Some results on Steiner decomposition number of graphs

E.Ebin Raja Merly[1], M.Mahiba[2,*]
[1,2] *Dept. of Mathematics, Nesamony Memorial Christian College,*
Manonmanium Sundaranar University, Tamilnadu, India.

**Corresponding author: mahibakala@gmail.com*

Abstract

Let G be a connected graph with Steiner number $s(G)$. A decomposition $\pi = \{G_1, G_2, \dots, G_n\}$ is said to be a Steiner decomposition if $s(G_i) = s(G)$ for all $i\ (1 \leq i \leq n)$. The maximum cardinality obtained for the Steiner decomposition π of G is called the Steiner decomposition number of G and is denoted by $\pi_{st}(G)$. In this paper we present a relation between Steiner decomposition number and independence number of G. Steiner decomposition number for some power of paths are discussed. It is also shown that given any pair m, n of positive integers with $m \geq 2$ there exists a connected graph G such that $s(G) = m$ and $\pi_{st}(G) = n$.

Keywords: Independence number; power of path; realization theorem; steiner decomposition number; steiner number.

1. Introduction

All graphs considered in this paper are connected, simple and undirected. For basic graph theoretic terminologies we refer to (Harary, 1988). The concept of Steiner number of a graph is introduced by Chartrand and Zhang (Chartrand & Zhang, 2002). Let G be a connected graph. For a set $W \subseteq V(G)$, a tree T contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all vertices in G that lie on some Steiner tree with respect to W. The set W is a Steiner set for G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the Steiner number $s(G)$. Steiner concept is considered to be the extension of geodesic concept and hence it provides a new way to study the structure of graphs based on distance. Further investigation on this concept is seen in the works (Pelayo, 2004; Hernando *et al.,* 2005; Yero & Rodriguez-Velazquez, 2015).

Decomposition of graphs is considered as one of the most prominent areas of research because of its significant contribution towards Structural graph theory and Combinatorics. A decomposition of graph G is the collection of connected edge disjoint subgraphs $G_1, G_2, \dots, G_n$ such that $E(G_1) \cup E(G_2) \cup \dots \cup E(G_n) = E(G)$. In literature, different types of decomposition of graph have been studied by imposing conditions on the subgraphs G_i such as decompositions given in (Merly & Jothi, 2018; Romero-Valencia *et al.,* 2019). A parameter called decomposition number is also studied along with the decomposition techniques. Some of these parameters are found in (Nagarajan *et al.,* 2009; Abraham & Hamid, 2010; Arumugam *et al.,* 2013; John & Stalin, 2021).

Motivated by the results and applications of the decomposition parameters stated in those papers, we introduced a new decomposition technique called Steiner decomposition of graphs (Merly & Mahiba, 2021a) and initiated the study of the parameter Steiner decomposition number of graphs. In (Merly & Mahiba, 2021b), Steiner decomposition number of Complete $n-$ Sun graph is presented. A Steiner decomposition is a decomposition $\pi = \{G_1, G_2, \dots, G_n\}$ such that $s(G_i) = s(G), (1 \leq i \leq n)$. The maximum cardinality of a Steiner decomposition π is called the Steiner decomposition number of G and is denoted as $\pi_{st}(G)$. A graph G is said to be Steiner decomposable graph if $\pi_{st}(G) \geq 2$. A graph G is said to be non Steiner decomposable graph if $\pi_{st}(G) = 1$.

For a connected graph G, a set $S \subseteq V(G)$ is said to be an independent set of G if no two vertices of S are adjacent in G. An independent set S is said to be maximum if G has no independent set S' with $|S'| > |S|$. The cardinality of the maximum independent set is called the independence number of G and is denoted by $\alpha(G)$. In a connected graph G, a vertex of degree one is said to be pendant vertex and a vertex whose removal makes the graph disconnected is said to be cutvertex. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The union of G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. Star graph $K_{1,n}$ is a tree of order $n+1$ with one vertex having degree n and all other vertices having degree one. Bistar denoted by $B_{m,n}(m, n \geq 2)$ is a graph obtained by joining the central vertices of star graphs $K_{1,m}$ and $K_{1,n}$ with an edge. A spider tree is a tree with atmost one vertex of degree ≥ 3 and the vertex of degree ≥ 3 is called as branch vertex. A leg of spider tree is a path from the branch vertex to a pendant vertex of the tree. $S_n(m)$ denote a spider tree of n legs with one leg having length $m \geq 2$ and other $(n-1)$ legs having length one. $U_3(k)$ denote a unicyclic graph created from the cycle C_3 by attaching k pendant vertices to a vertex of C_3. $U_3(k_1, k_2)$ denote a unicyclic graph created from the cycle C_3 by attaching k_1 pendant vertices to a vertex of C_3 and attaching k_2 pendant vertices to another vertex of C_3.

2. Main Results

In this section we derive a relation between $\pi_{st}(G)$ and $\alpha(G)$.

Theorem 2.1. *(Merly & Mahiba, 2021a) For any graph* G *with* q *edges,* $s(G) = 2$ *if and only if* $\pi_{st}(G) = q$.

Theorem 2.2. *(Merly & Mahiba, 2021a) For any Steiner decomposable graph* G *with* $s(G) \geq 3$, $\pi_{st}(G) \leq \left\lfloor \frac{q}{s(G)} \right\rfloor$.

Theorem 2.3. *(Merly & Mahiba, 2021a) Let* G *be a connected graph of size* q.

a) *For any Steiner decomposable graph* G *with* $s(G) > 3$, $\pi_{st}(G) = \frac{q}{s(G)}$ *if and only if* $G_i = K_{1,s(G)}\ \forall i$.
b) *For any Steiner decomposable graph* G *with* $s(G) = 3$, $\pi_{st}(G) = \frac{q}{3}$ *if and only if* $G_i = K_{1,3}$ *or* $K_3\ \forall i$.

Theorem 2.4. *Let G be a connected graph such that $|V(G)| = p, |E(G)| = q$ and $s(G) \geq 4$. If $\pi_{st}(G) = \frac{q}{s(G)}$ then $\alpha(G) \geq |V(G) - S|$ where S is the collection of cutvertices of all the subgraphs in the Steiner decomposition of maximum cardinality.*

Proof. Let G be a connected graph on p vertices, q edges and Steiner number $s(G) \geq 4$. Assume $\pi_{st}(G) = \frac{q}{s(G)}$. This implies that $\pi = \{G_i = K_{1,s(G)} \,/\, 1 \leq i \leq \frac{q}{s(G)}\}$ is the Steiner decomposition of maximum cardinality for G. Let S be the collection of all cutvertices of G_i, $1 \leq i \leq \frac{q}{s(G)}$. Any pair of vertices in $V(G) - S$ is non adjacent in G, if not it contradicts π is a decomposition for G. Therefore $V(G) - S$ is an independent set and hence $\alpha(G) \geq |V(G) - S|$. □

Corollary 2.5. *Let G be a connected graph with $p > \frac{q}{s(G)}$. Then $\pi_{st}(G) \neq \frac{q}{s(G)}$ if $\alpha(G) < p - \frac{q}{s(G)}$.*

Proof. Assume $\alpha(G) < p - \frac{q}{s(G)}$. To prove $\pi_{st}(G) \neq \frac{q}{s(G)}$. Suppose $\pi_{st}(G) = \frac{q}{s(G)}$ then $\pi = \{G_1, G_2, \ldots, G_{\frac{q}{s(G)}}\}$ is a Steiner decomposition for G. By the above theorem, $\alpha(G) \geq |V(G) - S|$ where S is the collection of all cutvertices in the decomposition π. Since $p > \frac{q}{s(G)}$, $|S| \leq \frac{q}{s(G)}$.

$$
\begin{aligned}
\alpha(G) &\geq |V(G) - S| \\
&= |V(G)| - |S| \;\; (since\ V(G) \supseteq S) \\
&= p - |S| \\
&\geq p - \frac{q}{s(G)}
\end{aligned}
$$

which is a contradiction to our assumption. Therefore $\pi_{st}(G) \neq \frac{q}{s(G)}$.

3. Steiner decomposition of power of path

Definition 3.1. *(Lin et al., 2011) The k^{th} power of the graph G denoted by G^k has the same vertex set as G and two distinct vertices u and v of G are adjacent in G^k if and only if their distance in G is atmost k.*

Definition 3.2. *Let G be a simple graph. For $S \subset V(G)$, graph $G - S$ is obtained by removing each vertex of S and all its associated incident edges from G. For $T \subset E(G)$, $G - T$ denote the graph obtained from G by deleting each edge of T.*

Let P_{n+1} denote the path of order $n + 1$. P_{n+1}^k denote the k^{th} power of path P_{n+1}. The number of edges of the graph P_{n+1}^k is $k\left((n+1) - \left(\frac{k+1}{2}\right)\right)$. If $k \geq n$ then P_{n+1}^k is the complete graph on $n + 1$ vertices. We proved that complete graph is non Steiner decomposable graph *(Merly & Mahiba, 2021a)*. Hence in this section we consider only the graphs P_{n+1}^k where $2 \leq k < n$ for our discussion.

Theorem 3.3. *(AbuGhneim et al., 2014) If* $n = qk + r$ *where* q *is a positive integer and* $0 < r \leq k$*, then* $s(P_{n+1}^k) = r + 1$.

Result 3.4. *If* P_{n+1}^k *is the graph with* $n = qk + 1$ *then* $\pi_{st}(P_{n+1}^k) = \frac{k}{2}(2n - k + 1)$.

Since $n = qk + 1$, $s(P_{n+1}^k) = 2$. By theorem 2.1, the result is attained.

Result 3.5. *For* P_{mk}^k *where* $m > 1$, $s(P_{mk}^k) = k$.

Since $mk - 1 = (m - 1)k + (k - 1)$ by theorem 3.3, $s(P_{mk}^k) = k$.

Lemma 3.6. $\alpha(P_{n+1}^k) = \left\lceil \frac{n+1}{k+1} \right\rceil$.

Proof. Let $V(P_{n+1}^k) = \{v_1, v_2, \dots, v_{n+1}\}$. Let $V_j = \{v_{(j-1)k+j}, v_{(j-1)k+j+1}, \dots, v_{jk+j}\}, 1 \leq j \leq \left\lceil \frac{n+1}{k+1} \right\rceil - 1$ and $V_{\lceil \frac{n+1}{k+1} \rceil} = \left\{ v_{(\lceil \frac{n+1}{k+1} \rceil - 1)k + \lceil \frac{n+1}{k+1} \rceil}, v_{(\lceil \frac{n+1}{k+1} \rceil - 1)k + \lceil \frac{n+1}{k+1} \rceil + 1}, \dots, v_{n+1} \right\}$. We have, $|V_j| = k + 1, 1 \leq j \leq \left\lceil \frac{n+1}{k+1} \right\rceil - 1$ and $\left| V_{\lceil \frac{n+1}{k+1} \rceil} \right| \leq k + 1$. Generate the set S by choosing the first vertex from the vertex subsets V_j, $1 \leq j \leq \left\lceil \frac{n+1}{k+1} \right\rceil$. The set thus formed will be $S = \left\{ v_1, v_{k+2}, v_{2k+3}, \dots, v_{(\lceil \frac{n+1}{k+1} \rceil - 1)k + \lceil \frac{n+1}{k+1} \rceil} \right\}$. For any two distinct vertices of S, their distance in P_{n+1} is atleast $k + 1$ and so they are non adjacent in P_{n+1}^k. Therefore S is an independent set. Suppose there exists an independent set S' with $|S'| > |S|$ then atleast two vertices of S' belong to the same vertex subset V_m (say). In P_{n+1}^k, any pair of vertices of V_j, $1 \leq j \leq \left\lceil \frac{n+1}{k+1} \right\rceil$ is adjacent. This contradicts that S' is an independent set. Hence S is a maximum independent set. Thus $\alpha(P_{n+1}^k) = \left\lceil \frac{n+1}{k+1} \right\rceil$.

Throughout the section we consider the vertex set of $G = P_{n+1}^k$ as $V(G) = \{v_1, v_2, \dots, v_{n+1}\}$. Define the set A_i for $1 \leq i \leq n$ as $A_i = \{v_{i+j} / 1 \leq j \leq k, i + j \leq n + 1\}$. Construct the decomposition $\psi = \{H_1, H_2, \dots, H_n\}$ such that H_i, $1 \leq i \leq n$ is a star graph with cut vertex as v_i and the vertices of A_i as pendant vertices. Construction of the subgraphs $H_i \in \psi, 1 \leq i \leq n$ is shown in figure 1. By making necessary alterations on H_i's belonging to ψ, we obtain the desired Steiner decomposition.

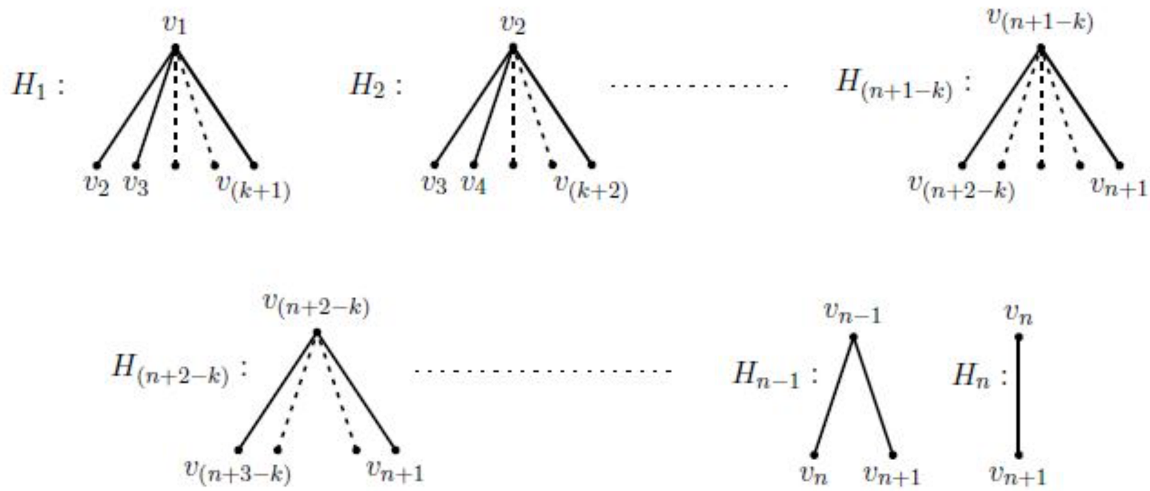

Fig. 1. Decomposition ψ of P_{n+1}^k

Theorem 3.7. *For the graph $G = P_{mk}^k$ with $k = 2n, n > 1, \pi_{st}(G) = mk - n - 1$.*

Proof. Let $G = P_{mk}^k$ and $k = 2n, n > 1$. The decomposition ψ can be reframed and written as $\psi = \{H_1, H_2, \dots, H_{mk-3n-1}\} \cup \{H_{mk-3n}, H_{mk-3n+1}, \dots, H_{mk-2n-1}\} \cup \{H_{mk-2n}, H_{mk-2n+1}, \dots, H_{mk-n-1}\} \cup \{H_{mk-n}, H_{mk-n+1}, \dots, H_{mk-1}\}$.

Let us define $G_s^* = H_{mk-3n+s} \cup H_{mk-n+s}, 0 \le s \le n-2$. Let $V'(G_s^*) \subset V(G_s^*), 0 \le s \le n-2$ such that $V'(G_s^*) = \{v_{mk-2n+(s+1)}, v_{mk-2n+(s+2)}, \dots, v_{mk-2n+(n-1)}\}$.

To obtain the Steiner decomposition of the graph, define

$$G_l = H_l, 1 \le l \le mk - 3n - 1$$

$$G_{mk-3n+s} = G_s^* - V'(G_s^*), 0 \le s \le n-2$$

$$G_{mk-2n-1} = H_{mk-2n-1} \cup H_{mk-1}$$
$$G_{mk-2n} = H_{mk-2n}$$

Construct $G_{mk-2n+r}, 1 \le r \le n-1$ from the graph $H_{mk-2n+r}$ by attaching the edges removed from $G_s^*, 0 \le s \le n-2$ in the process of constructing $G_{mk-3n+s}, 0 \le s \le n-2$ with one of the end vertex as $v_{mk-2n+r}$.

Consider the decomposition of P_{mk}^k, k even as $\pi = \{G_l / 1 \le l \le mk - 3n - 1\} \cup \{G_{mk-3n+s} / 0 \le s \le n-2\} \cup \{G_{mk-2n-1}, G_{mk-2n}\} \cup \{G_{mk-2n+r} / 1 \le r \le n-1\}$.

$$G_l, G_{mk-2n} \cong K_{1,k}, 1 \le l \le mk - 3n - 1$$

$$G_{mk-2n-1} \cong S_k(2)$$

$$G_{mk-3n+s} \cong B_{n+s,n-s}, 0 \le s \le n-2$$

$$G_{mk-2n+r} \cong K_{1,k}, 1 \le r \le n-1$$

By the result 3.5, Steiner number of G is k. Since $s(K_{1,k}) = s(S_k(2)) = k$ and $s(B_{n+s,n-s}) = 2n = k$, decomposition $\pi = \{G_1, G_2, \dots, G_{mk-n-1}\}$ is a Steiner decomposition for G. Now to prove $\pi_{st}(G) = mk - n - 1$. By theorem 2.2, $\pi_{st}(G) \le \left\lfloor \frac{q}{s(G)} \right\rfloor$. On calculating the value of $\left\lfloor \frac{q}{s(G)} \right\rfloor$,

$$\begin{aligned}\left\lfloor \frac{q}{s(G)} \right\rfloor &= \left\lfloor mk - \left(\frac{k+1}{2}\right) \right\rfloor \\ &= \left\lfloor \frac{2mk-(k+1)}{2} \right\rfloor \\ &= \frac{2mk-(k+1)-1}{2} \quad (since\ 2mk-(k+1)\ is\ odd) \\ &= \frac{4mn-(2n+1)-1}{2} \\ &= 2mn - n - 1 \\ &= mk - n - 1 \\ &= cardinality\ of\ \pi\end{aligned}$$

Therefore π is a Steiner decomposition of maximum cardinality for G and so $\pi_{st}(G) = mk - n - 1$.

Theorem 3.8. *Let* $G = P_{mk}^{k}$. *If* k *is odd and* $1 < m < \frac{k+1}{2}$ *then* $\pi_{st}(G) = mk - n - 1$.

Proof. Let $G = P_{mk}^{k}$, where $k = 2n-1, n \ge 3$ and $1 < m < \frac{k+1}{2}$. The decomposition ψ can be reframed and written as $\psi = \{H_1, H_2, \dots, H_{mk-3n}\} \cup \{H_{mk-3n+1}, H_{mk-3n+2}, \dots, H_{mk-2n-2}\} \cup \{H_{mk-2n-1}, H_{mk-2n}, H_{mk-2n+1}\} \cup \{H_{mk-2n+2}, H_{mk-2n+3}, \dots, H_{mk-n-1}\} \cup \{H_{mk-n}, H_{mk-n+1}, \dots, H_{mk-3}, H_{mk-2}, H_{mk-1}\}$.

Let us define $G_s^* = H_{mk-3n+(s+1)} \cup H_{mk-n+s}$, $0 \le s \le n-3$. Let $V'(G_s^*) \subset V(G_s^*)$, $0 \le s \le n-3$ such that $V'(G_s^*) = \{v_{mk-3n+(s+2)}, v_{mk-3n+(s+3)}, \dots, v_{mk-2n-1}, v_{mk-n-(s+1)}\}$.

To obtain the Steiner decomposition of the graph, define

$$G_l = H_l, 1 \le l \le mk - 3n$$

$$G_{mk-3n+(s+1)} = G_s^* - V'(G_s^*), 0 \le s \le n-3$$

$$\begin{aligned}G_{mk-2n-1} &= H_{mk-2n-1} \\ G_{mk-2n} &= H_{mk-2n} \cup H_{mk-2}\end{aligned}$$

$$G_{mk-2n+1} = H_{mk-2n+1} \cup H_{mk-1}$$

Let $E'(G_s^*)$, $0 \le s \le n-3$ be the set of edges removed from $E(G_s^*)$ while constructing $G_{mk-3n+(s+1)}$.

Construct $G_{mk-n-(s+1)}$, $0 \leq s \leq n-3$ from the graph $H_{mk-n-(s+1)}$ by attaching the edges in the set $E'(G_s^*)$.

Consider the decomposition of G as $\pi = \{G_l / 1 \leq l \leq mk-3n\} \cup \{G_{mk-3n+(s+1)} / 0 \leq s \leq n-3\} \cup \{G_{mk-2n-1}, G_{mk-2n}, G_{mk-2n+1}\} \cup \{G_{mk-n-(s+1)} / 0 \leq s \leq n-3\}$.

$$G_l, G_{mk-2n-1} \cong K_{1,k}, 1 \leq l \leq mk-3n$$

$$G_{mk-2n} \cong U_3(1, k-2)$$

$$G_{mk-2n+1} \cong U_3(k-2)$$

$$G_{mk-3n+(s+1)} \cong B_{n-s,n-1+s}, 0 \leq s \leq n-3$$

$$G_{mk-n-(s+1)} \cong B_{n-(s+2),n+(s+1)}, 0 \leq s \leq n-4$$

$$G_{mk-2n+2} \cong S_k(2)$$

Since $s(K_{1,k}) = s(U_3(1,k-2)) = s(U_3(k-2)) = s(B_{n-s,n-1+s}) = s(B_{n-(s+2),n+(s+1)}) = s(S_k(2)) = k = s(G)$, π is a Steiner decomposition for G. The cardinality of π is $mk-n-1$. Now, we have to prove $\pi_{st}(G) = mk-n-1$. From lemma 3.6, $S = \left\{v_1, v_{k+2}, v_{2k+3}, \dots, v_{\left(\left\lceil\frac{mk}{k+1}\right\rceil-1\right)k+\left\lceil\frac{mk}{k+1}\right\rceil}\right\}$ is a maximum independent set for G. We have, $(m-1)k + m = mk-(k-m)$. Since $m < \frac{k+1}{2}, k-m > k-\left(\frac{k+1}{2}\right)$. For $k > 1, k-\left(\frac{k+1}{2}\right) > 0$ and so $k-m > 0$. This implies $(m-1)k+m < mk$. We know that distance between any pair of vertices belonging to S is atleast $k+1$ in the graph G and clearly $mk+(m+1) > mk$. Hence we can conclude $\left\lceil\frac{mk}{k+1}\right\rceil = m$ and so $\alpha(G) = m$.

$$\frac{q}{s(G)} = mk - \left(\frac{k+1}{2}\right) \quad \left(since\ k\ is\ odd, mk-\left(\frac{k+1}{2}\right)\ is\ an\ integer\right)$$

$$p - \frac{q}{s(G)} = \frac{k+1}{2}$$

$$> m$$

$$= \alpha(G)$$

Therefore, $\alpha(G) < p - \frac{q}{s(G)}$

Also we have, $p > \frac{q}{s(G)}$. Hence by corollary 2.5, $\pi_{st}(G) \neq mk - \left(\frac{k+1}{2}\right)$. That is $\pi_{st}(G) \neq mk - n$. Hence π is a Steiner decomposition for G with maximum cardinality. Therefore $\pi_{st}(G) = mk-n-1$.

Theorem 3.9. *For* $G = P^4_{5^2+20m}$ *with* $m \geq 0, \pi_{st}(G) = 17+16m$.

Proof. Let $G = P^4_{5^2+20m}$, $m \geq 0$ be the graph with order p and size q.

$$p - 1 = 5^2 + 20m - 1$$
$$= 24 + 20m$$
$$= 4\big(5(1+m)\big) + 4$$

By theorem 3.3, $s(G) = 5$. The decomposition ψ can be reframed and written as $\psi = \{H_1, H_2, \dots, H_{20(1+m)}\} \cup \{H_{21+20m}, H_{22+20m}, H_{23+20m}, H_{24+20m}\}$

Define

$$G_{jk} = H_{k+5j} \cup < \{v_{k+5j} v_{1+5j}\} >;\ 0 \leq \mathrm{j} \leq 3 + 4m, k = 2,3,4,5$$
$$G^* = H_{21+20m} \cup\ H_{22+20m} \cup\ H_{23+20m} \cup\ H_{24+20m}$$

Clearly $\pi = \{G_{jk} /\ 0 \leq j \leq 3 + 4m, k = 2,3,4,5\} \cup \{\ G^*\}$ is a decomposition for G.

$$G_{jk} \cong K_{1,5}\ ;\ 0 \leq \mathrm{j} \leq 3 + 4m, k = 2,3,4,5$$
$$G^* \cong K_5$$

Since $s\big(K_{1,5}\big) = s(G^*) = 5, \pi$ is a Steiner decomposition for G. The cardinality of π is $17 + 16m$.

Now,

$$\frac{q}{s(G)} = \frac{4\big((5^2+20m) - \frac{5}{2}\big)}{5}$$
$$= 18 + 16m$$
$$\frac{q}{s(G)} = 18 + 16m < 5^2 + 20m = p$$

Therefore, $p > \frac{q}{s(G)}$.

$$\alpha(G) = \left\lceil \frac{5^2 + 20m}{5} \right\rceil$$
$$= 5 + 4m \qquad (1)$$
$$p - \frac{q}{s(G)} = 5^2 + 20m - (18 + 16m)$$
$$= 7 + 4m \qquad (2)$$

From Equations (1) & (2),

$$\alpha(G) < p - \frac{q}{s(G)}$$

Hence by corollary 2.5, $\pi_{st}(G) \neq 18 + 16m$. Therefore π is a Steiner decomposition with maximum cardinality and so $\pi_{st}(G) = 17 + 16m$.

4. Realization Theorem

Definition 4.1. *The contraction of pair of vertices v_i and v_j of a graph produces a graph in which the two vertices v_i and v_j are replaced by the new vertex v such that v is adjacent to the union of vertices to which v_i, v_j were originally adjacent.*

Definition 4.2. *(Ghosh et al., 2021) Globe graph (Gl_n) is obtained from two isolated vertices that are joined by n paths of length two.*

Theorem 4.3. *For any positive integer m, n $(m \geq 2)$ there exists a connected graph G such that $s(G) = m$ and $\pi_{st}(G) = n$.*

Proof. **Case 1:** $m \leq n$

Subcase 1: $m = 2$

Path graph on $n + 1$ vertices, P_{n+1} satisfies the required properties.

Subcase 2: $m > 2$

For $2 < m \leq n$, the Complete bipartite graph $G = K_{m,n}$ has the properties $s(G) = m$ and $\pi_{st}(G) = n$.

Case 2: $m > n$

Subcase 1: $n = 1$

Star graph $K_{1,m}$ is a non Steiner decomposable graph with $s(K_{1,m}) = m$. Therefore it satisfies the required properties.

Subcase 2: m, n both odd and $n \geq 3$

Construct the graph with the desired properties as follows:

- Take $\frac{n-1}{2}$ copies of the globe graph $Gl_{\frac{m+1}{2}}$. Label the two vertices of degree $\frac{m+1}{2}$ in each copy of $Gl_{\frac{m+1}{2}}$ as u_i and v_i, $1 \leq i \leq \frac{n-1}{2}$ respectively.
- Take $\frac{n-1}{2}$ copies of the globe graph $Gl_{\frac{m-1}{2}}$. Label the two vertices of degree $\frac{m-1}{2}$ in each copy of $Gl_{\frac{m-1}{2}}$ as x_i and y_i, $1 \leq i \leq \frac{n-1}{2}$ respectively.
- Consider the set $S = \{(v_i, x_i)/\ 1 \leq i \leq \frac{n-1}{2}\} \cup \{(y_i, u_{i+1})/\ 1 \leq i \leq \frac{n-3}{2}\}$. By vertex contraction process, contract the pair of vertices given in each ordered pair of S.

- Take a copy of the star graph $K_{1,\frac{m-1}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex u_1.
- Take a copy of the star graph $K_{1,\frac{m+1}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex $y_{\frac{n-1}{2}}$.

In figure 2, the resultant graph G and its Steiner decomposition indicated by horizantal lines is given.

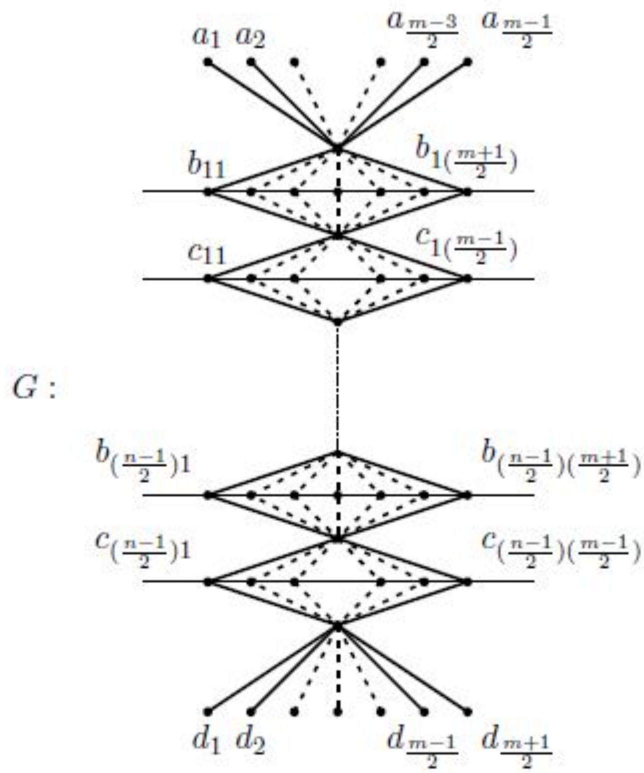

Fig. 2. Graph G with m, n both odd, $n \geq 3$ and $m > n$

Total number of edges of G is mn. Minimum Steiner set of $G = \left\{a_i / 1 \leq i \leq \frac{m-1}{2}\right\} \cup \{d_i / 1 \leq i \leq \frac{m+1}{2}\}$ and so $s(G) = m$. Since each subgraph in the decomposition is the star graph $K_{1,m}$by theorem 2.3, $\pi_{st}(G) = n$.

Subcase 3: m even

Construct the graph with the desired properties as follows:

- Take $(n-1)$ copies of the globe graph $Gl_{\frac{m}{2}}$. Label the two vertices of degree $\frac{m}{2}$ in each copy of $Gl_{\frac{m}{2}}$ as u_i and $v_i, 1 \leq i \leq n-1$ respectively.
- Consider the set $S = \{(v_i, u_{i+1})/\ 1 \leq i \leq n-2\}$. By vertex contraction process, contract the pair of vertices given in each ordered pair of S.
- Take a copy of the star graph $K_{1,\frac{m}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex u_1.
- Take another copy of the star graph $K_{1,\frac{m}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex v_{n-1}.

In figure 3, the resultant graph G and its Steiner decomposition indicated by horizantal lines is given.

Total number of edges of G is mn. Minimum Steiner set of $G = \{a_i/\ 1 \leq i \leq \frac{m}{2}\} \cup \{c_i/\ 1 \leq i \leq \frac{m}{2}\}$ and so $s(G) = m$. Since each subgraph in the decomposition is the star graph $K_{1,m}$ by theorem 2.3, $\pi_{st}(G) = n$.

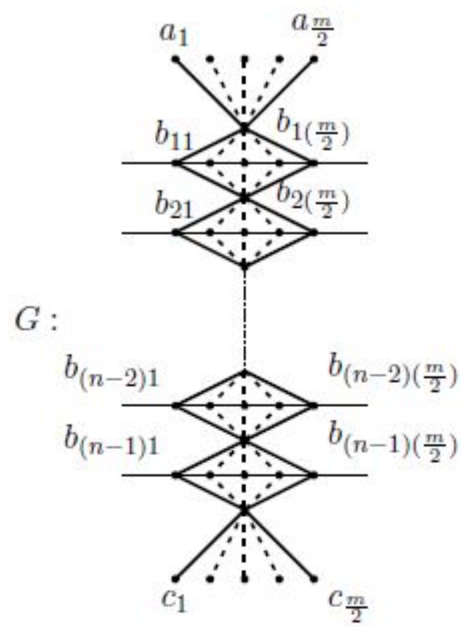

Fig. 3. Graph G with m even and $m > n$

Subcase 4: m odd and n even $(n > 2)$

Construct the graph with the desired properties as follows:

- Take a copy of the globe graph $Gl_{\frac{m+1}{2}}$. Label the two vertices of degree $\frac{m+1}{2}$ as u_1 and v_1 respectively.

- Take a copy of the star graph $K_{1,\frac{m-1}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex u_1. Label the new vertex as u_1^*.
- Take another copy of the star graph $K_{1,\frac{m-1}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex v_1. Label the new vertex as v_1^*.
- Take ($\frac{n}{2}-1$) copies of the globe graph Gl_m. Label the two vertices of degree m in each copy of Gl_m as x_i and y_i, $1 \leq \text{i} \leq \frac{n}{2}-1$ respectively.
- Consider the set $S = \{(y_i, x_{i+1})/\ 1 \leq i \leq \frac{n}{2}-2\} \cup \{(v_1^*, x_1)\}$. By vertex contraction process, contract the pair of vertices given in each ordered pair of S.

In figure 4, the resultant graph G and its Steiner decomposition indicated by horizantal lines is given.

Total number of edges of G is mn. Minimum Steiner set of $G = \{a_i/\ 1 \leq i \leq \frac{m-1}{2}\} \cup \{c_i/\ 1 \leq i \leq \frac{m-1}{2}\} \cup \{y_{\frac{n}{2}-1}\}$ and so $s(G) = m$. Since each subgraph in the decomposition is the star graph $K_{1,m}$ by theorem 2.3, $\pi_{st}(G) = n$.

Subcase 5: m odd and $n = 2$

Construct the graph with the desired properties as follows:

- Take a copy of the globe graph $Gl_{\frac{m+1}{2}}$. Label the two vertices of degree $\frac{m+1}{2}$ as u_1 and v_1 respectively.

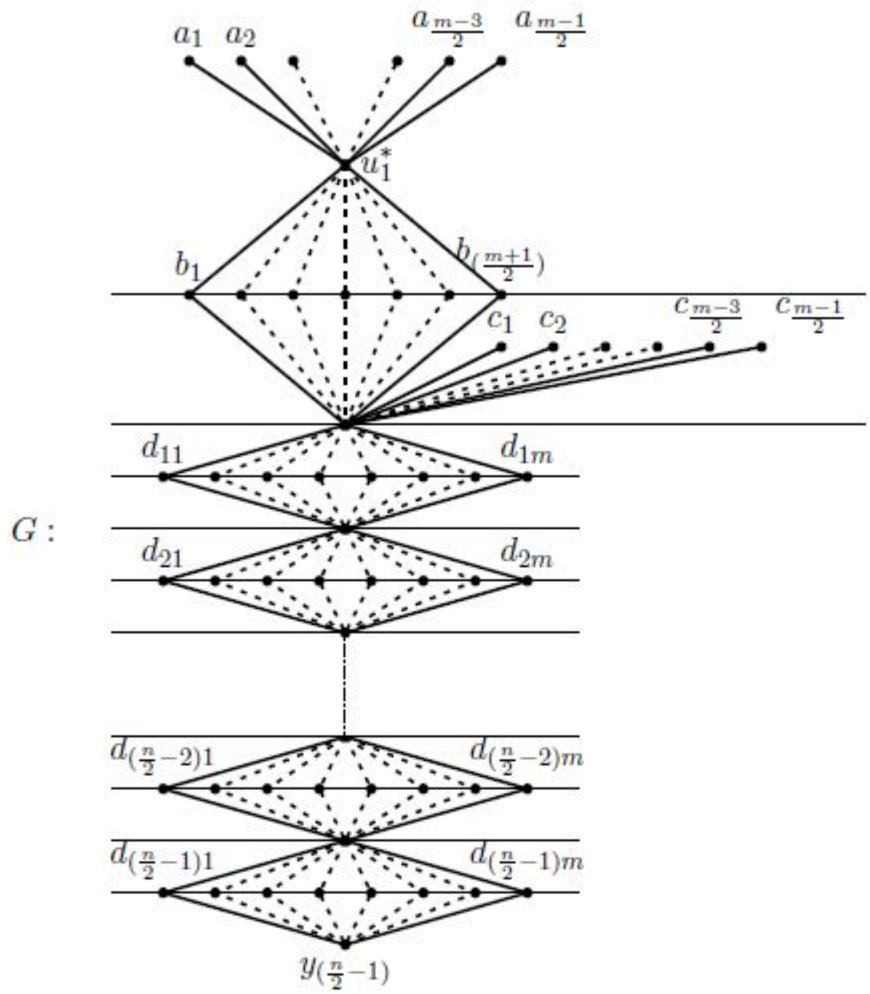

Fig. 4. Graph G with m odd, n even $(n > 2)$ and $m > n$

- Take a copy of the star graph $K_{1,\frac{m-1}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex u_1.
- Take a copy of the star graph $K_{1,\frac{m+1}{2}}$ and by vertex contraction process, contract its cut vertex with the vertex v_1.

In figure 5, the resultant graph G and its Steiner decomposition is given.

Total number of edges of G is $2m + 1$. Minimum Steiner set of $G = \{a_i / \ 1 \leq i \leq \frac{m-1}{2}\} \cup \{c_i / \ 1 \leq i \leq \frac{m+1}{2}\}$ and so $s(G) = m$. By theorem 2.2, $\pi_{st}(G) \leq 2$ and since $\pi = \{G_1, G_2\}$ is a Steiner decomposition of cardinality 2, $\pi_{st}(G) = 2$.

Thus for any positive integers m, n $(m \geq 2)$ there exists a connected graph G such that $s(G) = m$ and $\pi_{st}(G) = n$.

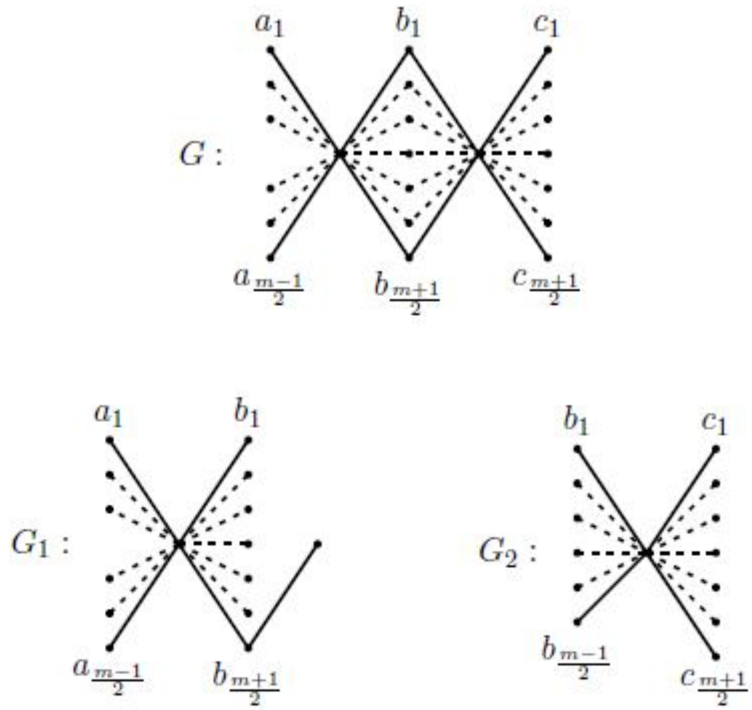

Fig. 5. Graph G with m odd, $n = 2$, $m > n$ and its Steiner decomposition

5. Conclusion

This paper is an extensive study of the decomposition parameter Steiner decomposition number of graphs. Here, a relation between independence number and Steiner decomposition number is obtained. This result plays a vital role in justifying the value of the parameter for some graph families. Also, Steiner decomposition number of some power of paths and a realization theorem is presented. Future works can be carried out on obtaining the Steiner decomposition number related bounds for any power of path and investigating the value of the parameter for other graph classes. Bounds of Steiner decomposition number of graphs based on various graph theoretical parameters can also be studied.

References

Abraham, V.M., & Hamid, I.S. (2010). Induced acyclic path decomposition in graphs. International Journal of Mathematical and Computer Sciences, **6(3)**, 166-169.

AbuGhneim, O. A., Al-Khamaiseh, B., & Al-Ezeh, H. (2014). The geodetic, hull, and Steiner numbers of powers of paths. Utilitas Mathematica, **95**, 289-294.

Arumugam, S., Hamid, I., & Abraham, V.M. (2013). Decomposition of graphs into paths and cycles. Journal of Discrete Mathematics, **2013**, DOI: https://doi.org/10.1155/2013/721051.

Chartrand, G., & Zhang, P. (2002). The Steiner number of a graph. Discrete Mathematics, **242**, 41-54.

Ghosh, P., Ghosh, S., & Pal, A. (2017). 3-Total Sum Cordial Labeling on Some New Graphs. Journal of Informatics and Mathematical Sciences, **9**, 665-673.

Harary, F. (1988). Graph Theory. Narosa Publishing House, New Delhi.

Hernando, C., Jiang, T., Mora, M., Pelayo, I.M., & Seara, C. (2005). On the Steiner, geodetic and hull numbers of graphs. Discrete Mathematics, **293**, 139-154.

John, J., & Stalin, D. (2021). The edge geodetic self decomposition number of a graph. RAIRO Oper. Res., **55**, S1935-S1947.

Lin, M. C., Rautenbach, D., Soulignac, F. J., & Szwarcfiter, J. L. (2021). Powers of cycles, powers of paths, and distance graphs. Discrete Applied Mathematics, **159**, 621-627.

Merly, E. E. R., & Jothi, D. J. (2018). Connected Domination decomposition of helm graph. International Journal of Scientific Research and Review, **7(10)**, 327-331.

Merly, E. E. R., & Mahiba, M. (2021a). Steiner decomposition number of graphs. Malaya Journal of Matematik, **Special Issue**, 560-563.

Merly, E. E. R., & Mahiba, M. (2021b). Steiner Decomposition Number of Complete $n -$ Sun graph. Journal of Physics: Conference series, **1947**, DOI: https://doi.org/10.1088/1742-6596/1947/1/012002.

Nagarajan, K., Nagarajan, A., & Hamid, I.S. (2009). Equiparity path decomposition number of a graph. International Journal of Mathematical Combinatorics, **1**, 61-76.

Pelayo, I.M. (2004). Comment on "The Steiner number of a graph" by G. Chartrand and P. Zhang:[Discrete Mathematics 242 (2002) 41-54]. Discrete Mathematics, **280**, 259-263.

Romero-Valencia, J., Hernández-Gómez, J.C., & Reyna-Hernández, G. (2019). On the inverse degree index and decompositions in graphs. Kuwait Journal of Science, **46(4)**, 14-22.

Yero, I.G., & Rodriguez-Velazquez, J.A. (2015). Analogies between the geodetic number and the Steiner number of some classes of graphs. Filomat, **29**, 1781-1788.

Submitted: 23/10/2021
Revised: 29/03/2022
Accepted: 10/04/2022
DOI: 10.48129/kjs.16863