

# Remarks on Green function of space-fractional biharmonic heat equation using Ramanujan’s master theorem

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## Abstract

In this paper, we show an addition formula for a class of Mellin transformable functions using Ramanujan’s master theorem. Also, we get the associated addition formula of Gaussian functions in terms of the Bessel functions and Green function of the space-fractional biharmonic heat equation is obtained with respect to the quartic Levy stable functions.

**Keywords:** Biharmonic heat equation; Green function; Mellin transform; Ramanujan’s master theorem.

**Mathematics Subject Classification (2010):** 34B27; 35K30;

## 1. Introduction

The Indian mathematician Srinivasa Ramanujan introduced the following theorem and used it widely in calculating definite integrals and infinite series. This theorem was named after him as Ramanujan’s master theorem and was proved by Hardy (1940) using the Cauchy’s residue theorem along with the Mellin inversion theorem.

Theorem 1.1. (Ramanujan’s master theorem) Amdeberhan *et al.* (2012) Let  $\lambda(z)$  be an analytic function (single-valued) on the half-plane

$$H(\delta) = \{z \in \mathbb{C} : \Re(z) \geq -\delta\}, 0 < \delta < 1,$$

and satisfy the following condition for

$$\begin{aligned} z \in H(\delta) \\ |\lambda(u + iv)| < e^{Pv + A|v|}, A < \pi, z = u + iv. \end{aligned}$$

Then, we have for  $0 < \Re(s) < \delta$

$$\begin{aligned} \int_0^\infty x^{s-1} (\lambda(0) - x\lambda(1) + x^2\lambda(2) - \dots) dx \\ = \frac{\pi}{\sin(\pi s)} \lambda(-s), \lambda(0) \neq 0. \end{aligned} \tag{1}$$

After substituting into  $\lambda(n) = \frac{\phi(n)}{\Gamma(1+n)}$  the above

relation and using a functional equation for Gamma function, an alternative form of Equation (1) can be obtained as follows

$$\begin{aligned} M\{f(x); s\} = \int_0^\infty x^{s-1} f(x) dx \\ = \Gamma(s)\phi(-s), 0 < \Re(s) < 1. \end{aligned} \tag{2}$$

For more applications of this theorem in applied mathematics, see the references Bertram (1997), Borosa & Moll (2001), Ding (1997), Gonzalez *et al.* (2015), Gorska *et al.* (2012) and Olafssona & Pasquale (2012, 2013). Now, in this paper we intend to introduce another application of this theorem for showing some addition formulas of elementary and special functions using the two-dimensional Mellin transform Apelblat (2008)

$$M_2\{f(x, y); s, t\} = \int_0^\infty \int_0^\infty f(x, y) x^{s-1} y^{t-1} dx dy. \tag{3}$$

For this purpose, first we state our main result for the addition formulas. In this sense, we consider the Mellin transformable function  $f$  of one variable and get the two dimensional Mellin transform of composite function  $f(a+b)$  in two variables. After applying the two-dimensional Mellin transform on the composite function  $f(a+b)$  and using the change of variables  $a = r \cos^2(\phi)$  and  $b = r \sin^2(\phi)$ , we obtain

$$\begin{aligned} \int_0^\infty \int_0^\infty f(a+b) a^{\alpha-1} b^{\beta-1} da db = B(\alpha, \beta) M\{f(r); \alpha + \beta\} \\ = \Gamma(\alpha)\Gamma(\beta)\phi(-(\alpha + \beta)), \end{aligned} \tag{4}$$

where B is the beta function given by

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \tag{5}$$

In the next section, as the special case of Ramanujan’s master theorem for the reciprocal gamma function, we state a theorem for showing an integral addition formula for the Gaussian functions. This representation is given in terms of the modified Bessel function of second kind.

We apply the obtained addition formula for showing the Green function of space-fractional biharmonic heat equation in the sense of Weyl fractional derivatives. This Green function is given in terms of the quartic Levy stable functions. Finally, the main conclusions are drawn.

## 2. Addition formula for Gaussian functions

In this section, we treat some mathematical aspects of Ramanujan's master theorem in the special case  $\phi(-n) = \frac{1}{\Gamma(d-cn)}$ . At the beginning we consider integral operators  $I_x$  and  $J_x$  (a linear combination of differentiation or integration operators) such that

$$M_2\{I_x[J_x\{f(a)\}J_x\{f(b)\}];\alpha,\beta\} = \Gamma(\alpha)\Gamma(\beta)\phi(-(\alpha+\beta)). \quad (6)$$

Next, we intend to demonstrate a representation for function  $f(a+b)$  in this special case as

$$f(a+b) = I_x[J_x\{f(a)\}J_x\{f(b)\}]. \quad (7)$$

Although, the obtained relation is formal, but in general, finding the operators  $I_x$  and  $J_x$  is not straightforward and solving the associated functional equation is the main problem of this paper.

Corollary 2.1. If we consider  $J_x$  as the differential operator  $J_x = \frac{d^v}{dx^v}$  (ordinary or fractional derivatives), and  $I_x$  as the beta function integral operator, then we can derive the following results.

1. For  $f(x) = \operatorname{Erfc}(\sqrt{2}x)$  and

$$\phi(-n) = 2^{-\frac{n}{2}} \frac{1}{\Gamma\left(\frac{n}{2}+1\right)},$$

we have

$$\operatorname{Erfc}(a+b) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-\csc^2(\theta)a^2 - \sec^2(\theta)b^2} d\theta, \quad (8)$$

$$|\arg(a)| < \frac{\pi}{2}, |\arg(b)| < \frac{\pi}{2}.$$

2. For  $f(x) = 3^{\frac{2}{3}} Ai\left(\frac{x}{3}\right)$  and,

$$\phi(-n) = \frac{1}{\Gamma\left(-\frac{n}{3} + \frac{2}{3}\right)}, \text{ we have Proskurin (1997)}$$

$$Ai(a+b) = \int_0^1 Ai'\left(\frac{a}{x^{\frac{1}{3}}}\right) Ai'\left(\frac{b}{(1-x)^{\frac{1}{3}}}\right) \frac{dx}{x^{\frac{2}{3}}(1-x)^{\frac{2}{3}}},$$

$$|\arg(a)| < \frac{\pi}{3}, |\arg(b)| < \frac{\pi}{3}, \quad (9)$$

where the Airy function of first kind is given by Vallee & Soares (2004)

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(xt + \frac{t^3}{3}\right) dt, \quad (10)$$

3. For  $v \in \mathbb{R}$ ,  $|\arg(a)| < |c|\pi$  and

$|\arg(b)| < |c|\pi$ , the following addition

formula holds

$$W(c, -2cv; -(a+b)) = \int_0^1 x^{-1-cv} (1-x)^{-1-cv} W(c, c-2cv^2; -ax^c) W(c, c-2cv^2; -b(1-x)^c) dx, \quad (11)$$

where the Wright function is given by Kilbas *et al.* (2006) and Mainardi (2010).

$$W(c, d; z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(cn+d)},$$

$$c > -1, d \in \mathbb{C}, z \in \mathbb{C}. \quad (12)$$

Theorem 2.2. For  $|\arg(a)| \leq \frac{\pi}{2}$  and  $|\arg(b)| \leq \frac{\pi}{2}$ ,

the following addition formula holds for the Gaussian function

$$e^{-(a+b)^2} = \frac{a^{\frac{3}{2}} b^{\frac{3}{2}}}{\pi 2^{\frac{7}{4}}} \int_0^1 x^{\frac{3}{2}} (1-x)^{-\frac{3}{2}} e^{\frac{a^2}{2x} + \frac{b^2}{2(1-x)}} dx,$$

$$\left[ K_{\frac{1}{4}}\left(\frac{b^2}{2(1-x)}\right) - K_{\frac{3}{4}}\left(\frac{b^2}{2(1-x)}\right) \right] dx, \quad (13)$$

where  $K_v$  is the modified Bessel function of second kind given by Gradshteyn & Ryzhik (2007).

$$K_v(z) = \int_0^{\infty} e^{-z \cosh(t)} \cos(vt) dt. \quad (14)$$

Proof: First, for the left hand side of Equation (13) we compute the two dimensional Mellin transform of function  $e^{-(a+b)^2}$ .

We apply the change of variables  $a = r \cos^2(\phi)$  and  $b = r \sin^2(\phi)$ , and use the Mellin transform of Gaussian function,

$$M\left\{e^{-r^2}; s\right\} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right)$$

to get

$$\int_0^{\infty} \int_0^{\infty} e^{-(a+b)^2} a^{\alpha-1} b^{\beta-1} da db = B(\alpha, \beta) M\left\{e^{-r^2}; \alpha + \beta\right\} \quad (15)$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\pi^{\frac{1}{2}} 2^{-(\alpha+\beta)}}{\Gamma\left(\frac{(\alpha+\beta+1)}{2}\right)},$$

where we used the well-known Legendre duplication formula Abramowitz & Stegan (1972)

$$\frac{1}{2}\Gamma\left(\frac{s}{2}\right) = \frac{\pi^{\frac{1}{2}}\Gamma(s)}{2^s\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}. \quad (16)$$

Now, for the right hand side of Equation (13), we employ the parabolic cylinder function  $D_\nu$ . Abramowitz & Stegan (1972).

$$D_{-\nu}(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(\nu)} \int e^{-zt + \frac{t^2}{2}} t^{\nu-1} dt, \quad \Re(\nu) > 0, \quad (17)$$

with its Mellin transform

$$M\left\{e^{\frac{x^2}{2}} D_{-\nu}(\sqrt{2}x); s\right\} = \pi^{\frac{1}{2}} 2^{-s-\frac{\nu}{2}} \frac{\Gamma(s)}{\Gamma\left(\frac{s+\nu+1}{2}\right)}. \quad (18)$$

We consider the following integral

$$I = \int_0^1 x^{\frac{\nu}{2}-1} (1-x)^{\frac{\nu}{2}-1} \frac{d}{d\xi} \left( e^{-\frac{\xi^2}{2}} D_{-\nu}(\sqrt{2}\xi) \right)_{\xi \rightarrow \frac{a}{x^{\frac{1}{2}}}} \frac{d}{d\xi} \left( e^{-\frac{\xi^2}{2}} D_{-\nu}(\sqrt{2}\xi) \right)_{\xi \rightarrow \frac{b}{(1-x)^{\frac{1}{2}}}} dx, \quad (19)$$

and compute its two dimensional Mellin transform as follows:

$$\left\{ \int_0^1 \left\{ \int_0^\infty \frac{d}{d\xi} \left( e^{-\frac{\xi^2}{2}} D_{-\nu}(\sqrt{2}\xi) \right)_{\xi \rightarrow \frac{a}{x^{\frac{1}{2}}}} a^{\alpha-1} da \right\} \right. \\ \left. \left\{ \int_0^\infty \frac{d}{d\xi} \left( e^{-\frac{\xi^2}{2}} D_{-\nu}(\sqrt{2}\xi) \right)_{\xi \rightarrow \frac{b}{(1-x)^{\frac{1}{2}}}} b^{\alpha-1} db \right\} \right. \quad (20)$$

$$\times x^{\frac{\nu}{2}-1} (1-x)^{\frac{\nu}{2}-1} dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\pi 2^{-\alpha-\beta-\nu+2}}{\Gamma\left(\frac{\alpha}{2} + \frac{\nu}{2}\right)\Gamma\left(\frac{\beta}{2} + \frac{\nu}{2}\right)}$$

$$\int_0^1 x^{\frac{\alpha+\nu}{2}-1} (1-x)^{\frac{\beta+\nu}{2}-1} dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\pi 2^{-\alpha-\beta-\nu+2}}{\Gamma\left(\frac{\alpha}{2} + \frac{\nu}{2}\right)\Gamma\left(\frac{\beta}{2} + \frac{\nu}{2}\right)} \\ \times B\left(\frac{\alpha}{2} + \frac{\nu}{2}, \frac{\beta}{2} + \frac{\nu}{2}\right) \\ = \frac{\Gamma(\alpha)\Gamma(\beta)\pi 2^{-\alpha-\beta-\nu+2}}{\Gamma\left(\frac{\alpha}{2} + \frac{\beta}{2} + \nu\right)}.$$

At this point, in the special case  $\nu = \frac{1}{2}$  for

$$D_{-\frac{1}{2}}(\xi) = \sqrt{\frac{\xi}{2\pi}} K_{\frac{1}{4}}\left(\frac{\xi^2}{4}\right),$$

we compare Equation (20) with (15) to obtain

$$e^{-(a+b)^2} = \frac{1}{\sqrt{8\pi}} \int_0^1 x^{-\frac{3}{4}} (1-x)^{-\frac{3}{4}} \\ \times \frac{d}{d\xi} \left( e^{-\frac{\xi^2}{2}} D_{-\frac{1}{2}}(\sqrt{2}\xi) \right)_{\xi \rightarrow \frac{a}{x^{\frac{1}{2}}}} \\ \times \frac{d}{d\xi} \left( e^{-\frac{\xi^2}{2}} D_{-\frac{1}{2}}(\sqrt{2}\xi) \right)_{\xi \rightarrow \frac{b}{(1-x)^{\frac{1}{2}}}} dx, \quad (21)$$

or equivalently

$$e^{-(a+b)^2} = \frac{1}{\pi 2^{\frac{7}{4}}} \int_0^1 x^{-\frac{3}{4}} (1-x)^{-\frac{3}{4}} \\ \times \left( \xi^{\frac{3}{2}} e^{\frac{\xi^2}{2}} \left[ K_{\frac{1}{4}}\left(\frac{\xi^2}{2}\right) - K_{\frac{3}{4}}\left(\frac{\xi^2}{2}\right) \right]_{\xi \rightarrow \frac{a}{x^{\frac{1}{2}}}} \right) \\ \times \left( \xi^{\frac{3}{2}} e^{\frac{\xi^2}{2}} \left[ K_{\frac{1}{4}}\left(\frac{\xi^2}{2}\right) - K_{\frac{3}{4}}\left(\frac{\xi^2}{2}\right) \right]_{\xi \rightarrow \frac{b}{(1-x)^{\frac{1}{2}}}} \right) dx, \quad (22)$$

or

$$e^{-(a+b)^2} = \frac{a^{\frac{3}{2}} b^{\frac{3}{2}}}{\pi 2^{\frac{7}{4}}} \int_0^1 x^{-\frac{3}{2}} (1-x)^{-\frac{3}{2}} e^{\frac{a^2}{2x} + \frac{b^2}{2(1-x)}} \\ \times \left[ K_{\frac{1}{4}}\left(\frac{a^2}{2x}\right) - K_{\frac{3}{4}}\left(\frac{a^2}{2x}\right) \right] \\ \times \left[ K_{\frac{1}{4}}\left(\frac{b^2}{2(1-x)}\right) - K_{\frac{3}{4}}\left(\frac{b^2}{2(1-x)}\right) \right] dx. \quad (23)$$

Corollary 2.3. We use the following integral representation for the modified Bessel function of second kind  $K_\nu$ , Gradshteyn & Ryzhik (2007).

$$K_{\nu-\frac{1}{2}}(\beta\mu) = \frac{\sqrt{\pi}}{\Gamma(\nu)} \left( \frac{2\beta}{\mu} \right)^{\frac{1}{2}-\nu} e^{-\beta\mu} \times \int_0^\infty (2\beta\xi + \xi^2)^{\nu-1} e^{-\mu\xi} d\xi, \quad (24)$$

$$\Re(\nu) > 0, \Re(\mu) > 0, |\arg(\beta)| < \pi,$$

and set  $\mu = a^2, b^2$  and  $\beta = \frac{1}{2x}, \frac{1}{2(1-x)}$  to rewrite the relation (23) as follows:

$$e^{-(a+b)^2} = \frac{a^2 b^2}{2^{\frac{7}{4}} \Gamma^2\left(\frac{3}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty x^{-\frac{5}{4}} (1-x)^{-\frac{5}{4}} \times \left( \frac{\xi}{x} + \xi^2 \right)^{\frac{1}{4}} \left( \frac{\tau}{1-x} + \tau^2 \right)^{\frac{1}{4}} e^{-a^2\xi - b^2\tau} d\xi d\tau dx - \frac{a^2 b^3}{2^{\frac{7}{4}} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty x^{-\frac{5}{4}} (1-x)^{-\frac{3}{4}} \times \left( \frac{\xi}{x} + \xi^2 \right)^{\frac{1}{4}} \left( \frac{\tau}{1-x} + \tau^2 \right)^{\frac{1}{4}} e^{-a^2\xi - b^2\tau} d\xi d\tau dx - \frac{a^3 b^2}{2^{\frac{7}{4}} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty x^{-\frac{3}{4}} (1-x)^{-\frac{5}{4}} \times \left( \frac{\xi}{x} + \xi^2 \right)^{\frac{1}{4}} \left( \frac{\tau}{1-x} + \tau^2 \right)^{\frac{1}{4}} e^{-a^2\xi - b^2\tau} d\xi d\tau dx + \frac{a^3 b^3}{2^{\frac{7}{4}} \Gamma^2\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty x^{-\frac{3}{4}} (1-x)^{-\frac{3}{4}} \times \left( \frac{\xi}{x} + \xi^2 \right)^{\frac{1}{4}} \left( \frac{\tau}{1-x} + \tau^2 \right)^{\frac{1}{4}} e^{-a^2\xi - b^2\tau} d\xi d\tau dx. \quad (25)$$

### 3. Green function of biharmonic heat equation

#### 3.1. Biharmonic heat equation with ordinary derivatives

The problem of determining the Green function of generalized heat equation

$$\frac{\partial}{\partial t} u(x, t) + \frac{\partial^4}{\partial x^4} u(x, t) = 0, \quad (26)$$

$$t > 0, x \in \mathbb{R}, u(x, 0) = \delta(x),$$

is referred to the fractional exponential operators as follows

$$u(x, t) = e^{-t \frac{\partial^4}{\partial x^4}} \delta(x) = L_4(x, t), \quad (27)$$

where  $\delta$  is the Dirac-delta function and  $L_4(x, t)$  is the quartic Levy stable function given by Gorska *et al.* (2013)

$$L_4(x, t) = \frac{1}{\pi} \int_0^\infty e^{-r^4} \cos(rx) dr. \quad (28)$$

For more complicated fractional exponential operators and their connections to heat equations, particularly in fractional calculus, see Ansari (2012, 2015) and Ansari *et al.* (2012a, 2012b, 2013). Now, in this section we intend to find the Green function of biharmonic heat equation Ferrero *et al.* (2008), Gazzola (2013) and Gazzola & Grunau (2008).

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^2 u(x, y, t) = 0,$$

$$t > 0, x, y \in \mathbb{R}, u(x, y, 0) = \delta(x) \delta(y), \quad (29)$$

where the operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

is the Laplacian operator. It is obvious that the solution of Equation (29) is obtained by the following fractional exponential operator

$$u(x, y, t) = e^{-t \Delta^2} \delta(x) \delta(y) \quad (30)$$

$$= e^{-t \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]^2} \delta(x) \delta(y),$$

which is stated in the next theorem. This theorem enables us to express the solution of biharmonic heat equation with respect to the solution of generalized heat Equation (26), that is the quartic Levy stable function.

Theorem 3.1. The Green function of biharmonic heat equation (29) is given by the following relation in terms of the derivatives of quartic Levy stable functions

$$\begin{aligned}
u(x, y, t) = & \frac{t^2}{2^{\frac{7}{4}} \Gamma^2\left(\frac{3}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{5}{4}} (1-\rho)^{\frac{5}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& \times L_4''(x, \xi t) L_4''(y, \tau t) d\xi d\tau d\rho \\
& - \frac{t^2}{2^{\frac{7}{4}} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{5}{4}} (1-\rho)^{\frac{3}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& \times L_4''(x, \xi t) L_4'''(y, \tau t) d\xi d\tau d\rho \\
& - \frac{t^{\frac{5}{2}}}{2^{\frac{7}{4}} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{3}{4}} (1-\rho)^{\frac{5}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& \times L_4'''(x, \xi t) L_4''(y, \tau t) d\xi d\tau d\rho \\
& + \frac{t^3}{2^{\frac{7}{4}} \Gamma^2\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{3}{4}} (1-\rho)^{\frac{3}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& \times L_4'''(x, \xi t) L_4'''(y, \tau t) d\xi d\tau d\rho.
\end{aligned} \tag{31}$$

Proof: By setting  $a = \sqrt{t} \frac{\partial^2}{\partial x^2}$  and  $b = \sqrt{t} \frac{\partial^2}{\partial y^2}$  in relation (25) and applying the identity (27), the result can be easily derived.

3.2. Biharmonic heat equation with space-fractional derivatives

Before stating our results, we mention that the problem of the fractional harmonic heat equation has been surveyed much more in the literature, for example see Furati *et al.* (2014), Garg & Manohar (2013) and Ghany & Hyder (2014). Now, in this section we intend to fractionalize the biharmonic heat equation with space-fractional derivative in the sense of Weyl fractional derivative

$$W_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty \frac{f(\tau) d\tau}{(x-\tau)^{\alpha+1-n}}, \tag{32}$$

$$n-1 < \alpha < n.$$

For this purpose, we consider the following equation which has been developed much less in the literature

$$\begin{aligned}
\frac{\partial}{\partial t} u(x, y, t) + \Delta_\alpha^2 u(x, y, t) &= 0, \\
t > 0, x, y \in \mathbb{R}, 0 < \alpha \leq 1, & \tag{33}
\end{aligned}$$

where the operator

$$\Delta_\alpha = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}},$$

is the Laplacian operator with the Weyl fractional derivatives. By a similar procedure to previous case, after solving a first order differential equation with respect to  $t$ , we obtain the Green function of (33) as follows

$$\begin{aligned}
u(x, y, t) &= e^{-t \Delta_\alpha^2} \delta(x) \delta(y) \\
&= e^{-t \left[ \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \right]^2} \delta(x) \delta(y).
\end{aligned} \tag{34}$$

Now, in order to simplify the above relation, we use Theorem 3.1 and apply the following fractional exponential operator to obtain the associated Green function in terms of the Wright function Ansari (2012, 2015) and Ansari *et al.* (2012a, 2012b, 2013)

$$\begin{aligned}
L_4^\alpha(x, t) &= e^{-t \frac{\partial^{4\alpha}}{\partial x^{4\alpha}}} \delta(x) \\
&= -\frac{1}{x} \int_{-\infty}^\infty L_4(\tau, t) \mathcal{W}(-\alpha, 0; \tau (-x)^{-\alpha}) d\tau, n-1 < \alpha < n.
\end{aligned} \tag{35}$$

Theorem 3.2. The Green function of space-fractional biharmonic heat equation (33) is given by the following relation in terms of the derivatives of  $L_4^\alpha(x, t)$

$$\begin{aligned}
u(x, y, t) = & \frac{t^2}{2^{\frac{7}{4}} \Gamma^2\left(\frac{3}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{5}{4}} (1-\rho)^{\frac{5}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& L_4^{(\alpha)''}(x, \xi t) L_4^{(\alpha)''}(y, \tau t) d\xi d\tau d\rho \\
& - \frac{t^{\frac{5}{2}}}{2^{\frac{7}{4}} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{5}{4}} (1-\rho)^{\frac{3}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& L_4^{(\alpha)''}(x, \xi t) L_4^{(\alpha)''}(y, \tau t) d\xi d\tau d\rho \\
& - \frac{t^{\frac{5}{2}}}{2^{\frac{7}{4}} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{3}{4}} (1-\rho)^{\frac{5}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& \times L_4^{(\alpha)''}(x, \xi t) L_4^{(\alpha)''}(y, \tau t) d\xi d\tau d\rho \\
& + \frac{t^3}{2^{\frac{7}{4}} \Gamma^2\left(\frac{5}{4}\right)} \int_0^1 \int_0^\infty \int_0^\infty \rho^{-\frac{3}{4}} (1-\rho)^{\frac{3}{4}} \\
& \times \left(\frac{\xi}{\rho} + \xi^2\right)^{\frac{1}{4}} \left(\frac{\tau}{1-\rho} + \tau^2\right)^{\frac{1}{4}} \\
& \times L_4^{(\alpha)''}(x, \xi t) L_4^{(\alpha)''}(y, \tau t) d\xi d\tau d\rho.
\end{aligned}$$

where the function  $L_4^\alpha(x, t)$  is given by the relation (34).

#### 4. Concluding remarks

This paper provides new results for addition formulas of the Gaussian functions. These integral addition formulas that were obtained in terms of the Bessel functions, enable us to present the Green functions of biharmonic heat equation in the cases of ordinary and fractional derivatives. Although in this paper we studied several results for

Ramanujan's master theorem in a special case of  $\phi(n)$  (the reciprocal gamma function), but other cases for the function  $\phi(n)$  and demonstrating the associated addition formulas can be considered as an open question. Solving the corresponding functional equation with operators  $I_x$  and  $J_x$  is our fundamental problem for determining addition formulas in general cases.

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ملاحظات على دالة جرين Green للفضاء الكسري لمعادلة الحرارة ثنائية التوافق باستخدام نظرية  
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### خلاصة

في هذا البحث نوجد صيغة جمع لمجموعة من دوال ملين Mellin القابلة للتحويل باستخدام نظرية رامانوجان Ramanujan الرئيسية. بالإضافة إلى ذلك، نوجد صيغة الجمع المصاحبة لدوال جاوسيان Gaussian بدلالة دوال بسل Bessel ودوال جرين Green للفضاء الكسري لمعادلة الحرارة ثنائية التوافق التي يتم الحصول عليها بالنسبة إلى دوال ليفي Levy الرباعية المستقرة.