study variable and transforming the variances of both the variables. Moreover, it described usual unbiased estimator s_y^2, the estimators of Isaki (1983); Upadhyaya & Singh (1999); Kadilar & Cingi (2006) as special cases for certain values of the scalars and constants; and under certain conditions surpassed all these estimators by obtaining smaller bias and mean-squared error.

Some biased variance estimators of the study variable y, utilizing information of an auxiliary variable along with their bias, MSE are as under:

Isaki (1983) ratio estimator is

$$t_{Isaki} = s_y^2\left(S_x^2/s_x^2\right) \tag{2}$$

Bias of this estimator is $Bias\left(t_{Isaki}\right) \cong \lambda' S_y^2\left\{\beta_2'(x) - h'\right\}$ and mean-squared error.

$$MSE\left(t_{Isaki}\right) \cong \lambda' S_y^4\left\{\begin{matrix}\beta_2'(y) + \\ \beta_2'(x) - 2h'\end{matrix}\right\}.$$

Upadhyaya & Singh (1999) proposed ratio-type estimator for population variance as

$$t_{US} = s_y^2\left\{s_x^2 + \beta_2(x)\right\}^{-1}\left\{S_x^2 + \beta_2(x)\right\} \tag{3}$$

and has bias as,

$$Bias\left(t_{US}\right) \cong \left[\begin{matrix}\lambda' S_y^2\frac{S_x^2}{S_x^2 + \beta_2(x)} \\ \left\{\begin{matrix}\frac{S_x^2}{S_x^2 + \beta_2(x)}\beta_2'(x) \\ -h'\end{matrix}\right\}\end{matrix}\right],$$

mean-squared error as

$$MSE\left(t_{US}\right) \cong \lambda' S_y^4 \begin{bmatrix} \beta_2'\left(y\right) + \\ \left\{\dfrac{S_x^2}{S_x^2+\beta_2\left(x\right)}\right\}^2 \beta_2'\left(x\right) \\ -2\dfrac{S_x^2}{S_x^2+\beta_2\left(x\right)} h' \end{bmatrix}$$

Following are the ratio-type estimators suggested by Kadilar & Cingi (2006):

$$t_{KC1} = s_y^2 \left\{s_x^2 - \beta_2\left(x\right)\right\}^{-1} \left\{S_x^2 - \beta_2\left(x\right)\right\} \tag{4}$$

$$t_{KC2} = s_y^2 \begin{bmatrix} \left\{C_x s_x^2 - \beta_2\left(x\right)\right\}^{-1} \\ \left\{C_x S_x^2 - \beta_2\left(x\right)\right\} \end{bmatrix} \tag{5}$$

$$t_{KC3} = s_y^2 \left\{s_x^2 - C_x\right\}^{-1} \left\{S_x^2 - C_x\right\} \tag{6}$$

$$t_{KC4} = \begin{bmatrix} s_y^2 \left\{\beta_2\left(x\right) s_x^2 - C_x\right\}^{-1} \\ \left\{\beta_2\left(x\right) S_x^2 - C_x\right\} \end{bmatrix} \tag{7}$$

Their biases and mean-squared errors are:

$$Bias(t_{KCi}) \cong \lambda' S_y^2 \beta_2'\left(x\right) A_i \left(A_i - \upsilon'\right)$$

$$MSE(t_{KCi}) \cong \lambda' S_y^4 \begin{Bmatrix} \beta_2'\left(y\right) + \\ \beta_2'\left(x\right) \\ A_i\left(A_i - 2\upsilon'\right) \end{Bmatrix},$$

$i = 1,2,3,4$

where $A_1 = \dfrac{S_x^2}{S_x^2 - \beta_2\left(x\right)}$, $A_2 = \dfrac{C_x S_x^2}{C_x S_x^2 - \beta_2\left(x\right)}$,

$A_3 = \dfrac{S_x^2}{S_x^2 - C_x}$, $A_4 = \dfrac{\beta_2\left(x\right) S_x^2}{\beta_2\left(x\right) S_x^2 - C_x}$,

Yadav *et al*. (2013) defined the following transformed ratio-product-type family of estimators

$$t_{YrpG} = (s_y^2 + d) \begin{bmatrix} \omega\left(\dfrac{S_u^2}{s_u^2}\right)^{\tau} \\ +\left(1-\omega\right)\left(\dfrac{s_u^2}{S_u^2}\right)^{\chi} \end{bmatrix} - d \tag{8}$$

where $s_u^2 = b s_x^2 + c S_x^2$ and $S_u^2 = \left(b+c\right) S_x^2$ τ and χ are suitably chosen values.

$$Bias(t_{YrpG}) \cong \begin{pmatrix} \frac{1}{2}\lambda' S_y^2 \theta_0 \\ \beta_2'\left(x\right) w_1 \end{pmatrix} \begin{bmatrix} w_1 \begin{Bmatrix} \omega\tau^2 + \\ \left(1-\omega\right)\chi^2 \end{Bmatrix} \\ + \begin{Bmatrix} \omega\tau - \\ \left(1-\omega\right)\chi \end{Bmatrix} \\ \left\{w_1 - 2\upsilon'\right\} \end{bmatrix} \tag{9}$$

$$MSE\left(t_{YrpG}\right) \cong \lambda' S_y^4 \begin{bmatrix} \beta_2'\left(y\right) + \\ \beta_2'\left(x\right) A_{YrpG}\left(A_{YrpG} - 2\upsilon'\right) \end{bmatrix}$$

where $A_{YrpG} = w_1 \left\{\omega\tau - \left(1-\omega\right)\chi\right\}$ (10)

All the variance estimators with one auxiliary variable from Equation (3) to Equation (8) are minimized at the same point i.e.

$$MSE_{\min}(t_{..}) \cong \lambda' S_y^4 \left\{\beta_2'\left(y\right) - h'\upsilon'\right\} \tag{11}$$

2. New generalized ratio-product-type estimator using two auxiliary variables information

Taking motivation from Yadav *et al*. (2013) here is being presented a new generalized estimator based on the information obtained from two auxiliary variables and using the concept of transformation of variances.

$$t_{rpG} = (s_y^2 + d) \begin{bmatrix} \omega\left(\dfrac{S_u^2}{s_u^2}\right)^{\tau} + \\ \left(1-\omega-\psi\right)\left(\dfrac{s_u^2}{S_u^2}\right)^{\chi} \\ +\psi\left(\dfrac{S_w^2}{s_w^2}\right)^{\varphi} \end{bmatrix} - d \tag{12}$$

where $s_w^2 = m s_z^2 + p S_z^2$ and $S_w^2 = \left(m+p\right) S_z^2$

ω, ψ, τ, χ and φ are suitably chosen values.

can be simplified as:

$$t_{rpG} = \begin{Bmatrix} S_y^2\left(1+e_0\right) \\ +d \end{Bmatrix} \begin{bmatrix} \omega\left\{\dfrac{\left(b+c\right)S_x^2}{b s_x^2 + c S_x^2}\right\}^{\tau} + \\ \left(1-\omega-\psi\right) \\ \left\{\dfrac{b s_x^2 + c S_x^2}{\left(b+c\right)S_x^2}\right\}^{\chi} + \\ \psi\left\{\dfrac{\left(m+p\right)S_z^2}{m s_z^2 + p S_z^2}\right\}^{\varphi} (9) \end{bmatrix} - d$$

$$t_{rpG}=\begin{pmatrix}S_y^2+\\ d+\\ S_y^2e_0\end{pmatrix}\left[\begin{matrix}\omega\left\{\dfrac{b\left(1+e_1\right)S_x^2+cS_x^2}{\left(b+c\right)S_x^2}\right\}^{-\tau}+\\ \left(1-\omega-\psi\right)\\ \left\{\dfrac{b\left(1+e_1\right)S_x^2+cS_x^2}{\left(b+c\right)S_x^2}\right\}^{\chi}+\\ \psi\left\{\dfrac{m\left(1+e_2\right)S_z^2+pS_z^2}{\left(m+p\right)S_z^2}\right\}^{-\varphi}\end{matrix}\right]-d$$

$$t_{rpG}=\begin{pmatrix}S_y^2+\\ d\end{pmatrix}\begin{pmatrix}1+\\ \theta_0e_0\end{pmatrix}\left[\begin{matrix}\omega\left(1+\theta_1e_1\right)^{-\tau}+\\ \left(1-\omega-\psi\right)\\ \left(1+\theta_1e_1\right)^{\chi}+\\ \psi\left(1+\theta_2e_2\right)^{-\varphi}\end{matrix}\right]-d$$

expanding and simplifying the above expression using Taylor's series expansion assuming that $\left|\theta_1e_1\right|<1$ and $\left|\theta_2e_2\right|<1$ and ignoring throughout higher-order terms in e

$$t_{rpG}\cong\begin{pmatrix}S_y^2+\\ d\end{pmatrix}\begin{pmatrix}1+\\ \theta_0e_0\end{pmatrix}\left[\begin{matrix}1-\omega\left\{\begin{matrix}\tau\theta_1e_1-\\ \dfrac{\tau(\tau+1)}{2}\\ \left(\theta_1e_1\right)^2\end{matrix}\right\}\\ +\left(1-\omega-\psi\right)\\ \left\{\begin{matrix}\chi\theta_1e_1+\\ \dfrac{\chi(\chi-1)}{2}\left(\theta_1e_1\right)^2\end{matrix}\right\}-\\ \psi\left\{\begin{matrix}\varphi\theta_2e_2-\\ \dfrac{\varphi(\varphi+1)}{2}\\ \left(\theta_2e_2\right)^2\end{matrix}\right\}\end{matrix}\right]-d$$

$$t_{rpG}-S_y^2\cong\frac{S_y^2}{\theta_0}\left[\begin{matrix}\theta_0e_0-\omega\left\{\begin{matrix}\tau\theta_1e_1-\\ \dfrac{\tau(\tau+1)}{2}\left(\theta_1e_1\right)^2\end{matrix}\right\}\\ +\left(1-\omega-\psi\right)\\ \left\{\chi\theta_1e_1+\dfrac{\chi(\chi-1)}{2}\left(\theta_1e_1\right)^2\right\}\\ -\psi\left\{\begin{matrix}\varphi\theta_2e_2-\\ \dfrac{\varphi(\varphi+1)}{2}\left(\theta_2e_2\right)^2\end{matrix}\right\}\\ -\omega\tau\theta_0\theta_1e_0e_1+\\ \left(1-\omega-\psi\right)\chi\theta_0\theta_1e_0e_1\\ -\psi\varphi\theta_0\theta_2e_0e_2\end{matrix}\right] \tag{13}$$

$$Bias(t_{rpG})\cong\left[\begin{matrix}\frac{1}{2}\lambda'S_y^2\theta_0\\ \left\{\begin{matrix}\omega\tau\left(\tau+1\right)w_1^2\beta_2'\left(x\right)+\\ \left(1-\omega-\psi\right)\chi\left(\chi-1\right)w_1^2\beta_2'\left(x\right)\\ +\psi\varphi\left(\varphi+1\right)w_2^2\beta_2'\left(z\right)\\ -2\omega\tau w_1h'+\\ 2\left(1-\omega-\psi\right)\chi w_1h'\\ -2\psi\varphi w_2k'\end{matrix}\right\}\end{matrix}\right]$$

$$Bias(t_{rpG})\cong\frac{1}{2}\lambda'S_y^2\theta_0\left[\begin{matrix}\beta_2'\left(x\right)w_1^2\left\{\begin{matrix}\omega\tau^2+\\ \left(1-\omega-\psi\right)\chi^2\end{matrix}\right\}+\\ \beta_2'\left(x\right)\left\{\begin{matrix}\omega\tau-\\ \left(1-\omega-\psi\right)\chi\end{matrix}\right\}\\ w_1\left\{w_1-2\upsilon'\right\}\\ +\beta_2'\left(z\right)\psi\varphi^2w_2^2+\\ \beta_2'\left(z\right)\psi\varphi w_2\left(w_2-2\mu'\right)\end{matrix}\right] \tag{14}$$

Now for mean squared error, squaring Equation (13)

$$\left(t_{rpG}-S_y^2\right)^2\cong S_y^4\left[\begin{matrix}e_0^2+\left(\omega\tau w_1\right)^2e_1^2+\\ \left(1-\omega-\psi\right)^2\left(\chi w_1\right)^2e_1^2\\ +\left(\psi\varphi w_2\right)^2e_2^2\\ -2\omega\tau w_1e_0e_1\\ +2\left(1-\omega-\psi\right)\chi w_1e_0e_1\\ -2\psi\varphi w_2e_0e_2-\\ 2\omega\tau w_1\left(1-\omega-\psi\right)\chi w_1e_1^2\\ +2\omega\tau w_1\psi\varphi w_2e_1e_2-\\ 2\left(1-\omega-\psi\right)\\ \chi w_1\psi\varphi w_2e_1e_2\end{matrix}\right]$$

neglecting higher-order terms in e

$$\cong S_y^4\left[\begin{matrix}e_0^2+W_1^2e_1^2+W_{11}^2e_1^2+\\ M_2^2e_2^2-2W_1e_0e_1+\\ 2W_{11}e_0e_1-2M_2e_0e_2-\\ 2W_1W_{11}e_1^2+2W_1M_2e_1e_2\\ -2W_{11}M_2e_1e_2\end{matrix}\right]$$

where $W_1=\omega\tau w_1$,

$W_{11}=\left(1-\omega-\psi\right)\chi w_1$, $M_2=\psi\varphi w_2$,

$$MSE\left(t_{rpG}\right) \cong \lambda' S_y^4 \begin{bmatrix} \beta_2'(y)+ \\ \begin{pmatrix} W_1^2+W_{11}^2 \\ -2W_1W_{11} \end{pmatrix} \beta_2'(x) \\ +M_2^2\beta_2'(z)- \\ 2\left(W_1-W_{11}\right)h' \\ -2M_2k'+ \\ 2\left(W_1-W_{11}\right)M_2 l' \end{bmatrix}$$

$$MSE\left(t_{rpG}\right) \cong \lambda' S_y^4 \begin{bmatrix} \beta_2'(y)+ \\ \beta_2'(x)M_1\begin{pmatrix} M_1- \\ 2\upsilon' \end{pmatrix} \\ +\beta_2'(z)M_2\begin{pmatrix} M_2- \\ 2\mu' \end{pmatrix} \\ +2M_1M_2l' \end{bmatrix} \quad (15)$$

where $M_1 = W_1 - W_{11}$ minimized for

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} \left(\beta_2'(z)h'-k'l'\right)/D \\ \left(\beta_2'(x)k'-h'l'\right)/D \end{bmatrix}$$

where $D = \beta_2'(x)\beta_2'(z) - l'^2$ (16)

3. Special cases

Following are the previously existing variance estimators as special case of t_{rpG} :

Table 1. Existing Estimators as special case of Equation (12)

Estimator	ω	ψ	d	b	c	m	p	τ	χ	φ	Remarks
s_y^2	1	0	0	0	c	m	p	1	χ	φ	Equation (1)
t_{Isaki}	1	0	0	1	0	m	p	1	χ	φ	Equation (2)
t_{US}	1	0	0	1	$\frac{\beta_2(x)}{S_x^2}$	m	p	1	χ	φ	Equation (3)
t_{KC1}	1	0	0	1	$\frac{-\beta_2(x)}{S_x^2}$	m	p	1	χ	φ	Equation (4)
t_{KC2}	1	0	0	C_x	$\frac{-\beta_2(x)}{S_x^2}$	m	p	1	χ	φ	Equation (5)
t_{KC3}	1	0	0	1	$\frac{-C_x}{S_x^2}$	m	p	1	χ	φ	Equation (6)
t_{KC4}	1	0	0	$\beta_2(x)$	$\frac{-C_x}{S_x^2}$	m	p	1	χ	φ	Equation (7)
t_{YrpG}	ω	0	d	b	c	m	p	τ	χ	φ	Equation (8)

Some new proposed estimators from Equation (12) at $\tau = \tau$, $\chi=\chi$, and $\varphi = \varphi$ and for different values of d, b, c, m, p can be generated as:

Table 2. Estimators from t_{rpG}

S.No	Estimator	d	b	c	m	p	Remark
1.	$t_{rpG1}=\left\{s_y^2+\beta_2(y)\right\}\left[\begin{array}{l}\omega\left\{\frac{S_x^2-\beta_2(x)}{s_x^2-\beta_2(x)}\right\}^{\tau}\\+(1-\omega-\psi)\left\{\frac{s_x^2-\beta_2(x)}{S_x^2-\beta_2(x)}\right\}^{\chi}\\+\psi\left\{\frac{S_z^2+\beta_2(z)}{s_z^2+\beta_2(z)}\right\}^{\varphi}\end{array}\right]-\beta_2(y)$	$\beta_2(y)$	1	$\frac{-\beta_2(x)}{S_x^2}$	1	$\frac{\beta_2(z)}{S_z^2}$	(17)
2.	$t_{rpG2}=\left\{s_y^2-\beta_2(y)\right\}\left[\begin{array}{l}\omega\left\{\frac{S_x^2+\beta_2(x)}{s_x^2+\beta_2(x)}\right\}^{\tau}\\+(1-\omega-\psi)\left\{\frac{s_x^2+\beta_2(x)}{S_x^2+\beta_2(x)}\right\}^{\chi}\\+\psi\left\{\frac{S_z^2+\beta_2(z)}{s_z^2+\beta_2(z)}\right\}^{\varphi}\end{array}\right]+\beta_2(y)$	$-\beta_2(y)$	1	$\frac{\beta_2(x)}{S_x^2}$	1	$\frac{\beta_2(z)}{S_z^2}$	(18)
3.	$t_{rpG3}=\left\{s_y^2-\beta_2(y)\right\}\left[\begin{array}{l}\omega\left\{\frac{S_x^2+C_x}{s_x^2+C_x}\right\}^{\tau}\\+(1-\omega-\psi)\left\{\frac{s_x^2+C_x}{S_x^2+C_x}\right\}^{\chi}\\+\psi\left\{\frac{S_z^2+\beta_2(z)}{s_z^2+\beta_2(z)}\right\}^{\varphi}\end{array}\right]+\beta_2(y)$	$-\beta_2(y)$	1	$\frac{C_x}{S_x^2}$	1	$\frac{\beta_2(z)}{S_z^2}$	(19)
4.	$t_{rpG4}=\left\{s_y^2-\beta_2(y)\right\}\left[\begin{array}{l}\omega\left\{\frac{C_xS_x^2+\beta_2(x)}{C_xs_x^2+\beta_2(x)}\right\}^{\tau}\\+(1-\omega-\psi)\left\{\frac{C_xs_x^2+\beta_2(x)}{C_xS_x^2+\beta_2(x)}\right\}^{\chi}\\+\psi\left\{\frac{S_z^2+\beta_2(z)}{s_z^2+\beta_2(z)}\right\}^{\varphi}\end{array}\right]+\beta_2(y)$	$-\beta_2(y)$	C_x	$\frac{\beta_2(x)}{S_x^2}$	1	$\frac{\beta_2(z)}{S_z^2}$	(20)
5.	$t_{rpG5}=s_y^2\left\{\begin{array}{l}\omega\left\{\frac{C_xS_x^2-\beta_2(x)}{C_xs_x^2-\beta_2(x)}\right\}^{\tau}\\+(1-\omega-\psi)\left\{\frac{C_xs_x^2-\beta_2(x)}{C_xS_x^2-\beta_2(x)}\right\}^{\chi}\\+\psi\left\{\frac{C_zS_z^2+\beta_2(z)}{C_zs_z^2+\beta_2(z)}\right\}^{\varphi}\end{array}\right\}$	0	C_x	$-\frac{\beta_2(x)}{S_x^2}$	C_z	$\frac{\beta_2(z)}{S_z^2}$	(21)

4. Efficiency comparison

Comparison of generalized ratio-product-type estimator vs. Yadav *et al*. (2013) generalized ratio-product-type estimator

4.1. Comparison of estimators by Equation (12) and Equation (8) through MSE

$$MSE\left(t_{rpG}\right) < MSE\left(t_{YrpG}\right)$$

$$\lambda' S_y^4 \begin{bmatrix} \beta_2'(y) + \\ \beta_2'(x) M_1 \left(M_1 - 2\upsilon'\right) \\ +\beta_2'(z) M_2 \left(M_2 - 2\mu'\right) \\ +2M_1 M_2 l' \end{bmatrix} <$$

$$\lambda' S_y^4 \begin{bmatrix} \beta_2'(y) + \\ \beta_2'(x) A_{YrpG}\left(A_{YrpG} - 2\upsilon'\right) \end{bmatrix}$$

$$\begin{bmatrix} \beta_2'(x) M_1 \left(M_1 - 2\upsilon'\right) + \\ \beta_2'(z) M_2 \left(M_2 - 2\mu'\right) \\ +2M_1 M_2 l' \end{bmatrix} <$$

$$\begin{bmatrix} \beta_2'(x)\left\{M_1 - \psi\chi w_1\right\} \\ \left\{M_1 - \psi\chi w_1 - 2\upsilon'\right\} \end{bmatrix}$$

$$\text{If} \begin{bmatrix} \beta_2'(z) M_2 \begin{Bmatrix} M_2 - 2\mu' \\ +2M_1 l'/\beta_2'(z) \end{Bmatrix} \\ \begin{Bmatrix} -\beta_2'(x)\psi\chi w_1 \\ \left(-2M_1 + \psi\chi w_1 + 2\upsilon'\right) \end{Bmatrix} \end{bmatrix} < 0 \quad (22)$$

4.2. Comparison of estimators by Equation (12) and Equation (8) through bias

$$Bias\left(t_{rpG}\right) < Bias\left(t_{YrpG}\right)$$

$$\begin{bmatrix} \beta_2'(x) w_1^2 \begin{Bmatrix} \omega\tau^2 + \\ (1-\omega-\psi)\chi^2 \end{Bmatrix} + \\ \beta_2'(x)\left\{\omega\tau - (1-\omega-\psi)\chi\right\} \\ w_1\left\{w_1 - 2\upsilon'\right\} \\ +\beta_2'(z)\psi\varphi^2 w_2^2 + \\ \beta_2'(z)\psi\varphi w_2\left(w_2 - 2\mu'\right) \end{bmatrix} <$$

$$\beta_2'(x) w_1 \begin{bmatrix} w_1 \begin{Bmatrix} \omega\tau^2 + \\ (1-\omega)\chi^2 \end{Bmatrix} \\ + \begin{Bmatrix} \omega\tau - \\ (1-\omega)\chi \end{Bmatrix} \\ \left\{w_1 - 2\upsilon'\right\} \end{bmatrix}$$

$$\begin{Bmatrix} \omega\tau(\tau+1) w_1^2 \beta_2'(x) + \\ (1-\omega-\psi)\chi(\chi-1) w_1^2 \beta_2'(x) \\ +\psi\varphi(\varphi+1) w_2^2 \beta_2'(z) \\ -2\omega\tau w_1 h' + \\ 2(1-\omega-\psi)\chi w_1 h' \\ -2\psi\varphi w_2 k' \end{Bmatrix}$$

$$\begin{bmatrix} -\beta_2'(x)\psi\chi^2 w_1^2 \\ +\beta_2'(x)\psi\chi w_1^2 \\ -2\beta_2'(x)\psi\chi w_1\upsilon' \\ +\beta_2'(z)\psi\varphi^2 w_2^2 \\ +\beta_2'(z)\psi\varphi w_2^2 \\ -2\beta_2'(z)\psi\varphi w_2\mu' \end{bmatrix} < 0$$

$$\begin{bmatrix} -\beta_2'(x)\psi\chi(\chi-1) w_1^2 \\ -2\beta_2'(x)\psi\chi w_1\upsilon' \\ +\beta_2'(z)\psi\varphi(\varphi+1) w_2^2 \\ -2\beta_2'(z)\psi\varphi w_2\mu' \end{bmatrix} < 0$$

$$\begin{bmatrix} -\beta_2'(x)\psi\chi(\chi-1) w_1^2 \\ -2\psi\chi w_1 h' + \\ \beta_2'(z)\psi\varphi(\varphi+1) w_2^2 \\ -2\psi\varphi w_2 k' \end{bmatrix} < 0 \quad (23)$$

holds if Equation (8) is positively biased otherwise

$$\begin{bmatrix} -\beta_2'(x)\psi\chi(\chi-1) w_1^2 \\ -2\psi\chi w_1 h' + \\ \beta_2'(z)\psi\varphi(\varphi+1) w_2^2 \\ -2\psi\varphi w_2 k' \end{bmatrix} > 0 \quad (24)$$

(24) is the required condition

Therefore, Equation (12) is more efficient than Equation (8) if Equation (22) and Equation (23) / Equation (24) are satisfied accordingly.

5. Empirical study

To carry out empirical analysis, we use the following data.

Source: Basic Econometrics, Gujarati & Sangeetha (2007)

Data collected from 64 countries regarding fertility and other factors:

y :number of children died in a year under age 5 per one thousand live-births.

x : literacy rate of females in percentage.

z :total fertility rate during 1980-1985

Table 3. Data description

C_y	C_x	C_z	S_y^2	S_x^2	S_z^2	$\beta_2(y)$	$\beta_2(x)$
0.5369477	0.50809	0.271906	5772.6667	676.40873	2.27706	2.378336	1.657521

$\beta_2(z)$	h	k	l	ρ_{xy}	ρ_{zy}	ρ_{xz}
2.816908	1.437668	1.481965	1.0868699	-0.818285	0.671135	-0.625954

Table 4. Percentage relative efficiency (PRE) and bias of the estimators

Estimators	s_y^2	t_{Isaki}	t_{US}	t_{KC1}	t_{KC2}	t_{KC3}	t_{KC4}
PRE	100	118.76878	118.87848	118.65794	118.54952	118.73492	118.74837
Bias	0	0.439706	0.435424	0.444024	0.44824	0.441025	0.440501

Table 5. PRE and bias of t_{rpG} and the corresponding t_{YrpG} for different values of ω, ψ, τ, χ and φ

Estimators	Constants					PRE		Bias	
	ω	ψ	τ	χ	φ	t_{rpG}	t_{YrpG}	t_{rpG}	t_{YrpG}
Equation (17)	0.4	0.7	1.4	0.2	0.3	134.257512	123.05358	0.398888	0.43904
	0.5	0.6	1.4	0.2	0.3	133.381028	126.48823	0.496947	0.53137
	0.6	0.5	1.4	0.2	0.3	129.293611	126.08983	0.595006	0.62369
Equation (18)	0.4	0.7	1.4	0.2	0.3	134.210432	122.97152	0.391466	0.43185
	0.5	0.6	1.4	0.2	0.3	133.421843	126.45383	0.487710	0.52233
	0.6	0.5	1.4	0.2	0.3	129.449562	126.15214	0.583954	0.61280
Equation (19)	0.4	0.7	1.4	0.2	0.3	134.222990	122.99588	0.393629	0.43398
	0.5	0.6	1.4	0.2	0.3	133.408687	126.46419	0.490412	0.52499
	0.6	0.5	1.4	0.2	0.3	129.402671	126.13400	0.587195	0.61601
Equation (20)	0.4	0.7	1.4	0.2	0.3	134.192699	122.93755	0.388468	0.42891
	0.5	0.6	1.4	0.2	0.3	133.439720	126.43917	0.483964	0.51863
	0.6	0.5	1.4	0.2	0.3	129.514255	126.17695	0.579461	0.60835
Equation (21)	0.4	0.7	1.4	0.1	0.3	129.716915	124.79004	0.376540	0.41665
	0.5	0.6	1.4	0.1	0.65	132.729378	126.78881	0.467065	0.51376
	0.6	0.5	1.4	0.1	0.55	128.286721	125.35168	0.574156	0.61087

Table 6. Percentage relative efficiency of the estimators for minimized MSE

Estimators	s_y^2	t_{US} , t_{KCi}	t_{YrpG}	t_{rpG}
PRE	100	126.80077		139.59701

where Percentage Relative Efficiency is computed as $PRE\left(t_{..}\right)=\dfrac{Var(s_y^2)}{MSE(t_{..})}\times 100$

and for the sake of computational convenience in comparison, the absolute biases of all the estimators under consideration have been divided by $\frac{1}{2}\lambda' S_y^2 \theta_0$.

6. Conclusion

In this paper, generalized ratio-product-type variance estimator utilizing two auxiliary variables information has been proposed. The bias and mean of the squared sampling error have been derived. The efficiency of the estimator has also been compared with that of Yadav *et al*. (2013) generalized ratio-product-type estimator based on single auxiliary variable information mathematically. Moreover, all the mathematically expressed estimators mentioned in this paper are the off-shoots of t_{rpG}. Several new estimators can be generated from the generalized variance estimator of which some are depicted. It can also be observed that t_{rpG} at $\psi = 0$ produces similar estimator as t_{YrpG}. PRE of the new ratio-product-type variance estimators for different values of ω , ψ , τ , χ and φ shown in Table 5 are far higher than the PRE of the estimators of the corresponding Yadav *et al*. (2013) ratio-product-type estimators and the estimators in Table 4 as well, in addition to reduced bias. It can also be observed that the new generalized estimator not only leads in its optimum value from that defined in Equation (8) rather all the new estimators for chosen values of scalars have significantly higher PRE than the minimized PRE of t_{YrpG}.

References

Ahmed, M. S., Raman, M. S. & Hossain, M. I. (2000). Some competitive estimators of finite population variance using multivariate auxiliary information. Information and Management Sciences, **11**(1):49-54.

Arcos, A., Rueda, M., Martinez, M. D., Gonzalez, S. & Y. R. (2005). Incorporating the auxiliary information available in variance estimation. Applied Mathematics and Computation, **160**:387-399.

Gujarati, D. N. & Sangeetha. (2007). Basic econometrics (4th Ed.). New Delhi: TATA McGRAW HILL Companies. Pp. 189.

Gupta, S. & Shabbir, J. (2008). Variance estimation in simple random sampling using auxiliary information. Hacettepe Journal of Mathematics and Statistcs, **37**(1):57 -67.

Isaki, C. T. (1983). Variance estimation using auxiliary information. Journal of the American Statistical Association, **78**:117-123.

Jhajj, H. S., Sharma, M. K. & Grover, L. K. (2005). An efficient class of chain estimators of population variance under sub-sampling scheme. J. Soc.Japan Statist, **35**(2):273-286.

Kadilar, C. & Cingi, H. (2006). Ratio estimators for the population variance in simple and stratified random sampling. Applied Mathematics and Computation **173**:1047-1059.

Singh, H. P. & Solanki, R. S. (2013a). A new procedure for variance estimation in simple random sampling using auxiliary information. Stat Papers, **54**:479-497.

Singh, H. P. & Solanki, R. S. (2013b). Improved estimation of finite population variance using auxiliary information. Communicationsin Statistics - Theory and Methods, **42**:2718-2730.

Subramani, J. & Kumarapandiyan, G. (2013). Estimation of variance using known coefficient of variation and median of an auxiliary variable. Journal of Modern Applied Statistical Methods, **12** (1):58-64.

Subramani, J. & Kumarapandiyan, G. (2012a). Variance estimation using median of the auxiliary variable. International Journal of Probabiility and Statistics, **1**(3):62-66.

Subramani, J. & Kumarapandiyan, G. (2012b). Variance estimation using quartiles and their functions of an auxiliary variable. International Journal of Statistics and Applications, **2**(5):67-72.

Upadhyaya, L. N. & Singh, H. P. (1999) An estimator for population variance that utilizes the kurtosis of an auxiliary variable in survey sampling. Vikram Mathemat. Journal, **19**:14-17.

Yadav, R., Upadhyaya, L. N., Singh, H. P. & Chatterjee, S. (2013). A generalized family of transformed ratio-product estimators for variance in sample surveys. Communications in Statistics - Theory and Methods, **42**:1839-1850.

Submitted: 14/10/2015
Revised : 05/12/2016
Accepted : 14/12/2016

تقدير للتباين من نوع ناتج الضرب والقسمة المعمم باستخدام معلومات من متغيرات جانبية في العينة العشوائية البسيطة

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خلاصة

نقترح هذا البحث تقدير جديد للتباين من نوع ناتج الضرب والنسبة المعمم باستخدام معلومات من متغيرين جانبيين. قمنا بحساب درجة كفاءة التقدير الجديد بالمقارنة مع تقديرات أخرى من نوع ناتج الضرب والنسبة المعمم للعينة العشوائية البسيطة بدون إحلال. أثبت التقدير تجريبياً أنه أكثر كفاءة بالمقارنة مع التقدير غير المتحيز العادي، وكذلك مع بعض التقديرات المتحيزة المعروفة تحت الشروط المفروضة ولقيم مناسبة للثوابت يكون عندها مقدار التحيز صغير. ومن الجدير بالذكر أن جميع التقديرات المُستخدمة هي حالات خاصة من التقدير الجديد.