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On ϕ-recurrent almost Kenmotsu manifolds

YANING WANG* AND XIMIN LIU**

**College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, P. R. China.*

***School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, Liaoning, P. R. China*

**Corresponding author: Yaning Wang, e-mail: wyn051@163.com*

ABSTRACT

The object of this paper is to investigate ϕ-recurrent and ϕ-symmetric almost Kenmotsu manifolds with the characteristic vector fields belonging to some nullity distributions.

Keywords: Almost Kenmotsu manifold; ϕ-recurrence; ϕ-symmetry; generalized nullity distribution.

MSC Classification: 53C25, 53D15.

INTRODUCTION

Kenmotsu (1972) introduced a new class of almost contact metric manifolds, which are known as Kenmotsu manifolds nowadays, and proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature -1. Takahashi (1977) introduced the notion of local ϕ-symmetry, which is weaker than local symmetry in the context of Sasakian geometry. Generalizing the notion of local ϕ-symmetry, De *et al*. (2003) introduced the notion of ϕ-recurrence on Sasakian manifolds. Since then, many results on ϕ-recurrent and ϕ-symmetric Kenmotsu manifolds were obtained by some authors, for more related results in this framework we refer the reader to some recent papers by De (2008), De & Pathak (2004) and De *et al*. (2009a, 2009b).

On the other hand, the notion of k-nullity distribution was first introduced by Gray (1966) and Tanno (1978) in the study of Riemannian manifolds (M, g), which is defined for any $p \in M$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \qquad (1)$$

where X, Y denote arbitrary vectors in T_pM and $k \in \mathbb{R}$.

Recently, Blair *et al*. (1995) introduced a generalized notion of the k-nullity distribution named the (k, μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ as follows:

$$N_p(k,\mu)=\{Z\in T_pM : R(X,Y)Z = k[g(Y,Z)X-g(X,Z)Y] + \mu[g(Y,Z)hX-g(X,Z)hY]\}, \tag{2}$$

where $h=\frac{1}{2}\mathcal{L}_\xi\phi$ and $\mathcal{L}$ denotes the Lie differentiation and $(k,\mu)\in\mathbb{R}^2$.

Later, Dileo & Pastore (2009) introduced another generalized notion of the k-nullity distribution which is named the $(k,\mu)'$-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ and is defined for any $p\in M^{2n+1}$ as follows:

$$N_p(k,\mu)'=\{Z\in T_pM : R(X,Y)Z = k[g(Y,Z)X-g(X,Z)Y] + \mu[g(Y,Z)h'X-g(X,Z)h'Y]\}, \tag{3}$$

where $h'=h\circ\phi$, $\mathcal{L}$ denotes the Lie differentiation and $(k,\mu)\in\mathbb{R}^2$. Suppose that both k and μ in relation (2) (resp. (3)) are smooth functions on M^{2n+1}, then such a nullity distribution is called a generalized (k,μ) (resp. $(k,\mu)'$)-nullity distribution (Pastore & Saltarelli, 2011). For some recent results on almost Kenmotsu manifolds with the characteristic vector field belonging to some nullity distributions mentioned above, we refer the reader to Wang & Liu (2014a, 2014b).

The object of this paper is to investigate ϕ-recurrent and ϕ-symmetric almost Kenmotsu manifolds, obtaining a classification theorem of ϕ-recurrent almost Kenmotsu manifolds with the characteristic vector fields belonging to the $(k,\mu)'$-nullity distribution. It is well-known (Koufogiorgos & Tsichlias, 2000) that a generalized (k,μ)-contact metric manifold of dimension greater than 3 must be a (k,μ)-contact metric manifold. However, there exist non-trivial examples of almost Kenmotsu manifolds of dimension greater than 3 such that ξ belongs to the generalized (k,μ) or $(k,\mu)'$-nullity distribution (Pastore & Saltarelli, 2011). Under the assumption of ϕ-symmetry, in this paper, we prove that on an almost Kenmotsu manifold M^{2n+1} of dimension greater than 3, if ξ belongs to the generalized (k,μ)-nullity distribution then both k and μ are constants on M^{2n+1}.

The present paper is organized as follows. In the following section, we provide some basic formulas and properties of almost Kenmotsu manifolds according to Dileo & Pastore (2007, 2009) and Kenmotsu (1972). Later another section is devoted to presenting some well-known results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions. Finally, in the last section, some classification theorems of almost Kenmotsu manifolds such that ξ belongs to the (k,μ) and $(k,\mu)'$-nullity distribution are given respectively. Some corollaries of our main theorems are also presented.

ALMOST KENMOTSU MANIFOLDS

From Dileo & Pastore (2007, 2009), we shall recall some basic notions and properties of almost Kenmotsu manifolds. An almost contact structure (Blair, 2010) on a

$(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is an $(1,1)$-type tensor field, ξ a global vector field and η an 1-form, such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{4}$$

where id denotes the identity mapping, which imply that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. A Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}$$

for any vector fields X, Y on M^{2n+1}. An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. Moreover, a manifold endowed with an almost contact metric structure is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M^{2n+1}. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. It is well-known (Blair, 2010) that the normality of almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. From Kenmotsu (1972), we see that the normality of an almost Kenmotsu manifold is expressed by

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y on M^{2n+1}. According to Janssens & Vanhecke (1981), a normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

Next, we consider two tensor fields $l = R(\cdot\,, \xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi \phi$ on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. From Dileo & Pastore (2007, 2009) and Kim & Pak (2005), we know that the two $(1,1)$-type tensor fields l and h are symmetric and satisfy

$$h\xi = 0, \;\; l\xi = 0, \;\; \mathrm{tr} h = 0, \;\; \mathrm{tr}(h\phi) = 0, \;\; h\phi + \phi h = 0, \tag{6}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{7}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{8}$$

$$\mathrm{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \mathrm{tr} h^2, \tag{9}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{10}$$

for any vector fields $X, Y \in \Gamma(TM)$, where S, Q, ∇ and $\Gamma(TM)$ denote the

Ricci curvature tensor, the Ricci operator with respect to metric g, the Levi-Civita connection of g and the Lie algebra of all vector fields on M^{2n+1}, respectively. On the other hand, according to Takahashi (1977) and De *et al*. (2003), we have the following two definitions.

Definition 1. An almost Kenmotsu manifold is said to be ϕ-recurrent if it satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z \tag{11}$$

for any vector fields $X,Y,Z,W \in \Gamma(TM)$, where A is an 1-form on M^{2n+1}. If equation (11) holds for any vector fields X,Y,Z,W orthogonal to ξ, then the manifold is called a locally ϕ-recurrent manifold.

Definition 2. An almost Kenmotsu manifold is said to be ϕ-symmetric, if it satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{12}$$

for any vector fields$X,Y,Z,W \in \Gamma(TM)$. If relation (12) holds for any vector fields X,Y,Z,W orthogonal to ξ, then the manifold is called a locally ϕ-symmetric manifold.

ξ BELONGS TO THE NULLITY DISTRIBUTION

Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold for which ξ belongs to the generalized $(k,\mu)'$-nullity distribution, from (3) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{13}$$

where both k and μ are smooth functions on M^{2n+1}. Throughout the paper, we denote by $\mathcal{D}$ the distribution which is defined by $\mathcal{D} = \ker(\eta) = \mathrm{Im}(\phi)$. Replacing Y by ξ in equation (13) gives that $lX = k(X - \eta(X)\xi) + \mu h'X$, making using of equations (4) and (6) in this equation then we get $\phi l \phi X = -k(X - \eta(X)\xi) + \mu h'X$. Substituting the above equation into (8) we have

$$h'^2 = (k+1)\phi^2 \ (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{14}$$

Let $X \in \mathcal{D}$ be an eigenvector field of h' with the corresponding eigenvalue λ, from relation (14) we have that $\lambda^2 = -(k+1)$. It follows that $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. In what follows, we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces associated with h' corresponding to the eigenvalue $\lambda \neq 0$ and $-\lambda$ of h' respectively. Thus we have the following two lemmas.

Lemma 1 (Pastore & Saltarelli, 2011). Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost

Kenmotsu manifold with $h \neq 0$. If the generalized $(k, \mu)'$-nullity condition holds, then

$$\xi(\lambda) = -\lambda(\mu + 2), \quad \xi(k) = -2(k + 1)(\mu + 2). \tag{15}$$

Moreover, if $2n + 1 \geq 5$, then we have

$$X(\lambda) = 0, \quad X(k) = 0, \quad X(\mu) = 0 \tag{16}$$

for any $X \in \mathcal{D}$.

Lemma 2 (Dileo & Pastore, 2009). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h \neq 0$ and ξ belongs to the $(k, \mu)'$-nullity distributions. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
R(X_\lambda, Y_\lambda)Z_\lambda &= (k - 2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k + 2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],
\end{aligned}$$

where $\lambda^2 = -(k + 1)$.

Dileo & Pastore (2009) proved that an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution satisfies $\mu = -2$. Thus, making use of the above Lemma 2 and Theorem 5.1 of Pastore & Saltarelli (2011), we obtain the following lemma.

Lemma 3 (Wang & Liu). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost Kenmotsu manifold with ξ belonging to the generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then we have

$$QX = -2nX + 2n(k + 1)\eta(X)\xi + [\mu - 2(n - 1)]h'X \tag{17}$$

for any $X \in \Gamma(TM)$. Moreover, if both k and μ are constants, then we have

$$QX = -2nX + 2n(k + 1)\eta(X)\xi - 2nh'X \tag{18}$$

for any $X \in \Gamma(TM)$. In both cases, the scalar curvature of M^{2n+1} is $2n(k - 2n)$.

Similarly, making use of Theorem 4.1 of Pastore & Saltarelli (2011), by a straightforward computation the present authors also obtained the following lemma.

Lemma 4 (Wand & Liu). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost

Kenmotsu manifold with ξ belonging to the generalized $(k,\mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then we have

$$QX = -2nX + 2n(k+1)\eta(X)\xi - 2(n-1)h'X + \mu hX \tag{19}$$

for any $X \in \Gamma(TM)$. Moreover, the scalar curvature of of M^{2n+1} is $2n(k-2n)$.

ϕ-RECURRENT ALMOST KENMOTSU MANIFOLDS

We now give a classification result of a type of almost Kenmotsu manifolds.

Theorem 1. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional ϕ-recurrent almost Kenmotsu manifold with $h' \neq 0$. Suppose that the characteristic vector field ξ belongs to the $(k,\mu)'$-nullity distribution, then $k = -2$ and hence M^{2n+1} is locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Proof. Assume that M^{2n+1} is a ϕ-recurrent almost Kenmotsu manifold, by virtue of equations (4) and (11) we obtain

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z \tag{20}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$. Taking the inner product of relation (20) with arbitrary vector field $U \in \Gamma(TM)$ we get

$$-g((\nabla_W R)(X,Y)Z, U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z, U) \tag{21}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$. Considering a local orthonormal basis $\{E_i : i = 1, 2, \cdots, 2n+1\}$ of tangent space at each point of the manifold M^{2n+1}. By setting $X = U = E_i$ in equation (21) and taking summation over $i : 1 \leq i \leq 2n+1$, we obtain

$$-(\nabla_W S)(Y,Z) + \eta((\nabla_W R)(\xi,Y)Z) = A(W)S(Y,Z) \tag{22}$$

for any $Y, Z, W \in \Gamma(TM)$. In view of the skew-symmetry property of the curvature tensor R we conclude that $\eta((\nabla_W R)(\xi,Y)\xi) = 0$ for any $Y, W \in \Gamma(TM)$. Thus it follows from (22) that

$$-(\nabla_W S)(Y,\xi) = A(W)S(Y,\xi) \tag{23}$$

for any $Y, W \in \Gamma(TM)$.

Applying Lemma 3 in this context, we obtain from (18) that $Q\xi = 2nk\xi$. Replacing Y by ξ in relation (23) yields that $2nkA(W) = -g(\nabla_W S)(\xi,\xi) = 0$ for any $W \in \Gamma(TM)$, then we get $A = 0$ and hence M^{2n+1} is ϕ-symmetric. Also, it follows from (18) that

$$\begin{aligned}&(\nabla_Y Q)X + 2n(\nabla_Y h')X\\ =&2n(k+1)[\eta(X)Y - 2\eta(X)\eta(Y)\xi + \eta(X)h'Y + g(X,Y)\xi + g(h'X,Y)\xi]\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Noticing that $g((\nabla_X Q)Y, Z) = (\nabla_X S)(Y, Z)$ for any $X, Y, Z \in \Gamma(TM)$ and making use of (23) we obtain

$$\begin{aligned}&(\nabla_W S)(Y,\xi)\\ =&g((\nabla_W Q)Y,\xi)\\ =&-2n\{g(\nabla_W h'Y,\xi) - (k+1)[g(Y,W) + g(h'Y,W) - \eta(Y)\eta(W)]\}\\ =&-2n\left\{g(h'^2Y,W) + (k+1)[g(Y,W) - \eta(Y)\eta(W)] + (k+2)g(h'Y,W)\right\}\end{aligned}$$

for any vector fields $Y, W \in \Gamma(TM)$. Taking into account $A = 0$ and comparing the above equation with (23) we get

$$g(h'^2Y,W) + (k+1)[g(Y,W) - \eta(Y)\eta(W)] + (k+2)g(h'Y,W) = 0 \qquad (24)$$

for any $Y, W \in \Gamma(TM)$. Letting $Y \in [\lambda]'$ in relation (24) and applying Lemma 2 we obtain

$$\lambda^2 + (k+2)\lambda + k + 1 = 0. \qquad (25)$$

In view of the fact that $\lambda^2 = -(k+1)$ and the assumption that h is non-vanishing, then we see from (25) that $k = -2$ and hence $\lambda = \pm 1$. The remaining proof is easy to check. For the sake of completeness, we give the details of the remaining proof. Without losing the generality, we now choose $\lambda = 1$, then by Lemma 2 we get:

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda] \text{ and } R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0 \qquad (26)$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Moreover, noticing $\mu = -2$ then it follows from (13) that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. As shown in Dileo & Pastore (2009) that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Being $\lambda = 1$, we know that two orthogonal distribution $[\xi] \oplus [\lambda]'$ and $[-\lambda]$ are both integrable with totally geodesic leaves. This completes the proof.

Corollary 1. A locally symmetric almost Kenmotsu manifold with the characteristic vector field belonging to the $(k,\mu)'$-nullity distribution and the non-vanishing tensor h is locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

The above result was proved by Dileo & Pastore (2009).

Theorem 2. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional ϕ-recurrent almost Kenmotsu manifold with $n > 1$ and $h' \neq 0$. If the characteristic vector field ξ belongs to the generalized $(k, \mu)'$-nullity distribution, then the 1-form A is given by

$$A = -\frac{1}{k}\mathrm{d}k, \tag{27}$$

where k is a non-zero function on M^{2n+1}.

Proof. Proceeding similarly to proof of Theorem 1, we see that the relation (23) holds in this context. Since ξ belongs to the generalized $(k, \mu)'$-nullity distribution, then by applying Lemma 3 we may obtain

$$\begin{aligned}(\nabla_W Q)Y =& 2nW(k)\eta(Y)\xi + W(\mu)h'Y + [\mu - 2(n-1)](\nabla_W h')Y \\ &+ 2n(k+1)[\eta(Y)W - 2\eta(Y)\eta(W)\xi + \eta(Y)h'W + g(W,Y)\xi + g(h'Y,W)\xi]\end{aligned}$$

for any $Y, W \in \Gamma(TM)$. Taking the inner product of the above equation with ξ and making use of $g((\nabla_W Q)Y, Z) = (\nabla_W S)(Y, Z)$ we obtain

$$\begin{aligned}(\nabla_W S)(Y,\xi) =& 2nW(k)\eta(Y) + 2n(k+1)[g(Y,W) - \eta(W)\eta(Y)] \\ &+ (2n-2-\mu)g(h'^2Y, W) + (2nk+4n-\mu-2)g(h'W, Y)\end{aligned}$$

for any $Y, W \in \Gamma(TM)$. Replacing Y by ξ in the above equation and using the first term of (6) we see that $(\nabla_W S)(\xi,\xi) = 2nW(k)$ for any $W \in \Gamma(TM)$. On the other hand, using relation (17) we get $Q\xi = 2nk\xi$ then it follows from equation (23) that $(\nabla_W S)(\xi,\xi) = -A(W)S(\xi,\xi) = -2nkA(W)$ for any $W \in \Gamma(TM)$, this means that

$$kA(W) = -W(k) \tag{28}$$

for any $W \in \Gamma(TM)$. Moreover, from (14) we see that the smooth function k satisfies $k \leq -1$, hence, (27) follows from (28). This completes the proof.

Corollary 2. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional ϕ-recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then the following statements are equivalent:

(i): k is a constant;

(ii): M^{2n+1} is ϕ-symmetric;

(iii): ξ belongs to the $(k, \mu)'$-nullity distribution.

Moreover, if the dimension of M^{2n+1} is assumed to be greater than 5, then the above three assertions are equivalent to the following statement:

(iv): the vector field associated to the 1-form A is orthogonal to ξ.

Proof. The equivalence between (i) and (ii) follows from relation (27). If k is a constant, then by (14) we know that the eigenvalues λ and $-\lambda$ of h' are also non-zero constants. Using the first term of equation (15) we conclude that $\mu = -2$, that is, ξ belongs to the $(k, -2)'$-nullity distribution. Conversely, (iii)⇒(i) is obviously.

Next, we assume that $n > 1$ and prove (iv)⇒(i). If the vector field associated to the 1-form A is orthogonal to ξ, in view of (27) we obtain $\xi(k) = 0$. As the dimension of M^{2n+1} is greater than 5, then the second term of equation (16) and $\xi(k) = 0$ assure that k is a constant. Also, (i)⇒(iv) follows from (27). This completes the proof.

We now present some classification theorems of almost Kenmotsu manifolds for which ξ belongs to the (k, μ)-nullity distribution.

Theorem 3. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional ϕ-recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k, μ) -nullity distribution. Then M^{2n+1} is of constant sectional curvature -1, provided that the vector field associated to the 1-form A is not orthogonal to ξ.

Proof. If ξ belongs to the (k, μ)-nullity distribution, it follows from Dileo & Pastore (2009) that $k = -1$ and hence $h = 0$. Thus, equation (7) becomes $\nabla_X \xi = X - \eta(X)\xi$ for any $X \in \Gamma(TM)$ and from (10) we see that

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{29}$$

for any $X, Y \in \Gamma(TM)$. Taking the inner product of relation (29) with arbitrary vector field $Z \in \Gamma(TM)$ we get

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \tag{30}$$

for any $X, Y, Z \in \Gamma(TM)$. Moreover, taking the covariant differentiation along arbitrary vector field $W \in \Gamma(TM)$ on (29) and making use of (29) we get

$$\begin{aligned}(\nabla_W R)(X,Y)\xi =& \nabla_W R(X,Y)\xi - R(\nabla_W X, Y)\xi - R(X, \nabla_W Y)\xi - R(X,Y)\nabla_W \xi \\ =& - R(X,Y)W - g(Y,W)X + g(X,W)Y\end{aligned} \tag{31}$$

for any $X, Y, W \in \Gamma(TM)$. Replacing Z by ξ in relation (30) gives

$$\eta(R(X,Y)\xi) = \eta(X)\eta(Y) - \eta(Y)\eta(X) = 0$$

for any $X, Y \in \Gamma(TM)$, thus, using this equation and taking the inner product of relation (31) with ξ we obtain

$$\eta((\nabla_W R)(X,Y)\xi) = 0 \tag{32}$$

for any $X, Y, W \in \Gamma(TM)$. In view of (31) and (32) we obtain from (11) that

$$(\nabla_W R)(X,Y)\xi = -A(W)R(X,Y)\xi \tag{33}$$

for any $X, Y, W \in \Gamma(TM)$. Using (31) in the left hand side of equation (33) gives that

$$R(X,Y)W - g(X,W)Y + g(Y,W)X = A(W)R(X,Y)\xi \tag{34}$$

for any $X, Y, W \in \Gamma(TM)$. Replacing W by ξ in (34) and using (30) we obtain

$$A(\xi)R(X,Y)\xi = 0$$

for any $X, Y \in \Gamma(TM)$. Suppose that the vector field associated to the 1-form A is not orthogonal to ξ, we see from the above relation that

$$R(X,Y)\xi = 0$$

for any $X, Y \in \Gamma(TM)$. Substituting the above relation into (34) we get

$$R(X,Y)W = -[g(Y,W)X - g(X,W)Y]$$

for any $X, Y, W \in \Gamma(TM)$. Thus we complete the proof.

Corollary 3. A ϕ-symmetric almost Kenmotsu manifold with the characteristic vector field belonging to the (k, μ)-nullity distribution is of constant sectional curvature -1.

Obviously, the above result extends Corollary 6 of Kenmotsu (1972).

Theorem 4. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional ϕ-recurrent almost Kenmotsu manifold with $h \neq 0$. If the characteristic vector field ξ belongs to the generalized (k, μ)-nullity distribution and $n > 1$, then the 1-form A is given by

$$A = -\frac{1}{k}\mathrm{d}k, \tag{35}$$

where k is a non-zero function on M^{2n+1}.

Proof. Similarly as in the proof of Theorem 2, we obtain that relation (28) holds for any vector field $W \in \Gamma(TM)$. On the other hand, from a result of Pastore & Saltarelli (2011), we know that the smooth function k satisfies $k \leq -1$. Therefore, (35) follows from (28).

Corollary 4. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional ϕ-recurrent almost Kenmotsu manifold with $h \neq 0$. If the characteristic vector field ξ belongs to the generalized (k, μ)-nullity distribution and $n > 1$, then the vector field associated to the 1-form A is never orthogonal to the characteristic vector field ξ.

Proof. It is easy to see from (35) that $kA(\xi) = -\xi(k)$, using this equation in the second term of relation (15) we obtain

$$A(\xi) = \frac{4(k+1)}{k}. \tag{36}$$

Suppose that the vector field associated to the the 1-form A is orthogonal to ξ, that is, $A(\xi) = 0$. We observe from Proposition 3.1 of Pastore & Saltarelli (2011) that relation (14) holds in this context. Thus, it follows from (36) that $k = -1$ and hence by relation (14) we may get $h = 0$, we arrive at a contradiction. This completes the proof.

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حول منطويات قرب – الكنموتسو المعاودة

***يانينغ وانغ ، **كسيمن ليو**

*كلية الرياضيات وعلوم المعلومات – جامعة هنان عادي– شينشيانغ–453007 هنان
جمهورية الصين الشعبية.

** مدرسة العلوم الرياضية، جامعة داليان للتكنولوجيا – وداليان 116024، ليونينغ
جمهورية الصين الشعبية

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P.O. Box 26585 Safat, 13126 Kuwait - Tel.: (965) 24817689 - (965) 24815453 - Fax: (965) 24812514
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