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Some new regularity criterion for MHD three-dimensional flow

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Abstract

The aim of the paper is to establish regularity criteria for the weak solution of fluid passing through the porous media in R^3. We show that if $(\nabla_h u, \partial_3 b_3) \in L^{2\alpha,2\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3,\ 1\le\gamma\le\infty$, then the weak solution is regular and unique; if $(\nabla_h u, \nabla_h b) \in L^{2\alpha,2\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3,\ 1\le\gamma\le\infty$, then the weak solution is regular and unique; if $(\partial_3 u, \nabla u_3) \in L^{2\alpha,2\gamma}$ and $(u_3, b, \partial_3 b, \nabla b_3) \in L^{4\alpha,4\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3,\ 1\le\gamma\le\infty$, then the weak solution is regular and unique. Here we use the notation $\nabla_h = (\partial_1, \partial_2)$.

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1. Introduction

In this paper we consider 3D flows of an incompressible magnetohydrodynamics fluid passing through the porous medium. Let (u_1,u_2,u_3) and (b_1,b_2,b_3) be the components of velocity u and magnetic field b respectively. The fundamental equations, which governs 3D equations under the assumption of an incompressible and unsteady MHD fluid passing through the porous medium are

$$\frac{\partial u}{\partial t}+u\cdot\nabla u=\nu_1\Delta u-\nabla P+b\cdot\nabla b-mu, \tag{1}$$

$$\frac{\partial b}{\partial t}+u\cdot\nabla b=\nu_2\Delta b+b\cdot\nabla u, \tag{2}$$

$$\nabla\cdot u=\nabla\cdot b=0, \tag{3}$$

$$u(x,0)=u_0(x),\quad b(x,0)=b_0(x), \tag{4}$$

where u is the velocity, b is the magnetic field, ν_1 is the kinematic viscosity, ν_2 is the magnetic diffusivity, and P is the pressure of the medium. For simplicity we let $m=\frac{\phi}{K}$, where ϕ is the porosity of the medium and K is the permeability of the medium. We also take $\nu_1=\nu_2=1$.

The MHD fluid passing through the porous medium has many practical applications such as the flow of mercury amalgams, handling of biological fluids and flow of plasma. The work on porous medium was first started by Darcy (1856) and Forchheimer (1901). Sermange Teman (1983) proved the local well-posedness of weak solutions of MHD equations in the absence of porous medium for any given initial datum $u_0, b_0 \in H^s(R^3),\ s\ge 3$. But whether this unique local solution can exists globally is an outstanding challenging problem. Fundamental Serrien type regularity was given by He Xin (2005) and Zhou (2005) in terms of the velocity. Chen *et al*. (2007) derived regularity by adding the condition on $\nabla_j(\nabla\times u)$ and some further improvement was done by He & Wang (2008). Ni *et al*. (2012) developed some regularity criteria for 3D MHD equations when $u_3,\ \partial_3 u$ and $\partial_3 b$ are serrin type integrable class. It is also mentioned that logarithmicall regularity criteria was established in Fang *et al*. (2011) and Zhou Fang (2012). Gala *et al*. (2012) demonstrated Serrien›s uniqueness results of Leray weak solution for the 3D incompressible MHD equations in Orlicz-Morrey spaces.

The regularity criterion on ∇u was obtained by Beirao (1995) who showed that if a weak solution $u(x,t)$ satisfies $\nabla u \in L^{\alpha,\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 2,\ \frac{3}{2}<\gamma<\infty$, then $u(x,t)\in C^\infty(R^3\times(0,T))$. Chae Choe (1999) improved Beirao (1995) condiction by applying two components of the vorticity field. Further Zhou (2002) pointed that if the Leray-Hopf weak solution u satisfyies $\nabla u_3 \in L^{\alpha,\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}=\frac{3}{2},\ 3\le\gamma<\infty$ and $\|\nabla u_3\|$ is sufficiently small, then u is strong.

Regularity criteria for the 3D MHD equations in terms of pressure was obtained by Zhou (2006), who derived that, $\nabla P \in L^{\alpha,\gamma}$ if with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3$ provided $u_0, b_0 \in H^s(R^3)$ for $s \ge 3$, then the solution remains smooth on $[0,T]$. Later on Duan (2012) also improved the results of Zhou (2006) and proved that (u,b) can be extended smoothly beyond $t=T$, for any given $u_0, b_0 \in H^s(R^3)$ with $s\ge 3$ and $\nabla P \in L^{\frac{2\gamma}{2\gamma-3}}(0,T;L^\gamma(R^3))$ for $1<\gamma\le\infty$. Regularity criteria for MHD equations in term for gradient pressure is obtained by Rahman (2014). Recently Beg *et. al.* (2014) consider nonlinear stochastic equations and discussed the stability of the equations.

The objective of current paper is to establish regularity of weak solutions of 3D incompressible, MHD fluid passing through the porous medium. The main results are

Theorem 1. Suppose that the initial velocity and magnetic field $(u_0,b_0)\in H^s(R^3)$, $s\ge 3$ and $\nabla\cdot u_0 = 0 = \nabla\cdot b_0$ in the sense of distribution. Additionally assume that

$$(\nabla_h u, \partial_3 b_3) \in L^{2\alpha,2\gamma} \text{ with } \frac{2}{\alpha}+\frac{3}{\gamma}\le 3, \quad 1\le\gamma\le\infty, \quad (5)$$

or $\|\nabla_h u\|_{L^{1,\infty}}$, $\|\nabla_h u\|_{L^{4,2}}$, $\|\partial_3 b_3\|_{L^{1,\infty}}$ and $\|\partial_3 b_3\|_{L^{4,2}}$ are sufficiently small, then the corresponding solution (u,b) remains smooth on $[0,T]$.

Theorem 2. Assume $(u_0,b_0)\in H^s(R^3)$, $s\ge 3$ and $\nabla\cdot u_0 = 0 = \nabla\cdot b_0$ in the sense of distribution. Suppose that $(\nabla_h u, \nabla_h b)\in L^{2\alpha,2\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3$, $1\le\gamma\le\infty$, or $\|\nabla_h u\|_{L^{1,\infty}}$, $\|\nabla_h u\|_{L^{4,2}}$, $\|\nabla_h b\|_{L^{1,\infty}}$ and $\|\nabla_h b\|_{L^{4,2}}$ are sufficiently small, then (u,b) is smooth on $[0,T]$.

Theorem 3. Assume $(u_0,b_0)\in H^s(R^3)$, $s\ge 3$ and $\nabla\cdot u_0 = 0 = \nabla\cdot b_0$ in the sense of distribution. Suppose that $(\partial_3 u, \nabla u_3)\in L^{2\alpha,2\gamma}$ and $(u_3, b, \partial_3 b, \nabla b_3)\in L^{4\alpha,4\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3$, $1\le\gamma\le\infty$ or $\|\partial_3 u\|_{L^{1,\infty}}$, $\|\nabla u_3\|_{L^{1,\infty}}$, $\|u_3\|_{L^{1,\infty}}$, $\|b\|_{L^{1,\infty}}$, $\|\nabla b_3\|_{L^{1,\infty}}$, $\|\partial_3 u\|_{L^{1,\infty}}$, $\|\partial_3 u\|_{L^{4,2}}$, $\|\nabla u_3\|_{L^{4,2}}$, $\|u_3\|_{L^{\frac{8}{5},4}}$, $\|\nabla b_3\|_{L^{\frac{8}{5},4}}$, $\|\partial_3 b\|_{L^{\frac{8}{5},4}}$, and $\|b\|_{L^{\frac{8}{5},4}}$ are sufficiently small, then (u,b) remains smooth on $[0,T]$.

Proof of Theorem 1

To prove Theorem 1, it is enough to show

$$(u,b)\in L^\infty(0,T,H^1)\cap L^2(0,T,H^2),$$

if (5) holds. Firstly we need the following Lemma.

Lemma 1: Suppose that $(u_0,b_0)\in H^s(R^3)$, $s\ge 3$ and $\nabla\cdot u_0 = 0 = \nabla\cdot b_0$ in the sense of distribution. If $(\nabla_h u, \partial_3 b_3)\in L^{2\alpha,2\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3$, $1\le\gamma\le\infty$, or $\|\nabla_h u\|_{L^{1,\infty}}$, $\|\nabla_h u\|_{L^{4,2}}$, $|\partial_3 b_3\|_{L^{1,\infty}}$ and $\|\partial_3 b_3\|_{L^{4,2}}$ are sufficiently small, then

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2), \quad (6)$$

where C depends on the m, T, norms of $\nabla_h u$ and $\partial_3 b_3$.

Proof: Multiplying (1) by Δu and integrating over R^3, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2 &= \int_{R^3}\Delta u(u\cdot\nabla u)dx-\sum_{i,j,k=1}^{3}\int_{R^3}b_k\partial_i\partial_k u_j\partial_i b_j dx - m\|\nabla u\|_{L^2}^2 \\ &= \sum_{i=1}^{3}\int_{R^3}(u_i\partial_i u)\Delta u dx-\sum_{i,j,k=1}^{3}\int_{R^3}b_k\partial_i\partial_k u_j\partial_i b_j dx - m\|\nabla u\|_{L^2}^2.\end{aligned} \quad (7)$$

Similarly, multiplying (2) by Δb and integrating over R^3, it follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2+\|\Delta b\|_{L^2}^2 &= -\frac{1}{2}\sum_{i,j,k=1}^{3}\int_{R^3}\partial_i u_k\partial_k b_j\partial_i b_j dx+\sum_{i,j,k=1}^{3}\int_{R^3}\partial_i b_k\partial_k u_j\partial_i b_j dx \\ &\quad + \sum_{i,j,k=1}^{3}\int_{R^3}b_k\partial_i\partial_k u_j\partial b_j dx\end{aligned} \quad (8)$$

Combining (7) and (8), we get

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)=\sum_{i=1}^{2}\int_{R^3}(u_i\partial_i u)\Delta u dx+\int_{R^3}(u_3\partial_3 u)\Delta u dx$$

$$-\frac{1}{2}\sum_{i,j,k=1}^{3}\int_{R^3}\partial_i u_k\partial_k b_j\partial_i b_j dx+\sum_{i,j,k=1}^{3}\int_{R^3}\partial_i b_k\partial_k u_j\partial_i b_j dx-m\|\nabla u\|_{L^2}^2 \quad (9)$$

$$\leq |I_1|+|I_2|+|I_3|+|I_4|+m\|\nabla u\|_{L^2}^2,$$

where $I_1 = \sum_{i=1}^{2}\int_{R^3}(u_i\partial_i u)\Delta u dx$, $I_2=\int_{R^3}(u_3\partial_3 u)\Delta u dx$, $I_3=\frac{1}{2}\sum_{i,j,k=1}^{3}\int_{R^3}\partial_i u_k\partial_k b_j\partial_i b_j dx$

and $I_4 = \sum_{i,j,k=1}^{3}\int_{R^3}\partial_i b_k\partial_k u_j\partial_i b_j dx.$

Now let us estimate one by one $|I_i|$, $i=1,2,3,4$:

$$\begin{aligned} I_1 &= -\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{R^3}(\partial_j u_i\partial_i u+u_i\partial_j\partial_i u)\partial_j u dx \\ &= -\sum_{j=1}^{2}\sum_{i=1}^{2}\int_{R^3}\partial_j u_i\partial_i u\partial_j u dx-\sum_{i=1}^{2}\int_{R^3}\partial_3 u_i\partial_i u\partial_3 u dx-\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{R^3}u_i\partial_i(\partial_j u)^2 dx \\ &= -\sum_{j=1}^{2}\sum_{i=1}^{2}\int_{R^3}\partial_j u_i\partial_i u\partial_j u dx-\sum_{i=1}^{2}\int_{R^3}\partial_3 u_i\partial_i u\partial_3 u dx+\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{R^3}\partial_i u_i(\partial_j u)^2 dx \\ &= I_{11}+I_{12}+I_{13}. \end{aligned}$$

Now we estimate $|I_{11}|$ using Gagliardo Nirenberg›s inequality and Young›s inequality to get

$$\begin{aligned} |I_{11}| &\leq \sum_{i,j=1}^{2}\int_{R^3}|\partial_j u_i||\partial_i u||\partial_j u| dx \\ &\leq \int_{R^3}|\nabla_h u||\nabla u|^2 dx \\ &\leq \|\nabla_h u\|_{L^{2\gamma}}\|\nabla u\|^2_{L^{\frac{4\gamma}{2\gamma-1}}} \\ &\leq C_1\|\nabla_h u\|_{L^{2\gamma}}\|\Delta u\|_{L^2}^{\frac{3}{2\gamma}}\|\nabla u\|_{L^2}^{\frac{4\gamma-3}{2\gamma}} \\ &\leq \frac{1}{16}\|\Delta u\|_{L^2}^2+C_2\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla u\|_{L^2}^2 \\ &\leq \frac{1}{16}\|\Delta u\|_{L^2}^2+C_2\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla u\|_{L^2}^2, \end{aligned}$$

where C_2 is a positive constant. Similarly $|I_{12}+I_{13}|$ is estimated as

$$|I_{12}+I_{13}|\leq\frac{1}{16}\|\Delta u\|_{L^2}^2+C_2\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla u\|_{L^2}^2.$$

Therefore

$$|I_1|\leq\frac{1}{8}\|\Delta u\|_{L^2}^2+2C_2\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla u\|_{L^2}^2.$$

To we split it into two parts and obtain

$$\begin{aligned} I_2 &= -\sum_{j=1}^{3}\int_{R^3}\partial_j u_3\partial_3 u\partial_j u dx-\sum_{j=1}^{3}\int_{R^3}u_3\partial_j\partial_3 u\partial_j u dx \\ &= -\sum_{j=1}^{3}\int_{R^3}\partial_j u_3\partial_3 u\partial_j u dx+\frac{1}{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 u_3(\partial_j u)^2)dx \\ &= -\sum_{j=1}^{2}\int_{R^3}\partial_j u_3\partial_3 u\partial_j u dx-\int_{R^3}\partial_3 u_3(\partial_3 u)^2 dx+\frac{1}{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 u_3(\partial_j u)^2)dx \\ &= -\sum_{j=1}^{2}\int_{R^3}\partial_j u_3\partial_3 u\partial_j u dx-\int_{R^3}\partial_3 u_3(\partial_3 u)^2 dx-\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{R^3}\partial_i u_i(\partial_j u)^2)dx. \end{aligned}$$

With the same way to estimate I_1, it yields

$$|I_2|\le \frac{1}{8}\|\Delta u\|_{L^2}^2+2C_2\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla u\|_{L^2}^2.$$

Noting

$$\begin{aligned} I_3 &= -\frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}\partial_i u_k\partial_k b_j\partial_i b_j dx-\frac{1}{2}\sum_{j,k=1}^{3}\int_{R^3}\partial_3 u_k\partial_k b_j\partial_3 b_j dx \\ &= I_{31}+I_{32}, \end{aligned} \tag{10}$$

we have

$$\begin{aligned} |I_{31}| &\le \frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}|\partial_i u_k\|\partial_k b_j\|\partial_i b_j|dx \\ &\le \frac{1}{2}\int_{R^3}|\nabla_h u\|\nabla b|^2\,dx \\ &\le \frac{1}{2}\|\nabla_h u\|_{L^{2\gamma}}\|\nabla b\| \\ &\le C_3\|\nabla_h u\|_{L^{2\gamma}}\|\Delta b\|_{L^2}^{\frac{3}{2\gamma}}\|\nabla b\|_{L^2}^{\frac{4\gamma-3}{2\gamma}} \\ &\le \frac{1}{32}\|\Delta b\|_{L^2}^2+C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla b\|_{L^2}^2 \\ &\le \frac{1}{32}\|\Delta b\|_{L^2}^2+C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2, \end{aligned}$$

where C_4 is a positive constant constant;

$$\begin{aligned} |I_{32}| &\le \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{3}\int_{R^3}|\partial_3 u_k\|\partial_k b_j\|\partial_3 b_j|dx+\frac{1}{2}\sum_{k=1}^{3}\int_{R^3}|\partial_3 u_k\|\partial_k b_3\|\partial_3 b_3|dx \\ &\le \frac{1}{2}\int_{R^3}|\nabla_h u\|\nabla b|^2\,dx+\frac{1}{2}\int_{R^3}|\partial_3 b_3\|\nabla b\|\nabla u|dx \\ &= I_{321}+I_{322}. \end{aligned}$$

Since I_{321} is treated similarly to $|I_{31}|$, it gets

$$\begin{aligned} I_{321} &\le \frac{1}{32}\|\Delta b\|_{L^2}^2+C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2; \\ I_{322} &\le \frac{1}{2}\|\partial_3 b_3\|_{L^{2\gamma}}\|\nabla b\|_{L^{\frac{4\gamma}{2\gamma-1}}}\|\nabla u\|_{L^{\frac{4\gamma}{2\gamma-1}}} \\ &\le C_5\|\partial_3 b_3\|_{L^{2\gamma}}\|\Delta b\|_{L^2}^{\frac{3}{4\gamma}}\|\nabla b\|_{L^2}^{\frac{4\gamma-3}{4\gamma}}\|\Delta u\|_{L^2}^{\frac{3}{4\gamma}}\|\nabla u\|_{L^2}^{\frac{4\gamma-3}{4\gamma}} \\ &\le \frac{1}{8}\|\Delta u\|_{L^2}^2+\frac{1}{16}\|\Delta b\|_{L^2}^2+C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla u\|_{L^2}\|\nabla b\|_{L^2} \end{aligned}$$

$$\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{16}\|\Delta b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

Then we have

$$|I_{32}| \leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{3}{32}\|\Delta b\|_{L^2}^2 + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

Combining estimates of $|I_{31}|$ and $|I_{32}|$ with (10) yields

$$|I_3| \leq \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

Similarly, it follows

$$|I_4| \leq \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

Using estimates of $|I_i|\,(i=1,2,3,4)$ into (9), we have

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \\
\leq\ & 2(C_7\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}} + 2C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + 2m\|\nabla u\|_{L^2}^2 \\
\leq\ & 2(C_7\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}} + 2C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}} + m)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2),
\end{aligned}$$

where C_7 is a positive constant. Since $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 3$ implies $0 \leq \frac{2\gamma}{3\gamma-3} \leq \alpha$ we obtain from Gronwall›s inequality,

$$\begin{aligned}
&\sup_{0\leq t\leq T}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int_0^T(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)dt \\
\leq\ & (\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) exp\{2\int_0^T(C_7\|\nabla_h u\|_{L^{2\gamma}}^{2\alpha} + 2C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{2\alpha} + m)dt\}.
\end{aligned} \tag{11}$$

Noting $\nabla_h u, \partial_3 b_3 \in L^{2\alpha,2\gamma}$, we choose $\|\nabla_h u\|_{L^{2\gamma}}^{2\alpha}$ and $\|\partial_3 b_3\|_{L^{2\gamma}}^{2\alpha}$ being sufficiently small such that

$$exp\{2\int_0^T(C_7\|\nabla_h u\|_{L^{2\gamma}}^{2\alpha} + 2C_6\|\partial_3 b_3\|_{L^{2\gamma}}^{2\alpha} + m)dt\} \leq C.$$

Using it into (11), it follows the required (6).

If $\gamma = \infty$, then the corresponding terms $|I_i|\,(i=1,2,3,4)$ become

$$\begin{aligned}
|I_1| &\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + 2C_2\|\nabla_h u\|_{L^\infty}\|\nabla u\|_{L^2}^2, \\
|I_2| &\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + 2C_2\|\nabla_h u\|_{L^\infty}\|\nabla u\|_{L^2}^2,
\end{aligned}$$

$$|I_3| \le \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^\infty}\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^\infty}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2),$$

$$|I_4| \le \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^\infty}\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^\infty}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2);$$

From these estimates and (9), we have

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)$$

$$\le 2(C_7\|\nabla_h u\|_{L^\infty} + 2C_6\|\partial_3 b_3\|_{L^\infty} + m)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

Now using Gronwall›s inequality and choosing $\|\nabla_h u\|_{L^{1,\infty}}$ and $\|\partial_3 b_3\|_{L^{1,\infty}}$ being sufficient small, we get (6).

If $\gamma = 1$, then

$$I_1| \le \frac{1}{8}\|\Delta u\|_{L^2}^2 + 2C_2\|\nabla_h u\|_{L^2}^4\|\nabla u\|_{L^2}^2,$$

$$I_2| \le \frac{1}{8}\|\Delta u\|_{L^2}^2 + 2C_2\|\nabla_h u\|_{L^2}^4\|\nabla u\|_{L^2}^2,$$

$$I_3| \le \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^2}^4\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^2}^4(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2),$$

$$I_4| \le \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^2}^4\|\nabla b\|_{L^2}^2 + C_6\|\partial_3 b_3\|_{L^2}^4(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2);$$

Putting these estimates into (9), we have

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)$$

$$\le 2(C_7\|\nabla_h u\|_{L^2}^4 + 2C_6\|\partial_3 b_3\|_{L^2}^4 + m)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

Now using the Gronwall›s inequality and choosing $\|\nabla_h u\|_{L^{4,2}}$ and $\|\partial_3 b_3\|_{L^{4,2}}$ being sufficient small, we also get (6).

Proof of Theorem 1: Smoothness on $[0,T]$ of (u,b) is followed by (7).

Proof of Theorem 2

To prove Theorem 2, we first show the following lemma.

Lemma 2: Suppose that $(u_0, b_0) \in H^s(R^3), \quad s \ge 3$ and $\nabla \cdot u_0 = 0 = \nabla \cdot b_0$ in the sense of distribution. Assume that $(\nabla_h u, \nabla_h b) \in L^{2\alpha,2\gamma}$ with $\frac{2}{\alpha} + \frac{3}{\gamma} \le 3, \quad 1 \le \gamma \le \infty$, or $\|\nabla_h u\|_{L^{1,\infty}}$, $|\nabla_h u\|_{L^{4,2}}$, $\|\nabla_h b\|_{L^{1,\infty}}$ and $\|\nabla_h b\|_{L^{4,2}}$ are sufficiently small, then

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2), \tag{12}$$

where C depends on the $m, \ T$, norms of $\nabla_h u$ and $\nabla_h b$.

Proof: Now let us begin with (9), then I_1 and I_2 are estimated similarly to the way in Lemma 1. To I_3, we have

$$\begin{aligned} I_3 &= -\frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}\partial_i u_k \partial_k b_j \partial_i b_j dx - \frac{1}{2}\sum_{k=1}^{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 u_k \partial_k b_j \partial_3 b_j dx - \frac{1}{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 u_3 \partial_3 b_j \partial_3 b_j dx \\ &= -\frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}\partial_i u_k \partial_k b_j \partial_i b_j dx - \frac{1}{2}\sum_{k=1}^{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 u_k \partial_k b_j \partial_3 b_j dx + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{R^3}\partial_i u_i |\partial_3 b_j|^2 dx \\ &= I_{31} + I_{32} + I_{33}. \end{aligned} \tag{13}$$

Since

$$
\begin{aligned}
|I_{31}| &\le \frac{1}{2}\|\nabla_h u\|_{L^{2\gamma}}\|\nabla b\|^2_{L^{\frac{4\gamma}{2\gamma-1}}} \\
&\le C_3\|\nabla_h u\|_{L^{2\gamma}}\|\Delta b\|_{L^2}^{\frac{3}{2\gamma}}\|\nabla b\|_{L^2}^{\frac{4\gamma-3}{2\gamma}} \\
&\le \frac{1}{32}\|\Delta b\|_{L^2}^2 + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla b\|_{L^2}^2 \\
&\le \frac{1}{32}\|\Delta b\|_{L^2}^2 + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2,
\end{aligned}
$$

where C_4 is a constant;

$$
\begin{aligned}
|I_{32}| &\le \frac{1}{2}\sum_{k=1}^{2}\sum_{j=1}^{3}\int_{R^3}|\partial_3 u_k\|\partial_k b_j\|\partial_3 b_j|\,dx \\
&\le \frac{1}{2}\int_{R^3}|\nabla u|\nabla_h b\|\nabla b|\,dx \\
&\le \frac{1}{2}\|\nabla_h b\|_{L^{2\gamma}}\|\nabla b\|_{L^{\frac{4\gamma}{2\gamma-1}}}\|\nabla u\|_{L^{\frac{4\gamma}{2\gamma-1}}} \\
&\le C_5\|\nabla_h b\|_{L^{2\gamma}}\|\Delta b\|_{L^2}^{\frac{3}{4\gamma}}\|\nabla b\|_{L^2}^{\frac{4\gamma-3}{4\gamma}}\|\Delta u\|_{L^2}^{\frac{3}{4\gamma}}\|\nabla u\|_{L^2}^{\frac{4\gamma-3}{4\gamma}} \\
&\le \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{16}\|\Delta b\|_{L^2}^2 + C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla u\|_{L^2}\|\nabla b\|_{L^2} \\
&\le \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{16}\|\Delta b\|_{L^2}^2 + C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2),
\end{aligned}
$$

where C_6 is a constant; similarly to $|I_{31}|$,

$$
|I_{33}| \le \frac{1}{32}\|\Delta b\|_{L^2}^2 + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2,
$$

we have

$$
|I_3| \le \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + 2C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2 + C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
$$

For I_4, we note

$$
\begin{aligned}
I_4 &= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}\partial_i b_k \partial_k u_j \partial_i b_j\,dx + \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 b_k \partial_k u_j \partial_3 b_j\,dx \\
&+ \sum_{j=1}^{3}\int_{R^3}\partial_3 b_3 \partial_3 u_j \partial_3 b_j\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}\partial_i b_k \partial_k u_j \partial_i b_j dx + \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{R^3}\partial_3 b_k \partial_k u_j \partial_3 b_j dx \\
&- \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{R^3}\partial_i b_i \partial_3 u_j \partial_3 b_j dx \\
&= I_{41} + I_{42} + I_{43}.
\end{aligned} \tag{14}
$$

Since

$$
\begin{aligned}
|I_{41}| &\le \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}|\partial_i b_k||\partial_k u_j||\partial_i b_j|\,dx \\
&\le \int_{R^3}|\nabla_h b||\nabla u||\nabla b|\,dx \\
&\le \|\nabla_h b\|_{L^{2\gamma}}\|\nabla u\|_{L^{\frac{4\gamma}{2\gamma-1}}}\|\nabla b\|_{L^{\frac{4\gamma}{2\gamma-1}}} \\
&\le \frac{1}{16}\|\Delta u\|_{L^2}^2 + \frac{1}{32}\|\Delta b\|_{L^2}^2 + C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2); \\
|I_{42}| &\le \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{R^3}|\partial_3 b_k||\partial_k u_j||\partial_3 b_j|\,dx \\
&\le \int_{R^3}|\nabla_h u||\nabla b|^2\,dx \\
&\le C_3\|\nabla_h u\|_{L^{2\gamma}}\|\nabla b\|_{L^{\frac{4\gamma}{2\gamma-1}}}^2
\end{aligned}
$$

$$
\le \frac{1}{16}\|\Delta b\|_{L^2}^2 + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla b\|_{L^2}^2
$$

$$
\begin{aligned}
&\le \frac{1}{16}\|\Delta b\|_{L^2}^2 + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2; \\
|I_{43}| &\le \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{R^3}|\partial_i b_j||\partial_3 u_j||\partial_3 b_j|\,dx \\
&\le \int_{R^3}|\nabla_h b||\nabla u||\nabla b|\,dx \\
&\le \|\nabla_h b\|_{L^{2\gamma}}\|\nabla u\|_{L^{\frac{4\gamma}{2\gamma-1}}}\|\nabla b\|_{L^{\frac{4\gamma}{2\gamma-1}}} \\
&\le \frac{1}{16}\|\Delta u\|_{L^2}^2 + \frac{1}{32}\|\Delta b\|_{L^2}^2 + C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2),
\end{aligned}
$$

it yields by combining $|I_{41}|$, $|I_{42}|$, and $|I_{43}|$ with (14),

$$
|I_4| \le \frac{1}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + C_4\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla b\|_{L^2}^2 + 2C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
$$

According to these estimates for $|I_i|\,(i=1,2,3,4)$ into (9), we have

$$\begin{aligned}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 &+ \|\nabla b\|_{L^2}^2)+(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\\ &\le 2(C_7\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}+3C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+2m\|\nabla u\|_{L^2}^2\\ &\le 2(C_7\|\nabla_h u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}+3C_6\|\nabla_h b\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}+m)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2).\end{aligned}$$

Since $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3$ implies $0\le\frac{2\gamma}{3\gamma-3}\le\alpha$, we obtain by using the Gronwall›s inequality,

$$\begin{aligned}\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 &+ \|\nabla b\|_{L^2}^2)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)dt\\ &\le (\|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2)exp\{2\int_0^T(C_7\|\nabla_h u\|_{L^{2\gamma}}^{2\alpha}+3C_6\|\nabla_h b\|_{L^{2\gamma}}^{2\alpha}+m)dt\}.\end{aligned} \tag{15}$$

As $\nabla_h u,\nabla_h b\in L^{2\alpha,2\gamma}$, we choose $\|\nabla_h u\|_{L^{2\gamma}}^{2\alpha}$ and $\|\nabla_h b\|_{L^{2\gamma}}^{2\alpha}$ being sufficiently small such that

$$exp\{2\int_0^T(C_7\|\nabla_h u\|_{L^{2\gamma}}^{2\alpha}+3C_6\|\nabla_h b\|_{L^{2\gamma}}^{2\alpha}+m)dt\}\le C.$$

From this, we get (12). For the cases $\gamma=1$ and $\gamma=\infty$, the estimates are the same as Lemma 1. So (13) is proved.

Proof of Theorem 2: Conclusions are followed by Lemma 2.

Proof of Theorem 3

In this section, we denote $\Delta_2 f=\sum_{i=1}^2\frac{\partial^2 f}{\partial x^i}$.

Lemma 3: Suppose $(u_0,b_0)\in H^s(R^3),\ \ s\ge 3$ and $\nabla\cdot u_0=0=\nabla\cdot b_0$, in the sense of distribution.

Assume that $(\partial_3 u,\nabla u_3)\in L^{2\alpha,2\gamma}$ and $(b,u_3,\partial_3 b,\nabla b_3)\in L^{4\alpha,4\gamma}$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3,\quad 1\le\gamma\le\infty$, or

$\|\partial_3 u\|_{L^{1,\infty}},\quad \|\nabla u_3\|_{L^{1,\infty}},\quad \|u_3\|_{L^{1,\infty}},\quad \|b\|_{L^{1,\infty}},\quad \|\nabla b_3\|_{L^{1,\infty}},\quad \|\partial_3 u\|_{L^{1,\infty}},\quad \|\partial_3 u\|_{L^{4,2}},\quad \|\nabla u_3\|_{L^{4,2}},\quad \|u_3\|_{L^{\frac{8}{5},4}},$

$\|\nabla b_3\|_{L^{\frac{8}{5},4}},\quad \|\partial_3 b\|_{L^{\frac{8}{5},4}}$, and $\|b\|_{L^{\frac{8}{5},4}}$ are sufficiently small, then

$$\sup_{0\le t\le T}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+\int_0^T(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)dt\le C(\|\nabla_h u_0\|_{L^2}^2+\|\nabla_h b_0\|_{L^2}^2), \tag{16}$$

where C depends on the $m,\ \ T$, norms of $\nabla u_3,\ \ \partial_3 u,\ \ b,\ \ u_3,\ \ \nabla b_3$ and $\partial_3 b$.

Proof: Multiplying (1) by $\Delta_2 u$ and integrating over R^3, it follows

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h u\|_{L^2}^2=\int_{R^3}(u\cdot\nabla u)\Delta_2 u dx-\int_{R^3}(b\cdot\nabla b)\Delta_2 u dx-m\|\nabla_h u\|_{L^2}^2. \tag{17}$$

Similarly, multiplying (2) by $\Delta_2 b$ and integrating over R^3, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h b\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2=\int_{R^3}(u\cdot\nabla b)\Delta_2 b dx-\int_{R^3}(b\cdot\nabla u)\Delta_2 b dx. \tag{18}$$

Combining (17) and (18), we get

$$\frac{1}{2}\frac{d}{dt}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2) + (\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)=\int_{R^3}(u\cdot\nabla u)\Delta_2 u dx-\int_{R^3}(b\cdot\nabla b)\Delta_2 u dx$$

$$+ \int_{R^3}(u\cdot\nabla b)\Delta_2 b dx-\int_{R^3}(b\cdot\nabla u)\Delta_2 b dx - m\|\nabla_h u\|_{L^2}^2$$

$$\leq |J_1|+|J_2|+|J_3|+|J_4|+m\|\nabla_h u\|_{L^2}^2, \tag{19}$$

where $J_1 = \int_{R^3}(u\cdot\nabla u)\Delta_2 u dx$, $J_2=\int_{R^3}(b\cdot\nabla b)\Delta_2 u dx$, $J_3=\int_{R^3}(u\cdot\nabla b)\Delta_2 b dx$ and $J_4 = \int_{R^3}(b\cdot\nabla u)\Delta_2 b dx.$

Now let us estimate $|J_i|$, where $i=1,2,3,4$:

$$\begin{aligned} J_1 &= \sum_{i,j=1}^{3}\int_{R^3}(u_i\partial_i u_j)\Delta_2 u_j dx \\ &= \sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}(u_i\partial_i u_j)\Delta_2 u_j dx+\sum_{i=1}^{3}\int_{R^3}(u_i\partial_i u_3)\Delta_2 u_3 dx \\ &= \sum_{i,j=1}^{2}\int_{R^3}(u_i\partial_i u_j)\Delta_2 u_j dx+\sum_{j=1}^{2}\int_{R^3}(u_3\partial_3 u_j)\Delta_2 u_j dx+\sum_{i=1}^{3}\int_{R^3}(u_i\partial_i u_3)\Delta_2 u_3 dx \\ &= J_{11}+J_{12}+J_{13}. \end{aligned}$$

We first estimate J_{12}. After the integration by parts , we have

$$\begin{aligned} J_{12} &= -\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3}\partial_i(u_3\partial_3 u_j)\partial_i u_j dx \\ &= -\sum_{i,j=1}^{2}\int_{R^3}\partial_i u_3\partial_3 u_j\partial_i u_j dx-\sum_{i,j=1}^{2}\int_{R^3}u_3\partial_i\partial_3 u_j\partial_i u_j dx \\ &= -\sum_{i,j=1}^{2}\int_{R^3}\partial_i u_3\partial_3 u_j\partial_i u_j dx-\frac{1}{2}\sum_{i,j=1}^{2}\int_{R^3}u_3\partial_3(\partial_i u_j)^2 dx \end{aligned}$$

$$\begin{aligned} &= -\sum_{i,j=1}^{2}\int_{R^3}\partial_i u_3\partial_3 u_j\partial_i u_j dx+\frac{1}{2}\sum_{i,j=1}^{2}\int_{R^3}\partial_3 u_3(\partial_i u_j)^2 dx \\ &= J_{121}+J_{122}. \end{aligned} \tag{20}$$

Using GagliardoNirenberg›s inequality and Young›s inequality, it implies

$$\begin{aligned} |J_{121}| &\leq \sum_{i,j=1}^{2}\int_{R^3}|\partial_i u_3\|\partial_3 u_j\|\partial_i u_j|dx \\ &\leq \int_{R^3}|\partial_3 u\|\nabla_h u|^2 dx \\ &\leq \|\partial_3 u\|_{L^{2\gamma}}\|\nabla_h u\|^2_{L^{\frac{4\gamma}{2\gamma-1}}} \\ &\leq C_1\|\partial_3 u\|_{L^{2\gamma}}\|\nabla\nabla_h u\|_{L^2}^{\frac{3}{2\gamma}}\|\nabla_h u\|_{L^2}^{\frac{4\gamma-3}{2\gamma}} \end{aligned}$$

$$\leq \frac{1}{64}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla_h u\|_{L^2}^2$$

$$\leq \frac{1}{64}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2,$$

where C_2 is positive a constant. Also we have

$$\begin{aligned} |J_{122}| &\leq \sum_{i,j=1}^{2}\int_{R^3}|\partial_3 u_3\|\partial_i u_j|^2\,dx \\ &\leq \int_{R^3}|\partial_3 u\|\nabla_h u|^2\,dx \\ &\leq \frac{1}{64}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2. \end{aligned}$$

Combining $|J_{121}$ and $|J_{122}|$ with (20), it yields

$$|J_{12}| \leq \frac{1}{32}\|\nabla\nabla_h u\|_{L^2}^2 + 2C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2.$$

For J_{11}, we have

$$\begin{aligned} J_{11} &= -\sum_{k=1}^{2}\sum_{i,j=1}^{2}\int_{R^3}\partial_k(u_i\partial_i u_j)\partial_k u_j dx \\ &= -\sum_{i,j,k=1}^{2}\int_{R^3}\partial_k u_i\partial_i u_j\partial_k u_j dx - \sum_{i,j,k=1}^{2}\int_{R^3}u_i\partial_k\partial_i u_j\partial_k u_j dx \\ &= -\sum_{i,j,k=1}^{2}\int_{R^3}\partial_k u_i\partial_i u_j\partial_k u_j dx + \sum_{i,j,k=1}^{2}\int_{R^3}\partial_i(u_i\partial_k u_j)\partial_k u_j dx \end{aligned}$$

$$\begin{aligned} &= -\sum_{i,j,k=1}^{2}\int_{R^3}\partial_k u_i\partial_i u_j\partial_k u_j dx + \sum_{i,j,k=1}^{2}\int_{R^3}\partial_i u_i\partial_k u_j\partial_k u_j dx + \sum_{i,j,k=1}^{2}\int_{R^3}u_i\partial_i\partial_k u_j\partial_k u_j dx \\ &= -\sum_{i,j=1}^{2}\int_{R^3}\partial_1 u_i\partial_i u_j\partial_1 u_j dx - \sum_{i,j=1}^{2}\int_{R^3}\partial_2 u_i\partial_i u_j\partial_2 u_j dx + \sum_{i,j=1}^{2}\int_{R^3}\partial_i u_i\partial_1 u_j\partial_1 u_j dx \\ &+ \sum_{i,j=1}^{2}\int_{R^3}\partial_i u_i\partial_2 u_j\partial_2 u_j dx + \frac{1}{2}\sum_{i,j,k=1}^{2}\int_{R^3}u_i\partial_i(\partial_k u_j)^2 dx \\ &= \int_{R^3}\partial_3 u_3\partial_1 u_2\partial_2 u_1 dx - \int_{R^3}\partial_3 u_3\partial_1 u_1\partial_2 u_2 dx - \frac{1}{2}\sum_{i,j,k=1}^{2}\int_{R^3}\partial_i u_i(\partial_k u_j)^2 dx \\ &= \int_{R^3}\partial_3 u_3\partial_1 u_2\partial_2 u_1 dx - \int_{R^3}\partial_3 u_3\partial_1 u_1\partial_2 u_2 dx + \frac{1}{2}\sum_{j,k=1}^{2}\int_{R^3}\partial_3 u_3(\partial_k u_j)^2 dx \\ &= J_{111} + J_{112} + J_{113}. \end{aligned} \tag{21}$$

Since

$$|J_{111}| \leq \int_{R^3} |\partial_3 u_3| |\nabla_h u|^2 \, dx$$
$$\leq \frac{1}{96}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2;$$

$$|J_{112}| \leq \frac{1}{96}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2;$$

$$|J_{113}| \leq \frac{1}{96}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2,$$

we obtain

$$|J_{11}| \leq \frac{1}{32}\|\nabla\nabla_h u\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2.$$

Noting

$$\begin{aligned}
J_{13} &\leq -\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}\partial_j(u_i\partial_i u_3)\partial_j u_3 dx \\
&= -\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}\partial_j u_i\partial_i u_3\partial_j u_3 dx - \sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}u_i\partial_j\partial_i u_3\partial_j u_3 dx \\
&= -\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}\partial_j u_i\partial_i u_3\partial_j u_3 dx - \frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}u_i\partial_i(\partial_j u_3)^2 dx \\
&= -\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}\partial_j u_i\partial_i u_3\partial_j u_3 dx + \frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}\partial_i u_i(\partial_j u_3)^2 dx \\
&= -\sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}\partial_j u_i\partial_i u_3\partial_j u_3 dx,
\end{aligned}$$

it follows

$$\begin{aligned}
|J_{13}| &\leq \sum_{j=1}^{2}\sum_{i=1}^{3}\int_{R^3}|\partial_j u_i||\partial_i u_3||\partial_j u_3| dx \\
&\leq \int_{R^3}|\nabla u_3||\nabla_h u|^2 dx \\
&\leq C_3\|\nabla u_3\|_{L^{2\gamma}}\|\nabla\nabla_h u\|_{L^2}^{\frac{3}{2\gamma}}\|\nabla_h u\|_{L^2}^{\frac{4\gamma-3}{2\gamma}} \\
&\leq \frac{1}{16}\|\nabla\nabla_h u\|_{L^2}^2 + C_4\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{4\gamma-3}}\|\nabla_h u\|_{L^2}^2 \\
&\leq \frac{1}{16}\|\nabla\nabla_h u\|_{L^2}^2 + C_4\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2.
\end{aligned}$$

Using estimates for $|J_{11}|$, $|J_{12}|$ and $|J_{13}|$, it yields

$$|J_1| \le \frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2 + (5C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}} + C_4\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}})\|\nabla_h u\|_{L^2}^2.$$

For J_2, we have

$$\begin{aligned} J_2 &= -\sum_{i,j=1}^{2}\int_{R^3} b_i\partial_i b_j \Delta_2 u_j dx - \sum_{j=1}^{2}\int_{R^3} b_3\partial_3 b_j \Delta_2 u_j dx - \sum_{i=1}^{3}\int_{R^3} b_i\partial_i b_3 \Delta_2 u_3 dx \\ &= J_{21} + J_{22} + J_{23}. \end{aligned} \tag{22}$$

Since

$$\begin{aligned} J_{21} &= \int_{R^3}\partial_2 b_2 \partial_1 b_1 \partial_3 u_3 dx - \int_{R^3}\partial_1 b_2 \partial_2 b_1 \partial_3 u_3 dx - \sum_{i,j,k=1}^{2}\int_{R^3} b_i \partial_k b_j \partial_i \partial_k u_j dx \\ &= J_{211} + J_{212} + J_{213}, \end{aligned} \tag{23}$$

it follows

$$\begin{aligned} |J_{211}| &\le \int_{R^3} |\partial_2 b_2||\partial_1 b_1||\partial_3 u_3| dx \\ &\le \int_{R^3} |\nabla u_3||\nabla_h b|^2 dx \\ &\le \|\nabla u_3\|_{2\gamma}\|\nabla_h b\|_{L^{\frac{4\gamma}{2\gamma-1}}}^2 \\ &\le \|\nabla u_3\|_{L^{2\gamma}}\|\nabla\nabla_h b\|_{L^2}^{\frac{3}{2\gamma}}\|\nabla_h b\|_{L^2}^{\frac{4\gamma-3}{2\gamma}} \\ &\le \frac{1}{96}\|\nabla\nabla_h b\|_{L^2}^2 + C_6\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2; \end{aligned}$$

$$\begin{aligned} |J_{212}| &\le \int_{R^3} |\partial_1 b_2||\partial_2 b_1||\partial_3 u_3| dx \\ &\le \int_{R^3} |\nabla u_3||\nabla_h b|^2 dx \\ &\le \frac{1}{96}\|\nabla\nabla_h b\|_{L^2}^2 + C_6\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2; \end{aligned}$$

$$\begin{aligned} |J_{213}| &\le \sum_{i,j,k=1}^{2}\int_{R^3} |b_i||\partial_k b_j||\partial_i\partial_k u_j| dx \\ &\le \int_{R^3} |b||\nabla_h b||\nabla\nabla_h u| dx \\ &\le \|b\|_{L^{4\gamma}}\|\nabla\nabla_h u\|_{L^2}\|\nabla_h b\|_{L^{\frac{4\gamma}{2\gamma-1}}} \\ &\le C_7\|b\|_{L^{4\gamma}}\|\nabla\nabla_h u\|_{L^2}\|\nabla\nabla_h b\|_{L^2}^{\frac{3}{4\gamma}}\|\nabla_h b\|_{L^2}^{\frac{4\gamma-3}{4\gamma}} \\ &\le \frac{1}{24}\|\nabla\nabla_h u\|_{L^2}^2 + \frac{1}{48}\|\nabla\nabla_h b\|_{L^2}^2 + C_8\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{4\gamma-3}}\|\nabla_h b\|_{L^2}^2 \\ &\le \frac{1}{24}\|\nabla\nabla_h u\|_{L^2}^2 + \frac{1}{48}\|\nabla\nabla_h b\|_{L^2}^2 + C_8\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2, \end{aligned}$$

and then

$$|J_{21}|\le\frac{1}{24}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+(2C_6\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}+C_8\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}})\|\nabla_h b\|_{L^2}^2.$$

Noting

$$\begin{aligned}
J_{22} &= \sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3}\partial_i b_3\partial_3 b_j\partial_i u_j dx-\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3}b_3\partial_i\partial_3 b_j\partial_i u_j dx\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3}\partial_i b_3\partial_3 b_j\partial_i u_j dx-\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3}\partial_3(b_3\partial_i u_j)\partial_i b_j dx\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3}\partial_i b_3\partial_3 b_j\partial_i u_j dx-\sum_{i,j=1}^{2}\int_{R^3}\partial_3 b_3\partial_i u_j\partial_i b_j dx-\sum_{i,j=1}^{2}\int_{R^3}b_3\partial_3\partial_i u_j\partial_i b_j dx\\
&= J_{221}+J_{222}+J_{223},
\end{aligned}\tag{24}$$

we estimate one by one

$$\begin{aligned}
|J_{221}| &\le \sum_{i,j=1}^{2}\int_{R^3}|\partial_i b_3\|\partial_3 b_j\|\partial_i u_j|dx\\
&\le \int_{R^3}|\partial_3 b_j\|\nabla_h b\|\nabla_h u|dx\\
&\le \|\partial_3 b\|_{L^{4\gamma}}\|\nabla_h b\|_{L^{\frac{8\gamma}{4\gamma-1}}}\|\nabla_h u\|_{L^{\frac{8\gamma}{4\gamma-1}}}\\
&\le C_9\|\partial_3 b\|_{L^{4\gamma}}\|\nabla\nabla_h b\|_{L^2}^{\frac{3}{8\gamma}}\|\nabla_h b\|_{L^2}^{1-\frac{3}{8\gamma}}\|\nabla\nabla_h u\|_{L^2}^{\frac{3}{8\gamma}}\|\nabla_h u\|_{L^2}^{1-\frac{3}{8\gamma}}\\
&\le \frac{1}{96}\|\nabla\nabla_h u\|_{L^2}^2+\frac{1}{96}\|\nabla\nabla_h b\|_{L^2}^2+C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{8\gamma-3}}\|\nabla_h u\|_{L^2}\|\nabla_h b\|_{L^2}\\
&\le \frac{1}{96}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2),
\end{aligned}$$

where C_{10} is a positive constant; similarly to $|J_{221}|$,

$$\begin{aligned}
|J_{222}| &\le \frac{1}{96}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2);\\
|J_{223}| &\le \sum_{i,j=1}^{2}\int_{R^3}|b_3\|\partial_i b_j\|\partial_3\partial_i u_j|dx\\
&\le \int_{R^3}|b\|\nabla_h b\|\nabla\nabla_h u|dx\\
&\le \frac{1}{48}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+C_{11}\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2,
\end{aligned}$$

and obtain

$$|J_{22}| \le \frac{1}{24}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+2C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+C_{11}\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2.$$

Since

$$J_{23} = \sum_{i=1}^{3}\sum_{j=1}^{2}\int_{R^3} u_3 \partial_i \partial_j b_3 \partial_j b_i dx,$$

it follows

$$\begin{aligned}
|J_{23}| &\leq \sum_{i=1}^{3}\sum_{j=1}^{2}\int_{R^3} |u_3|\,|\partial_i \partial_j b_3|\,|\partial_j b_i|\, dx \\
&\leq \int_{R^3} |u_3|\,|\nabla\nabla_h b|\,|\nabla_h b|\, dx \\
&\leq \|u_3\|_{L^{4\gamma}} \|\nabla\nabla_h b\|_{L^2} \|\nabla_h b\|_{L^{\frac{4\gamma}{2\gamma-1}}} \\
&\leq C_{12} \|u_3\|_{L^{4\gamma}} \|\nabla\nabla_h b\|_{L^2}^{\frac{4\gamma+3}{4\gamma}} \|\nabla_h b\|_{L^2}^{\frac{4\gamma-3}{4\gamma}} \\
&\leq \frac{1}{24}\|\nabla\nabla_h b\|_{L^2}^2 + C_{13}\|u_3\|_{L^{4\gamma}}^{\frac{8\gamma}{4\gamma-3}} \|\nabla_h b\|_{L^2}^2 \\
&\leq \frac{1}{24}\|\nabla\nabla_h b\|_{L^2}^2 + C_{13}\|u_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}} \|\nabla_h b\|_{L^2}^2,
\end{aligned}$$

where C_{13} is a positive constant.

Combining estimates for $|J_{21}|, |J_{22}|$ and $|J_{23}|$ in (22), we have

$$\begin{aligned}
|J_2| &\leq \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_6\|\nabla u_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}} + C_{14}\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}} \\
&+ C_{12}\|u_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}})\|\nabla_h b\|_{L^2}^2 + 2C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2),
\end{aligned}$$

where $C_{14} = C_8 + C_{11}$. Similar to J_1, we can divide J_3 into the following form

$$\begin{aligned}
J_3 &= \sum_{j=1}^{2}\int_{R^3} u_3 \partial_3 b_j \Delta_2 b_j dx + \sum_{i,j=1}^{2}\int_{R^3} u_i \partial_i b_j \Delta_2 b_j dx + \sum_{i=1}^{3}\int_{R^3} u_i \partial_i b_3 \Delta_2 b_3 dx \\
&= J_{31} + J_{32} + J_{33}.
\end{aligned} \tag{25}$$

Integrating by parts of J_{31}, it gets

$$\begin{aligned}
J_{31} &= -\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3} \partial_i u_3 \partial_3 b_j \partial_i b_j dx + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{R^3} \partial_3 u_3 (\partial_i b_j)^2 dx \\
&= J_{311} + J_{312}.
\end{aligned}$$

Since

$$\begin{aligned}
|J_{311}| &\leq \int_{R^3} |\nabla_h u\|\partial_3 b\|\nabla_h b|\, dx \\
&\leq \frac{1}{64}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2); \\
|J_{312}| &\leq \frac{1}{2}\int_{R^3} |\partial_3 u\|\nabla_h b|^2\, dx \\
&\leq \frac{1}{32}\|\nabla\nabla_h b\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2,
\end{aligned}$$

we see

$$\begin{aligned}
|J_{31}| &\leq \frac{1}{64}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) \\
&+ \frac{1}{32}\|\nabla\nabla_h b\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|J_{32}| &\leq \frac{1}{64}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) \\
&+ \frac{1}{32}\|\nabla\nabla_h b\|_{L^2}^2 + C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2;
\end{aligned}$$

$$\begin{aligned}
|J_{33}| &\leq \int_{R^3} |\nabla_h u\|\nabla b_3\|\nabla_h b|\, dx \\
&\leq \frac{1}{32}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + C_{10}\|\nabla b_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2).
\end{aligned}$$

Combining these estimates with (25), we have

$$\begin{aligned}
|J_3| &\leq \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}} + C_{10}\|\nabla b_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}})(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) \\
&+ 2C_2\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2.
\end{aligned}$$

Now we estimate J_4 and have

$$\begin{aligned} J_4 &= -\sum_{i,j=1}^{2}\int_{R^3} b_i\partial_i u_j \Delta_2 b_j dx - \sum_{i=1}^{3}\int_{R^3} b_i\partial_i u_3 \Delta_2 b_3 dx - \sum_{j=1}^{2}\int_{R^3} b_3\partial_3 u_j \Delta_2 b_j dx \\ &= J_{41}+J_{42}+J_{43}. \end{aligned}$$

It follows

$$|J_{41}| \le \frac{1}{24}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+(2C_{15}\|\nabla b_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}+C_{16}\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}})\|\nabla_h b\|_{L^2}^2,$$

$$|J_{42}| \le \frac{1}{24}\|\nabla\nabla_h b\|_{L^2}^2+C_{13}\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2,$$

$$\begin{aligned} |J_{43}| &\le \frac{1}{24}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2) \\ &+ C_8\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}\|\nabla_h u\|_{L^2}^2+C_6\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}\|\nabla_h b\|_{L^2}^2, \end{aligned}$$

which implies

$$\begin{aligned} |J_4| &\le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+(2C_{15}\|\nabla b_3\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}+C_{17}\|b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}+C_6\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}})\|\nabla_h b\|_{L^2}^2 \\ &+ C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2), \end{aligned}$$

where $C_{17}=C_{13}+C_{16}$. Using estimates for $|J_i|\,(i=1,2,3,4)$ into we have

$$\begin{aligned} &\frac{d}{dt}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2)+(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2) \\ &\le 2(C_{19}\|\partial_3 u\|_{L^{2\gamma}}^{\frac{4\gamma}{3\gamma-3}}+C_{20}\|\nabla u_3\|_{L^{4\gamma}}^{2\alpha}+C_{21}\|b\|_{L^{4\gamma}}^{\frac{4\gamma}{3\gamma-3}}+C_{12}\|u_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}} \\ &+ C_{18}\|\nabla b_3\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}+5C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{\frac{8\gamma}{3\gamma-3}}+m)(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2), \end{aligned}$$

where $C_{18}=C_{10}+2C_{15}$, $C_{19}=7C_2+C_5$, $C_{20}=2C_4+2C_6$, and $C_{21}=2C_{14}+C_{17}$. Since $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3$ implies $0\le\frac{2\gamma}{3\gamma-3}\le\alpha$, we obtain

$$\begin{aligned} &\sup_{0\le t\le T}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2)+\int_0^T(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)dt \\ &\le (\|\nabla_h u_0\|_{L^2}^2+\|\nabla_h b_0\|_{L^2}^2)exp\{2\int_0^T(C_{19}\|\partial_3 u\|_{L^{2\gamma}}^{2\alpha}+C_{20}\|\nabla u_3\|_{L^{2\gamma}}^{2\alpha} \\ &+ C_{21}\|b\|_{L^{4\gamma}}^{4\alpha}+C_{12}\|u_3\|_{L^{4\gamma}}^{4\alpha}+C_{18}\|\nabla b_3\|_{L^{4\gamma}}^{4\alpha}+5C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{4\alpha}+m)dt\}. \end{aligned} \tag{26}$$

As $(\partial_3 u, \nabla u_3) \in L^{2\alpha,2\gamma}$ and $(u_3, b, \partial_3 b, \nabla b_3) \in L^{4\alpha,4\gamma}$, we choose $\|\partial_3 u\|_{L^{2\gamma}}^{2\alpha}$, $\|\nabla u_3\|_{L^{2\gamma}}^{2\alpha}$, $\|u_3\|_{L^{4\gamma}}^{4\alpha}$, $\|b\|_{L^{4\gamma}}^{4\alpha}$, $\|\partial_3 b\|_{L^{4\gamma}}^{4\alpha}$ and $\|\nabla b_3\|_{L^{4\gamma}}^{4\alpha}$ being sufficiently small such that

$$exp\{2\int_0^T (C_{19}\|\partial_3 u\|_{L^{2\gamma}}^{2\alpha} + C_{20}\|\nabla u_3\|_{L^{2\gamma}}^{2\alpha} + C_{21}\|b\|_{L^{4\gamma}}^{4\alpha} + C_{12}\|u_3\|_{L^{4\gamma}}^{4\alpha} + C_{18}\|\nabla b_3\|_{L^{2\gamma}}^{2\alpha}$$
$$+ 5C_{10}\|\partial_3 b\|_{L^{4\gamma}}^{4\alpha} + m)dt\} \le C.$$

Using into (26) , we get (16).

If $\gamma = \infty$, then

$$J_1| \le \frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2 + (5C_2\|\partial_3 u\|_{L^\infty} + C_4\|\nabla u_3\|_{L^\infty})\|\nabla_h u\|_{L^2}^2,$$

$$J_2| \le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_6\|\nabla u_3\|_{L^\infty} + C_{13}\|\nabla b\|_{L^\infty} + C_{12}\|u_3\|_{L^\infty})\|\nabla_h b\|_{L^2}^2$$
$$+ C_{10}\|\partial_3 b\|_{L^\infty}^8(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2),$$

$$J_3| \le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_{10}\|\partial_3 b\|_{L^\infty} + C_{10}\|\nabla b_3\|_{L^\infty})(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2)$$
$$+ 2C_2\|\partial_3 u\|_{L^\infty}\|\nabla_h u\|_{L^2}^2,$$

$$J_4| \le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_{15}\|\nabla b\|_{L^\infty} + C_{17}\|b\|_{L^\infty} + C_6\|\partial_3 u\|_{L^\infty})\|\nabla_h b\|_{L^2}^2$$
$$+ C_{10}\|\partial_3 b\|_{L^\infty}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2).$$

From these estimates for $|J_i|(i = 1,2,3,4)$ and (19), we have

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + (\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) \le 2(C_{19}\|\partial_3 u\|_{L^\infty} + C_{20}\|\nabla u_3\|_{L^\infty} + C_{21}\|b\|_{L^\infty}$$
$$+ C_{12}\|u_3\|_{L^\infty}^2 + C_{18}\|\nabla b_3\|_{L^\infty} + 5C_{10}\|\partial_3 b\|_{L^\infty} + m)(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2).$$

Now using Gronwall's inequality, choosing $\|\partial_3 u\|_{L^{1,\infty}}$, $\|\nabla u_3\|_{L^{1,\infty}}$, $\|u_3\|_{L^{1,\infty}}$, $\|b\|_{L^{1,\infty}}$, $\|\nabla b_3\|_{L^{1,\infty}}$ and being $\|\partial_3 b\|_{L^{1,\infty}}$ sufficient small, we get (16).

If $\gamma = 1$, then

$$|J_1| \le \frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2 + (5C_2\|\partial_3 u\|_{L^2}^4 + C_4\|\nabla u_3\|_{L^2}^4)\|\nabla_h u\|_{L^2}^2,$$

$$|J_2| \le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_6\|\nabla u_3\|_{L^2}^4 + C_{13}\|\nabla b\|_{L^4}^{\frac{8}{5}} + C_{12}\|u_3\|_{L^4}^{\frac{8}{5}})\|\nabla_h b\|_{L^2}^2$$
$$+ C_{10}\|\partial_3 b\|_{L^4}^{\frac{8}{5}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2),$$

$$|J_3| \le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (C_{10}\|\partial_3 b\|_{L^4}^{\frac{8}{5}} + C_{10}\|\nabla b_3\|_{L^4}^{\frac{8}{5}})(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2)$$
$$+ 2C_2\|\partial_3 u\|_{L^2}^4\|\nabla_h u\|_{L^2}^2,$$

$$|J_4| \le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + (2C_{15}\|\nabla b_3\|_{L^4}^{\frac{8}{5}} + C_{17}\|b\|_{L^4}^{\frac{8}{5}} + C_6\|\partial_3 u\|_{L^2}^4)\|\nabla_h b\|_{L^2}^2$$

$$+ \quad C_{10}\|\partial_3 b\|_{L^4}^{\frac{8}{5}}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2).$$

Using these estimates of $|J_i|\,(i=1,2,3,4)$ into (19), it follows

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + (\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) \le 2(C_{19}\|\partial_3 u\|_{L^2}^4 + C_{20}\|\nabla u_3\|_{L^2}^4 + C_{21}\|b\|_{L^4}^{\frac{8}{5}}$$
$$+ \quad C_{12}\|u_3\|_{L^4}^{\frac{8}{5}} + C_{18}\|\nabla b_3\|_{L^4}^{\frac{8}{5}} + 5C_{10}\|\partial_3 b\|_{L^4}^{\frac{8}{5}} + m)(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2).$$

Now choosing $\|\partial_3 u\|_{L^{4,2}}$, $\|\nabla u_3\|_{L^{4,2}}$, $\|u_3\|_{L^{\frac{8}{5},4}}$, $\|b\|_{L^{\frac{8}{5},4}}$, $\|\nabla b_3\|_{L^{\frac{8}{5},4}}$, and $\|\partial_3 b\|_{L^{\frac{8}{5},4}}$ being sufficient small, we obtain (16).

Proof of Theorem 3: From Lemma 3, we immediately prove Theorem 3.

2. Conclusion

Here three new regularity criteria for three-dimensional flow of MHD fluid filling the porous medium are established. Assuming $(\nabla_h u, \partial_3 b_3) \in L^{2\alpha,2\gamma}$ (with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3,\ \ 1\le\gamma\le\infty$) the corresponding solution (u,b) remain smooth on $[0,T]$. Next it is seen that replacing $\partial_3 b_3$ by $\nabla_h b$ one also has the regularity criteria for solution (u,b) on $[0,T]$. Finally it is observed that letting $(\partial_3 u, \nabla u_3) \in L^{2\alpha,2\gamma}$ and $(u_3, b, \partial_3 b, \nabla b_3) \in L^{4\alpha,4\gamma}$ (with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 3,\ 1\le\gamma\le\infty$), the solution (u,b) remain smooth on $[0,T]$.

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بعض الشروط المنظمة لتدفق MHD ثلاثي الأبعاد

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خلاصة

يهدف هذا البحث إلى إيجاد بعض الشروط التنظيمية للحل الضعيف لمرور السوائل في وسط مسامي ثلاثي الأبعاد R3. يكون الحل الضعيف منتظم ووحيد إذا تحققت أحد الشروط التالية: $(\nabla_h u, \partial_3 b_3) \in L^{2\alpha,2\gamma}$ أو $(\nabla_h u, \nabla_h b) \in L^{2\alpha,2\gamma}$ أو $(\partial_3 u, \nabla u_3) \in L^{2\alpha,2\gamma}$ و $(u_3, b, \partial_3 b, \nabla b_3) \in L^{4\alpha,4\gamma}$ حيث. $\nabla_h = (\partial_1, \partial_2)$.