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# On left restriction semigroups with zero

Baddi-Ul-Zaman*
*School of Mathematical Sciences, South China Normal University,*
*Guangzhou 510631, P. R. China*
**Corresponding author: amath1141@gmail.com*

## Abstract

In this article, we give the notion of left restriction meet-semigroup, and establish some results regarding atomistic left restriction semigroups. Then we discuss decompositions of (non-zero) semigroups with zero by proving a decomposition theorem. We also show that every atomistic left restriction semigroup $S$ can be decomposed as an orthogonal sum of atomistic left restriction semigroups $N_i$, where each summand $N_i$ is an irreducible ideal of $S$. Finally, properties of the summands $N_i$, when $S$ embeds in some $\mathcal{PT}_X$ the partial transformation monoid on a set $X$, are investigated.

**Keywords:** Atomistic left restriction semigroup; irreducible ideal; left restriction meet-semigroup; left restriction semigroup; orthogonal sum.

## 1. Introduction

A semigroup $T$ is an inverse semigroup, if for all $v \in T$, there is a unique element $w$ in $T$ such that $vwv = v$ and $wvw = w$. Recently, in (FitzGerald, 2020), the author presented the theory of representations of inverse semigroups via homomorphisms into complete atomistic inverse meet-semigroups. The class of inverse meet-semigroups contains $\mathcal{I}_X$ the symmetric inverse monoid on $X$, $\mathcal{I}_X^*$ (the dual of $\mathcal{I}_X$) and partial automorphism monoids of structures, namely modules, vector spaces and graphs. Some remarkable theorems of decompositions of various representations were proved in (FitzGerald, 2020). Another motivation of the FitzGerald's work is that the representation of $T$ in $\mathcal{I}_X^*$ is more influential than that of in $\mathcal{I}_X$ (see, e.g., (FitzGerald, 2020)).

Left restriction semigroups are non-regular semigroups and are generalizations of inverse semigroups. They arise very naturally from partial transformation monoids in the same way that inverse semigroups arise from symmetric inverse monoids. Since the 1960s, left restriction semigroups occurred with various names and from diverse points of view in literature. For the first time in 1973, left restriction semigroups appeared in their own right in the paper (Trokhimenko, 1973). Also, they were studied in the setting of *SL2 $\gamma$-semigroups* in (Batbedat, 1981; Batbedat & Fountain, 1981). These semigroups were also studied as the idempotent connected Ehresmann semigroups in (Lawson, 1991). Later, left restriction semigroups arose in (Jackson & Stokes, 2001) as *(left) twisted C-semigroups*. In (Manes, 2006), they were studied as *guarded semigroups*, which appeared from the restriction categories in (Cockett & Lack, 2002). Recall that for any set $X$, the partial transformation monoid $\mathcal{PT}_X$ becomes left restriction semigroup under the unary operation $\alpha \mapsto I_{\text{dom}\,\alpha}$. We also recall that left restriction semigroups are precisely the $(2, 1)$-subalgebras of some $\mathcal{PT}_X$. Left restriction semigroups were termed as weakly left $E$-ample semigroups—the (former) York terminology. For weakly left $E$-ample semigroups, see, e.g., (Hollings, 2007). The reader is referred to (Gould, 2010) for the history of (left) restriction semigroups and their basic properties.

We shall make use of LR-semigroup, ALR-semigroup, LR-meet-semigroup and CALR-meet-semigroup as the abbreviations of left restriction semigroup, atomistic left restriction semigroup, left restriction meet-semigroup and complete atomistic left restriction meet-semigroup respectively unless stated otherwise.

The remaining article is adorned with four more sections. In Section 2, some helpful definitions, related facts are provided. In Section 3, the notion of LR-meet-semigroup is given, and some results associated with ALR-semigroups are proved. Note that LR-semigroups and LR-meet-semigroups generalize inverse semigroups and inverse meet-semigroups respectively. In Section 4, we establish a decomposition theorem for (non-zero) semigroups with zero, and then we prove that every ALR-semigroup $S$ can be decomposed as an orthogonal sum of ALR-semigroups $N_i$, where each summand $N_i$ is an irreducible ideal of $S$. In Section 5, we explore properties of the summands $N_i$, when $S$ is an LR-subsemigroup of some $\mathcal{PT}_X$.

## 2. Preliminaries

For rudimentary notions related to semigroup theory, and Green's relations $\mathcal{R}$, $\mathcal{L}$, we suggest (Howie, 1995). First, we recall generalized Green's relations.

In (Lawson, 1991), the author introduced the *generalized Green's relations*, i.e., $\widetilde{\mathcal{R}}_F$, $\widetilde{\mathcal{L}}_F$ on a semigroup $S$, where $F$ is a subset of $E(S)$ the set of idempotents of $S$. For any $v, w \in S$, $\widetilde{\mathcal{R}}_F$ can be defined as:

$$v \,\widetilde{\mathcal{R}}_F\, w \Longleftrightarrow \Big[(\,\forall\, f \in F\,)\; fv = v \Leftrightarrow fw = w\Big].$$

The relation $\widetilde{\mathcal{L}}_F$ is defined dually. The relation $\widetilde{\mathcal{R}}_F$ ($\widetilde{\mathcal{L}}_F$) is an equivalence relation. Green's relation $\mathcal{R}$ ($\mathcal{L}$) is left (right) compatible. On the contrary, $\widetilde{\mathcal{R}}_F$ ($\widetilde{\mathcal{L}}_F$) needs not be left (right) compatible. Note that $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_F$ ($\mathcal{L} \subseteq \widetilde{\mathcal{L}}_F$).

Let $v \in S$ and $f \in F$. Let $v \,\widetilde{\mathcal{R}}_F\, f$. Then as $f \in F$,

$$ff = f \Rightarrow fv = v. \tag{1}$$

Moreover, for any $v \in S$, $f \in F$,

$$v \,\widetilde{\mathcal{R}}_F\, f \Longleftrightarrow fv = v \text{ and } \forall\, h \in F\; [hv = v \Rightarrow hf = f]. \tag{2}$$

Therefore, $f$ is the minimum element of $\mathsf{LI}_v(F)$, where $\mathsf{LI}_v(F)$ is the set of all left identities of $v$ belonging to $F$.

Let $F$ be a semilattice (a semigroup of idempotents in which every two elements commute) such that $f, g \in F$. If $v \,\widetilde{\mathcal{R}}_F\, f$ and $v \,\widetilde{\mathcal{R}}_F\, g$, then $f \,\widetilde{\mathcal{R}}_F\, g$. Since $gg = g$, by Equation 1, we have $gf = f$. Since $g \,\widetilde{\mathcal{R}}_F\, f$ and $ff = f$, by Equation 1, we have $fg = g$. Since $gf = fg$, we deduce $f = g$. Therefore, $f$ is unique in the $\widetilde{\mathcal{R}}_F$-class of $v$ if $F$ is a semilattice. For $\widetilde{\mathcal{R}}_F$, $\widetilde{\mathcal{L}}_F$, see, e.g., (Zenab, 2018).

Second, our necessity is to remind the notion of LR-semigroup and related facts. For LR-semigroups, their right sided and two-sided versions, we prescribe (Gould, 2010; Zenab, 2018).

**Definition 2.1.** *(Zenab, 2018) An* LR*-semigroup is a unary semigroup* $(S, \cdot, ^\dagger)$ *such that the unary operation* $^\dagger$ *satisfies the following identities:*

$$v^\dagger v = v, \tag{3}$$

$$v^\dagger w^\dagger = w^\dagger v^\dagger, \tag{4}$$

$$(v^\dagger w)^\dagger = v^\dagger w^\dagger, \tag{5}$$

$$vw^\dagger = (vw)^\dagger v. \tag{6}$$

If we put $E_S = S^\dagger = \{w^\dagger \mid w \in S\}$, then one can check that $E_S$ is a semilattice. For every $w^\dagger \in E_S$, $(w^\dagger)^\dagger = w^\dagger$. Each element of $E_S$ is called a projection of $S$. The set $E_S$ is known as the *semilattice of projections* of $S$. A partial order $\leq$ on S is defined by the rule that for all $v, w \in S$, $v \leq w$ if and only if $v = v^\dagger w$. This relation is the natural partial order on S, and restricts to the usual partial order on $E_S$. Moreover, $\leq$ is compatible with multiplication. If $\mathscr{V}$ is the class of all LR-semigroups, then $\mathscr{V}$ is a variety of algebras of type $(2, 1)$. An inverse semigroup $Y$ is an LR-semigroup, if $^\dagger$ is defined by $y^\dagger = yy^{-1}$.

Now we define LR-semigroup with zero as follows.

**Definition 2.2.** *An* LR*-semigroup with zero is a unary semigroup $(S, \cdot, ^\dagger)$, where $(S, \cdot)$ is a semigroup with zero $0_S$, $^\dagger$ is a unary operation with $0_S^\dagger = 0_S$, and $^\dagger$ satisfies Equation 3–Equation 6.*

In the above definition, for all $w \in S$ such that $w \neq 0_S$, $w^\dagger \neq 0_S$. Also, for all $w \in S$, $0_S \leq w$.

An alternative characterization for LR-semigroups is given by Lemma 2.3.

**Lemma 2.3.** *(Zenab, 2018) Suppose that $(S, \cdot, ^\dagger)$ is a unary semigroup. Then $S$ is an* LR*-semigroup with semilattice of projections $E_S$ if and only if*

(i) *$E_S$ is a semilattice;*

(ii) *every $\widetilde{\mathcal{R}}_{E_S}$-class has an idempotent of $E_S$;*

(iii) *$\widetilde{\mathcal{R}}_{E_S}$ is a left congruence;*

(iv) *the left ample condition holds, i.e., for all $t \in S$, $e \in E_S$, $te = (te)^\dagger t$.*

Note that, by Lemma 2.3, the LR-semigroup $S$ with semilattice of projections $E_S$ is a weakly left $E_S$-ample semigroup, and vice versa. Also, in $S$, for any $t \in S$, the $\widetilde{\mathcal{R}}_{E_S}$-class of $t$ contains a unique idempotent of $E_S$, which we denote by $t^\dagger$. Then by Equation 2, $t^\dagger t = t$. Remember that $t^\dagger$ is the minimum element of $\mathsf{LI}_t(E_S)$ the set of all left identities of $t$ in $E_S$. It can be observed that in $S$,

$$s \,\widetilde{\mathcal{R}}_{E_S}\, t \iff s^\dagger = t^\dagger. \tag{7}$$

**Example 2.4.** (Hollings, 2007) Suppose that $T$ is a weakly left $E$-ample semigroup, namely LR-semigroup $T$ with semilattice of projections $E$, and suppose that $J$ is a non-empty set. Denote by $P$ the $J \times J$ identity matrix and consider the Rees matrix semigroup $\mathcal{M} := \mathcal{M}^0(T; J, J; P)$. Define a multiplication on $\mathcal{M}$ by

$$(j, t, k)0 = 0(j, t, k) = 00 = 0$$

and

$$(j, t, k)(l, u, m) = \begin{cases} (j, tu, m) & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

The set of idempotents of $\mathcal{M}$ is $E(\mathcal{M}) = \{(j, f, j) \mid f \in E(T)\} \cup \{0\}$. In (Hollings, 2007), Example 2.7.3 shows that $\mathcal{M}$ is a weakly left $\mathcal{E}$-ample semigroup such that $0^\dagger = 0$ and $(j, t, k)^\dagger = (j, t^\dagger, j)$, where $\mathcal{E} = \{(j, f, j) \in E(\mathcal{M}) \mid f \in E\} \cup \{0\}$.

**Definition 2.5.** *(FitzGerald, 2020; Petrich, 1984) Let $W$ be a semigroup containing zero. Let $\{W_\lambda\}_{\lambda \in I}$ be the class of subsemigroups such that $W = \bigcup_{\lambda \in I} W_\lambda$. If for all $\lambda, \mu \in I$ with $\lambda \neq \mu$, $W_\lambda \cap W_\mu = W_\lambda W_\mu = \{0\}$, then $W$ is an orthogonal sum of subsemigroups $W_\lambda$, denoted by $W = \sum_{\lambda \in I} W_\lambda$.*

In the above definition, each $W_\lambda$ is said to be a *summand* in the orthogonal sum $W$.

Next we remind the following definitions, utmost useful, and taken from (Erné & Joshi, 2015; Howie, 1995).

Let $(P, \leq)$ be a partial ordered set (poset). Then $P$ is called a *meet-semilattice* if for any $m, n \in P$, $m \wedge n$ (meet of $m$ and $n$) exists in $P$. Let $\overline{P} = P \cup \{0\}$ be a poset, where $0$ is the least element of $\overline{P}$.

If $0 \neq w \in \overline{P}$, then $w$ is called an *atom* if $w$ is a minimal element of $\overline{P} \setminus \{0\}$. The set $\overline{P}$ is an *atomistic poset* if for all $0 \neq w \in \overline{P}$, $w$ is a join of a set of atoms (i.e., of the set of all atoms it dominates).

In the rest of the paper, every LR-semigroup $S$ is an LR-semigroup with zero $0_S$ unless explicitly stated. Denote by $E_S$ the semilattice of projections of an LR-semigroup $S$. Moreover, $r \wedge s$ ($r \vee s$) means the meet (join) of a set $\{r, s\}$, while $\bigwedge A$ ($\bigvee A$) means the meet (join) of a non-empty set $A$.

## 3. Left restriction meet-semigroups

We furnish the notion of LR-meet-semigroup, and prove some results associated with ALR-semigroups.

In the beginning, let us define the following.

**Definition 3.1.** *An* LR*-meet-semigroup* $(M, \cdot, ^\dagger, \wedge)$ *is an* LR*-semigroup* $(M, \cdot, ^\dagger)$ *such that* $M$ *is a meet-semilattice with respect to (w.r.t.) the natural partial order* $\leq$ *on* $(M, \cdot, ^\dagger)$*.*

In the above definition, $(M, \wedge)$ is a semilattice, and for any $u_1, u_2 \in M$,

$$u_1 \leq u_2 \Longleftrightarrow u_1 \wedge u_2 = u_1.$$

Hence, $\leq$ is also a natural ordering on $(M, \wedge)$.

**Definition 3.2.** *A complete left restriction meet-semigroup* $(M, \cdot, ^\dagger, \wedge)$ *is an* LR*-semigroup* $(M, \cdot, ^\dagger)$ *such that for any* $\emptyset \neq B \subseteq M$, $\bigwedge B$ *exists w.r.t.* $\leq$ *on* $(M, \cdot, ^\dagger)$*.*

**Definition 3.3.** *An* LR*-semigroup* $(M, \cdot, ^\dagger)$ *is an* ALR*-semigroup if* $M$ *is an atomistic poset w.r.t. its natural partial order.*

**Definition 3.4.** *Let* $(M, \cdot, ^\dagger)$ *be an* ALR*-semigroup. If for any* $\emptyset \neq B \subseteq M$, $\bigwedge B$ *exists w.r.t.* $\leq$ *on* $(M, \cdot, ^\dagger)$*, then* $M$ *is called a* CALR*-meet-semigroup.*

**Proposition 3.5.** *Let* $M$ *be an* ALR*-semigroup with zero* $0_M$*. Let* $P_{t^\dagger} = t^\dagger M t^\dagger$*, where* $t^\dagger \in E_M \setminus \{0_M\}$*. Then*

(i) $P_{t^\dagger}$ *is an* LR*-subsemigroup of* $M$ *with zero, and containing an identity* $t^\dagger$*;*

(ii) *every non-zero element of* $P_{t^\dagger}$ *dominates an atom of* $P_{t^\dagger}$*;*

(iii) *for all non-zero* $x, y \in P_{t^\dagger}$ *such that* $x \nleq y$*, a non-zero element* $k$ *exists in* $P_{t^\dagger}$ *such that* $k \leq x$ *and* $k \wedge y = 0_{P_{t^\dagger}}$*;*

(iv) $P_{t^\dagger}$ *is an* ALR*-subsemigroup of* $M$ *with zero, and containing an identity* $t^\dagger$*.*

*Proof.* (i) It is simple to verify that $P_{t^\dagger}$ is a subsemigroup of $M$ with zero $0_{P_{t^\dagger}} = 0_M$. We put $0 = 0_{P_{t^\dagger}} = 0_M$. It can be seen that $t^\dagger$ is an identity element of $P_{t^\dagger}$. Now we show that $P_{t^\dagger}$ is closed under $^\dagger$. If $d \in P_{t^\dagger}$ is such that $d \neq 0$, then $t^\dagger d = d$. We can write $(t^\dagger d)^\dagger = d^\dagger$. Since $M$ is an LR-semigroup, by Equation 5, we deduce $t^\dagger d^\dagger = d^\dagger$. Then we have $d^\dagger = t^\dagger t^\dagger d^\dagger$. Since projections of $M$ commute, we deduce $d^\dagger = t^\dagger d^\dagger t^\dagger$. Therefore, $d^\dagger \in P_{t^\dagger}$. Also, $0^\dagger = 0$. So $P_{t^\dagger}$ is closed under $^\dagger$. Hence, $P_{t^\dagger}$ is an LR-subsemigroup of $M$ with zero, and containing an identity $t^\dagger$.

(ii) Let $x \in P_{t^\dagger}$ be such that $x \neq 0$. Since $x \in M$ and $M$ is atomistic, there exists an atom $a$ of $M$ such that $a \leq x$. Since $\leq$ is compatible with multiplication, we obtain $t^\dagger a t^\dagger \leq t^\dagger x t^\dagger$. Since $x \in P_{t^\dagger}$, we have $t^\dagger a t^\dagger \leq x$. Now we prove that $t^\dagger a t^\dagger$ is an atom of $P_{t^\dagger}$. As $a \leq x$, we have $t^\dagger a t^\dagger = t^\dagger a^\dagger x t^\dagger$. Then $t^\dagger a t^\dagger = a^\dagger t^\dagger x t^\dagger = a^\dagger x = a$. Since $a > 0$, $t^\dagger a t^\dagger > 0$. Suppose that for all $r \in P_{t^\dagger}$, $0 \leq r < t^\dagger a t^\dagger$. Since $a = t^\dagger a t^\dagger$, we have $0 \leq r < a$. Since $a$ is an atom of $M$ and $r \in M$, we obtain $r = 0$. Consequently, $t^\dagger a t^\dagger$ is an atom of $P_{t^\dagger}$. Thus, every non-zero element of $P_{t^\dagger}$ dominates an atom of $P_{t^\dagger}$.

(iii) For any non-zero $v \in M$, let $M_v = \{m \mid m \text{ is an atom of } M, m \leq v\}$. Let $x, y \in P_{t^\dagger}$ be such that $x, y \neq 0$ and $x \nleq y$. Since $x, y \in M$, there exists an atom (a non-zero element) $c \in M$ such that $c \in M_x$ and $c \notin M_y$. Therefore, we have $c \leq x$ and $c \wedge y = 0$. Since $c \leq x$, by compatibility, we have $t^\dagger c t^\dagger \leq t^\dagger x t^\dagger$. As $x \in P_{t^\dagger}$, we obtain $t^\dagger c t^\dagger \leq x$. Now we prove that $t^\dagger c t^\dagger \neq 0$. Suppose that

$t^\dagger ct^\dagger = 0$. As $c \leq x$, we obtain $t^\dagger c^\dagger xt^\dagger = 0$. Then we have $c^\dagger t^\dagger xt^\dagger = 0$. Then $c^\dagger x = 0$. Therefore, $c = 0$—a contradiction. Hence, $t^\dagger ct^\dagger \neq 0$. Next we prove that $t^\dagger ct^\dagger \wedge y = 0$. Certainly, one lower bound of $\{t^\dagger ct^\dagger, y\}$ is 0. If $\ell$ is any lower bound of $\{t^\dagger ct^\dagger, y\}$, then $\ell \leq t^\dagger ct^\dagger$ and $\ell \leq y$. By Equation 5, $(t^\dagger ct^\dagger)^\dagger c = t^\dagger (ct^\dagger)^\dagger c$. By Equation 6, we have $(t^\dagger ct^\dagger)^\dagger c = t^\dagger c(t^\dagger)^\dagger$. Then $(t^\dagger ct^\dagger)^\dagger c = t^\dagger ct^\dagger$. So $t^\dagger ct^\dagger \leq c$. Since $\ell \leq t^\dagger ct^\dagger$, we have $\ell \leq c$. Since $\ell$ is the lower bound of $\{c, y\}$ and $c \wedge y = 0$, we deduce $\ell = 0$. Thus, $t^\dagger ct^\dagger \wedge y = 0$. Hence, for all non-zero $x, y \in P_{t^\dagger}$ such that $x \nleq y$, a non-zero element $k$ exists in $P_{t^\dagger}$ such that $k \leq x$ and $k \wedge y = 0$.

(iv) By (i), $P_{t^\dagger}$ is an LR-subsemigroup of $M$ with zero, and containing an identity $t^\dagger$. Now we prove that $P_{t^\dagger}$ is atomistic. For this purpose, we show that every non-zero element of $P_{t^\dagger}$ is a join of a set of atoms of $P_{t^\dagger}$. For any non-zero $x \in P_{t^\dagger}$, let $\mathscr{P}_x = \{p \mid p \text{ is an atom of } P_{t^\dagger}, p \leq x\}$. We require to show that for any non-zero $x \in P_{t^\dagger}$, $x \leq y$, where $y \in P_{t^\dagger}$ such that $y$ is any upper bound of $\mathscr{P}_x$. On the contrary, suppose that $x \nleq y$. By (iii), there exists a non-zero $c \in P_{t^\dagger}$ such that $c \leq x$ and $c \wedge y = 0$. By (ii), there exists an atom $\overline{p}$ of $P_{t^\dagger}$ such that $\overline{p} \leq c$. Then we have $\overline{p} \leq x$. Therefore, $\overline{p} \in \mathscr{P}_x$. Since $c \wedge y = 0$, we deduce $\overline{p} \wedge y = 0$. Since $\overline{p} \in \mathscr{P}_x$ and $y$ is any upper bound of $\mathscr{P}_x$, we deduce $\overline{p} \leq y$. Since $\overline{p} \wedge y = 0$, we have $\overline{p} \nleq y$—a contradiction. Hence, $x \leq y$. Therefore, $x = \bigvee \mathscr{P}_x$. Therefore, $P_{t^\dagger}$ is atomistic. Thus, $P_{t^\dagger}$ is an ALR-subsemigroup of $M$ with zero, and containing an identity $t^\dagger$. □

**Proposition 3.6.** *Let $M$ be a* CALR*-meet-semigroup with zero $0_M$. Let $P_{t^\dagger} = t^\dagger M t^\dagger$, where $t^\dagger \in E_M \setminus \{0_M\}$. Then $P_{t^\dagger}$ is a* CALR*-meet-subsemigroup of $M$ with zero, and containing an identity $t^\dagger$.*

*Proof.* By Proposition 3.5 (iv), $P_{t^\dagger}$ is an ALR-subsemigroup of $M$ with zero $0_{P_{t^\dagger}} = 0_M$ and an identity $t^\dagger$. We put $0 = 0_{P_{t^\dagger}} = 0_M$. Let $\emptyset \neq B \subseteq P_{t^\dagger}$. If $0 \in B$, then $\bigwedge B = 0$. Suppose that $0 \notin B$. Since $P_{t^\dagger} \subseteq M$ and $M$ is a CALR-meet-semigroup with zero, it follows that $\bigwedge B$ exists in $M$. Let $g = \bigwedge B$, where $g \in M$. Then for all $b \in B$, $g \leq b$. Since $\leq$ is compatible with multiplication, we obtain $t^\dagger gt^\dagger \leq t^\dagger bt^\dagger$. Since $b \in P_{t^\dagger}$, we have $t^\dagger gt^\dagger \leq b$. Accordingly, $t^\dagger gt^\dagger$ is a lower bound of $B$, belonging to $P_{t^\dagger}$. Let $\ell$ be any lower bound of $B$ such that $\ell \in P_{t^\dagger}$. Since $\ell \in M$ and $g$ is a meet of $B$ in $M$, we deduce $\ell \leq g$. By compatibility, we have $t^\dagger \ell t^\dagger \leq t^\dagger gt^\dagger$. Since $\ell \in P_{t^\dagger}$, we have $\ell \leq t^\dagger gt^\dagger$. Consequently, $t^\dagger gt^\dagger = \bigwedge B$. Hence, $P_{t^\dagger}$ is a CALR-meet-subsemigroup of $M$ with zero, and containing an identity $t^\dagger$. □

From now on, for ease of notation, for any semigroup $A$ with zero, we will drop the subscript from zero element $0_A$ and write simply 0.

## 4. Decompositions of semigroups with zero

In this section, we prove a theorem of decomposition for (non-zero) semigroups with zero.

Let us define the following.

**Definition 4.1.** *Let $S$ be a semigroup with zero. Let $N$ be a non-zero ideal of $S$. Then $N$ is called reducible if there exist non-zero ideals $N_1, N_2$ of $S$ such that $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \{0\}$, in this case, we denote it by $N = N_1 \overset{0}{\coprod} N_2$; otherwise $N$ is called irreducible.*

**Lemma 4.2.** *Let $S$ be a semigroup with zero. Let $\{N_i\}_{i \in I}$ be a family of irreducible ideals of $S$. Suppose that $\bigcap\limits_{i \in I} N_i \neq \{0\}$. Then $\bigcup\limits_{i \in I} N_i$ is an irreducible ideal of $S$.*

*Proof.* Clearly, $\bigcup\limits_{i \in I} N_i$ is an ideal of $S$. On the contrary, suppose that $\bigcup\limits_{i \in I} N_i = C \overset{0}{\coprod} D$ such that $C$ and $D$ are non-zero ideals of $S$. By Definition 4.1, we have $\bigcup\limits_{i \in I} N_i = C \cup D$ and $C \cap D = \{0\}$. Take $N_0 \in \{N_i \mid i \in I\}$. This implies that $N_0 = N_0 \cap \left[\bigcup\limits_{i \in I} N_i\right]$. Since $\bigcup\limits_{i \in I} N_i = C \cup D$, we have $N_0 = N_0 \cap (C \cup D)$. Then we have $N_0 = (N_0 \cap C) \cup (N_0 \cap D)$. Since $N_0$ is irreducible, it follows that either $N_0 \cap C = \{0\}$ or $N_0 \cap D = \{0\}$. Assume that $N_0 \cap D = \{0\}$. Then $N_0 = N_0 \cap C$. Then $N_0 \subseteq C$. Now assume that there exist $i, j$ such that $i \neq j$ with $N_i \subseteq C$ and $N_j \subseteq D$. Then we have

$\{0\} \neq \bigcap_{i\in I} N_i \subseteq N_i \cap N_j \subseteq C \cap D = \{0\}$—a contradiction. Then either $\bigcup_{i\in I} N_i \subseteq C$ or $\bigcup_{i\in I} N_i \subseteq D$. So either $\bigcup_{i\in I} N_i = C$ or $\bigcup_{i\in I} N_i = D$. If $\bigcup_{i\in I} N_i = C$, then $D = 0$, which is a contradiction, or if $\bigcup_{i\in I} N_i = D$, then $C = 0$—a contradiction. Thus, $\bigcup_{i\in I} N_i$ is an irreducible ideal of $S$. □

**Theorem 4.3.** *Let $S$ be a semigroup with zero. Then $S$ has a unique decomposition $S = \sum_{i\in I} N_i$, where each $N_i$ is an irreducible ideal of $S$.*

*Proof.* We divide our proof into the following steps.

**Step (1).** We know that for all $0 \neq x \in S$, $\langle x\rangle := \{x\} \cup xS \cup Sx \cup SxS$ is the ideal of $S$ generated by $x$. First, we need to show that $\langle x\rangle$ is irreducible. On the contrary, suppose that $\langle x\rangle = A \coprod^{0} B$, where $A$ and $B$ are non-zero ideals of $S$. Then $x \in A \cup B$ and either $x \in A$ or $x \in B$. Without loss of generality, assume that $x \in A$. As $A$ is an ideal of $S$, it follows that $\{x\}, xS, Sx, SxS \subseteq A$. Therefore, $\langle x\rangle \subseteq A$. Since $A \cap B = \{0\}$, we obtain $B = \{0\}$—a contradiction. Hence, $\langle x\rangle$ is irreducible.

**Step (2).** For all $0 \neq x \in S$, define

$$\Omega_x = \{V \mid x \in V \text{ and } V \text{ is an irreducible ideal of } S\}.$$

By the proof of Step (1), $\langle x\rangle \in \Omega_x$. Therefore, $\Omega_x \neq \emptyset$. Let $T_x = \bigcup_{V\in\Omega_x} V$. Since $\bigcap_{V\in\Omega_x} V \neq \{0\}$, by Lemma 4.2, $T_x$ is an irreducible ideal of $S$.

**Step (3).** Now we show that for all $x, y \in S$, either $T_x \cap T_y = \{0\}$ or $T_x = T_y$. If $T_x \cap T_y = \{0\}$, then we are done. If $T_x \cap T_y \neq \{0\}$, then by Lemma 4.2, $T_x \cup T_y$ is an irreducible ideal of $S$. Since $x \in T_x \cup T_y$, it follows that $T_x \cup T_y \in \Omega_x$. Since $T_x = \bigcup_{V\in\Omega_x} V$, we have $T_x \cup T_y \subseteq T_x$. As $T_x \subseteq T_x \cup T_y$, we obtain $T_x = T_x \cup T_y$. Similarly, $T_y = T_x \cup T_y$. Hence $T_x = T_y$.

**Step (4).** By the proof of Step (3), there exists an index set $I$ such that $S = \bigcup_{i\in I} T_{x_i}$ and for any $i, j \in I$ with $i \neq j$, $T_{x_i} \cap T_{x_j} = \{0\}$. In particular, for $i \neq j$, we have $T_{x_i} T_{x_j} \subseteq T_{x_i} \cap T_{x_j} = \{0\}$. Thus,

$$S = \sum_{i\in I} T_{x_i}.$$

**Step (5).** Suppose that $S$ has another decomposition $S = \sum_{j\in J} M_j$. For all $i \in I$, $T_{x_i} = T_{x_i} \cap S = T_{x_i} \cap \left[\bigcup_{j\in J} M_j\right] = \bigcup_{j\in J} (T_{x_i} \cap M_j)$. Since $T_{x_i}$ is irreducible, it follows that there exists exactly one $k \in J$ such that

$$T_{x_i} \cap M_k \neq \{0\}. \tag{8}$$

Then we have $T_{x_i} = T_{x_i} \cap M_k$. Then $T_{x_i} \subseteq M_k$. Now $M_k = M_k \cap S = \bigcup_{i\in I} (M_k \cap T_{x_i})$. Since $M_k$ is irreducible, it follows that there exists exactly one $l \in I$ such that $M_k \cap T_{x_l} \neq \{0\}$. By Equation 8, we deduce $l = i$. Thus, $M_k = M_k \cap T_{x_i}$. Then we have $M_k \subseteq T_{x_i}$. Hence $T_{x_i} = M_k$. The proof is completed. □

Now we explore some properties of the orthogonal sum $S = \sum_{i\in I} N_i$ as in the above theorem when $S$ is an LR-semigroup.

**Proposition 4.4.** *Suppose that $S$ is an* LR*-semigroup with zero, where $S = \sum_{i\in I} N_i$, the orthogonal sum as in Theorem 4.3. Then the following hold:*

(i) *every $N_i$ is an* LR*-subsemigroup of $S$;*

(ii) *for all $i \in I$, $0 \neq x \in N_i$ and $0 \neq y \in S$, if $y \leq x$, then $y \in N_i$;*

(iii) *for all $i \in I$ and $0 \neq c \in N_i$, $c$ is an atom of $N_i$ if and only if $c$ is an atom of $S$;*

(iv) *for all* $i \in I$ *and* $0 \neq x \in N_i$*, define* $A_x = \{c \mid c \text{ is an atom of } S, c \leq x\}$ *and* $B_x = \{c \mid c \text{ is an atom of } N_i, c \leq x\}$*. Then* $A_x = B_x$*.*

*Proof.* (i) It is clear that every $N_i$ is a subsemigroup of $S$. Now we prove that every $N_i$ is an LR-subsemigroup of $S$. We need to prove that for any $i \in I$, and for any $0 \neq x \in N_i$, $x^\dagger \in N_i$. On the contrary, suppose that for $i \neq k$, $x^\dagger \in N_k$. Since $N_k N_i = \{0\}$, we deduce $x = x^\dagger x = 0$—a contradiction. Therefore, $x^\dagger \in N_i$. Also, $0^\dagger = 0$. Hence, every $N_i$ is an LR-subsemigroup of $S$.

(ii) On the contrary, assume that for $i \neq k$, $y \in N_k$. As $y \leq x$, we have $y = y^\dagger x$. By (i), $y^\dagger \in N_k$. Since $N_k N_i = \{0\}$, we deduce $y = y^\dagger x = 0$—a contradiction. Hence, $y \in N_i$.

(iii) Let $c$ be any non-zero element of $N_i$. Suppose that $c$ is an atom of $S$. Then it is clear that $c$ is an atom of $N_i$. Conversely, suppose that $c$ is an atom of $N_i$. For every non-zero $s \in S$ such that $0 < s \leq c$, by (ii), $s \in N_i$. As $c$ is an atom of $N_i$, it follows that $s = c$. Thus, $c$ is an atom of $S$.

(iv) Let $a \in A_x$. Then $a$ is an atom of $S$ with $a \leq x$. Since $x \in N_i$, by (ii), it follows that $a \in N_i$. So $a$ is also an atom of $N_i$. Therefore, $a \in B_x$. So $A_x \subseteq B_x$. If $b \in B_x$, then $b$ is an atom of $N_i$ with $b \leq x$. By (iii), $b$ is also an atom of $S$. Therefore, $b \in A_x$. So $A_x = B_x$. □

As a corollary of Theorem 4.3 and Proposition 4.4, we obtain the following theorem.

**Theorem 4.5.** *Let* $S$ *be a semigroup with zero. Let* $S = \sum_{i \in I} N_i$ *be as in Theorem 4.3. Then*

(a) $S$ *is an* LR*-semigroup if and only if every* $N_i$ $(i \in I)$ *is an* LR*-semigroup;*

(b) $S$ *is an* ALR*-semigroup if and only if every* $N_i$ $(i \in I)$ *is an* ALR*-semigroup.*
*In particular, every* ALR*-semigroup* $S$ *is an orthogonal sum of* ALR*-subsemigroups such that each summand is an irreducible ideal of* $S$*.*

*Proof.* (a) If $S$ is an LR-semigroup, then by Proposition 4.4 (i), each $N_i$ is an LR-semigroup. Conversely, if each $N_i$ is an LR-semigroup, then we need to show that Equation 3−Equation 6 hold in $S$. If all the letters involved lie in the same $N_i$ for some $i \in I$, then Equation 3−Equation 6 hold. On the other hand, in Equation 4−Equation 6, if $v$ and $w$ lie in $N_i$ and $N_j$ $(i \neq j)$ respectively, then all the involved products are zero. Therefore, $S$ is an LR-semigroup.
(b) Let $S$ be an ALR-semigroup. By (a), each $N_i$ is an LR-semigroup. Now we show that $N_i$ is atomistic. For all $i \in I$ and $0 \neq x \in N_i$, define $A_x = \{c \mid c \text{ is an atom of } S, c \leq x\}$ and $B_x = \{c \mid c \text{ is an atom of } N_i, c \leq x\}$. By Proposition 4.4 (iv), $A_x = B_x$. Since $S$ is atomistic, it follows that $x = \bigvee A_x = \bigvee B_x$. Hence, each $N_i$ is an ALR-semigroup. Conversely, suppose that each $N_i$ is an ALR-semigroup. Then by (a), $S$ is an LR-semigroup. For every $0 \neq x \in S$, we have $x \in N_i$ for some $i \in I$. Let $A_x$, $B_x$ be as above. Then we have $x = \bigvee B_x = \bigvee A_x$. Hence, $S$ is atomistic. The proof is completed. □

## 5. Properties of the $N_i$ when $S$ embeds in some $\mathcal{PT}_X$

It is known that any LR-semigroup $S$ embeds in some $\mathcal{PT}_X$, which is an ALR-semigroup, and that in any such embedding, for $\sigma \in S$, $\sigma^\dagger$ is the identity map on the domain $\mathrm{d}(\sigma)$ of $\sigma$.

Therefore it is of interest to examine the properties of the $N_i$ when $S = \sum_{i \in I} N_i$ is an LR-subsemigroup of $\mathcal{PT}_X$. Without loss of generality, we need consider only the case where the zero of $S$ is the zero of $\mathcal{PT}_X$, namely the empty partial mapping $\emptyset$. This is because of the Proposition 5.2.

**Lemma 5.1.** *If* $S$ *is an* LR*-subsemigroup of* $\mathcal{PT}_X$ *with zero element* $\zeta$*, and suppose that* $\alpha \in S$*, and if* $(x, y) \in \alpha$ *and* $x \in \mathrm{d}(\zeta)$*, then* $x = y$*.*

*Proof.* Since $\zeta = \zeta^\dagger$ is the identity map on its domain, it follows that $(x, y) \in \zeta \circ \alpha = \zeta$ whence $x = y$. □

**Proposition 5.2.** *If $S$ is an* LR*-subsemigroup of $\mathcal{PT}_X$ with zero element $\zeta$, then the map*

$$\alpha \mapsto \alpha \setminus \zeta$$

*is an injective morphism of $S$ into $\mathcal{PT}_Y$ such that $\zeta \mapsto \emptyset$, where $Y = X \setminus \mathrm{d}(\zeta)$.*

*Proof.* Since $\zeta \leq \alpha$, i.e., $\zeta \subseteq \alpha$, the map is injective, and clearly $\zeta \mapsto \emptyset$. Then $(\alpha\setminus\zeta)\circ(\beta\setminus\zeta) = \alpha\circ\beta\setminus\zeta$, as can be shown in the usual manner, together with the aid of the Lemma 5.1. □

If we put $D_i = \bigcup\{\mathrm{d}(\alpha) \colon \alpha \in N_i\}$, $R_i = \bigcup\{\mathrm{r}(\alpha) \colon \alpha \in N_i\}$ and $X_i = D_i \cup R_i$, then we see that $N_i$ is an LR-subsemigroup of $\mathcal{PT}_{X_i}$; and $N_i$ is irreducible since Theorem 4.3 still applies. For $i \neq j$, the sets $X_i$ and $X_j$ need not be disjoint, but must be distinct.

Next, if $R_i \cap D_j \neq \emptyset$, then there are $\alpha \in N_i$, $\beta \in N_j$ such that $\alpha\beta \neq \emptyset$, thus, $i = j$. The converse is true since non-trivial $N_i$ always contains a non-zero $\alpha^\dagger$ and $\mathrm{d}(\alpha^\dagger) = \mathrm{r}(\alpha^\dagger)$ whence $R_i \cap D_i \neq \emptyset$.

In fact, if $\mathrm{r}(\alpha) = \mathrm{d}(\beta)$, then $\alpha$ and $\beta$ are in the same component, $N_i$ say. So if we say that $\alpha$,$\beta$ are $\Phi$-related if $\mathrm{r}(\alpha) = \mathrm{d}(\beta)$, and let $\Psi$ be the smallest equivalence relation containing $\Phi$, then $\Psi$ must partition $S$ into its irreducible components $N_i$.

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