On $\phi$-recurrent almost Kenmotsu manifolds

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ABSTRACT

The object of this paper is to investigate $\phi$-recurrent and $\phi$-symmetric almost Kenmotsu manifolds with the characteristic vector fields belonging to some nullity distributions.

Keywords: Almost Kenmotsu manifold; $\phi$-recurrence; $\phi$-symmetry; generalized nullity distribution.

MSC Classification: 53C25, 53D15.

INTRODUCTION

Kenmotsu (1972) introduced a new class of almost contact metric manifolds, which are known as Kenmotsu manifolds nowadays, and proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature $-1$. Takahashi (1977) introduced the notion of local $\phi$-symmetry, which is weaker than local symmetry in the context of Sasakian geometry. Generalizing the notion of local $\phi$-symmetry, De et al. (2003) introduced the notion of $\phi$-recurrence on Sasakian manifolds. Since then, many results on $\phi$-recurrent and $\phi$-symmetric Kenmotsu manifolds were obtained by some authors, for more related results in this framework we refer the reader to some recent papers by De (2008), De & Pathak (2004) and De et al. (2009a, 2009b).

On the other hand, the notion of $k$-nullity distribution was first introduced by Gray (1966) and Tanno (1978) in the study of Riemannian manifolds $(\bar{M}, g)$, which is defined for any $p \in \bar{M}$ as follows:

$$N_p(k) = \{Z \in T_p\bar{M} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

(1)

where $X, Y$ denote arbitrary vectors in $T_p\bar{M}$ and $k \in \mathbb{R}$.

Recently, Blair et al. (1995) introduced a generalized notion of the $k$-nullity distribution named the $(k, \mu)$-nullity distribution on a contact metric manifold $(\bar{M}^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in \bar{M}^{2n+1}$ as follows:
\[ N_p(k, \mu) = \{ Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY] \}. \tag{2} \]

\[ N_p(k, \mu)' = \{ Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)h'X - g(X, Z)h'Y] \}. \tag{3} \]

where \( h = \frac{1}{2} \mathcal{L}_\xi \phi \) and \( \mathcal{L} \) denotes the Lie differentiation and \((k, \mu) \in \mathbb{R}^2\).

Later, Dileo & Pastore (2009) introduced another generalized notion of the \( k \)-nullity distribution which is named the \((k, \mu)'\)-nullity distribution on an almost Kenmotsu manifold \((M^{2n+1}, \phi, \xi, \eta, g)\) and is defined for any \( p \in M^{2n+1} \) as follows:

\[ N_p(k, \mu)' = \{ Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)h'X - g(X, Z)h'Y] \}, \]

where \( h' = h \circ \phi \), \( \mathcal{L} \) denotes the Lie differentiation and \((k, \mu) \in \mathbb{R}^2\). Suppose that both \( k \) and \( \mu \) in relation (2) (resp. (3)) are smooth functions on \( M^{2n+1} \), then such a nullity distribution is called a generalized \((k, \mu)\) (resp. \((k, \mu)')\)-nullity distribution (Pastore & Saltarelli, 2011). For some recent results on almost Kenmotsu manifolds with the characteristic vector field belonging to some nullity distributions mentioned above, we refer the reader to Wang & Liu (2014a, 2014b).

The object of this paper is to investigate \( \phi \)-recurrent and \( \phi \)-symmetric almost Kenmotsu manifolds, obtaining a classification theorem of \( \phi \)-recurrent almost Kenmotsu manifolds with the characteristic vector fields belonging to the \((k, \mu)'\)-nullity distribution. It is well-known (Koufogiorgos & Tsichlias, 2000) that a generalized \((k, \mu)'\)-contact metric manifold of dimension greater than 3 must be a \((k, \mu)'\)-contact metric manifold. However, there exist non-trivial examples of almost Kenmotsu manifolds of dimension greater than 3 such that \( \xi \) belongs to the generalized \((k, \mu)'\)-nullity distribution (Pastore & Saltarelli, 2011). Under the assumption of \( \phi \)-symmetry, in this paper, we prove that on an almost Kenmotsu manifold \( M^{2n+1} \) of dimension greater than 3, if \( \xi \) belongs to the generalized \((k, \mu)'\)-nullity distribution then both \( k \) and \( \mu \) are constants on \( M^{2n+1} \).

The present paper is organized as follows. In the following section, we provide some basic formulas and properties of almost Kenmotsu manifolds according to Dileo & Pastore (2007, 2009) and Kenmotsu (1972). Later another section is devoted to presenting some well-known results on almost Kenmotsu manifolds with \( \xi \) belonging to some nullity distributions. Finally, in the last section, some classification theorems of almost Kenmotsu manifolds such that \( \xi \) belongs to the \((k, \mu)\) and \((k, \mu)'\)-nullity distribution are given respectively. Some corollaries of our main theorems are also presented.

**ALMOST KENMOTSU MANIFOLDS**

From Dileo & Pastore (2007, 2009), we shall recall some basic notions and properties of almost Kenmotsu manifolds. An almost contact structure (Blair, 2010) on a
(2n + 1)-dimensional smooth manifold \( M^{2n+1} \) is a triplet \((\phi, \xi, \eta)\), where \(\phi\) is an \((1, 1)\)-type tensor field, \(\xi\) a global vector field and \(\eta\) an 1-form, such that

\[
\phi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{4}
\]

where \([\cdot, \cdot]\) denotes the identity mapping, which imply that \(\phi(\xi) = 0\), \(\eta \circ \phi = 0\) and \(\text{rank}(\phi) = 2n\). A Riemannian metric \(g\) on \(M^{2n+1}\) is said to be compatible with the almost contact structure \((\phi, \xi, \eta)\) if

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}
\]

for any vector fields \(X, Y\) on \(M^{2n+1}\). An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. Moreover, a manifold endowed with an almost contact metric structure is said to be an almost contact metric manifold. The fundamental 2-form \(\Omega\) on an almost contact metric manifold \(M^{2n+1}\) is defined by

\[
\Omega(X, Y) = g(\phi X, \phi Y) - \eta(X)\eta(Y) \tag{5}
\]

for any vector fields \(X, Y\) on \(M^{2n+1}\). An almost Kenmotsu manifold is defined as an almost contact metric manifold such that \(\phi^2 = -\text{id} + \eta \otimes \xi\), \(\eta(\xi) = 1\), and \(\phi(\xi) = 0\). It is well-known (Blair, 2010) that the normality of almost contact structure is expressed by the vanishing of the tensor \(N_\phi = [\phi, \phi] + 2\eta \otimes \xi\), where \([\phi, \phi]\) is the Nijenhuis tensor of \(\phi\). From Kenmotsu (1972), we see that the normality of an almost Kenmotsu manifold is expressed by

\[
(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X
\]

for any vector fields \(X, Y\) on \(M^{2n+1}\). According to Janssens & Vanhecke (1981), a normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

Next, we consider two tensor fields \(l = R(\cdot, \xi)\xi\) and \(h = \frac{1}{2}L_\xi \phi\) on an almost Kenmotsu manifold \((M^{2n+1}, \phi, \xi, \eta, g)\), where \(R\) is the Riemannian curvature tensor of \(\eta\) and \(L\) is the Lie differentiation. From Dileo & Pastore (2007, 2009) and Kim & Pak (2005), we know that the two \((1, 1)\)-type tensor fields \(l\) and \(h\) are symmetric and satisfy

\[
h\xi = 0, \quad l\xi = 0, \quad \text{tr}h = 0, \quad \text{tr}(h\phi) = 0, \quad h\phi + \phi h = 0, \tag{6}
\]

\[
\nabla_X \xi = -\phi^2 X - \phi h X, \tag{7}
\]

\[
\phi l\phi - l = 2(h^2 - \phi^2), \tag{8}
\]

\[
\text{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \text{tr}h^2, \tag{9}
\]

\[
R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{10}
\]

for any vector fields \(X, Y \in \Gamma(TM)\), where \(S, Q, \nabla\) and \(\Gamma(TM)\) denote the
Ricci curvature tensor, the Ricci operator with respect to metric $g$, the Levi-Civita connection of $g$ and the Lie algebra of all vector fields on $M^{2n+1}$, respectively. On the other hand, according to Takahashi (1977) and De et al. (2003), we have the following two definitions.

**Definition 1.** An almost Kenmotsu manifold is said to be $\phi$-recurrent if it satisfies
\[
\phi^2(\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z
\]
for any vector fields $X, Y, Z, W \in \Gamma(TM)$, where $A$ is an 1-form on $M^{2n+1}$. If equation (11) holds for any vector fields $X, Y, Z, W$ orthogonal to $\xi$, then the manifold is called a locally $\phi$-recurrent manifold.

**Definition 2.** An almost Kenmotsu manifold is said to be $\phi$-symmetric, if it satisfies
\[
\phi^2(\nabla_W R)(X, Y)Z = 0
\]
for any vector fields $X, Y, Z, W \in \Gamma(TM)$. If relation (12) holds for any vector fields $X, Y, Z, W$ orthogonal to $\xi$, then the manifold is called a locally $\phi$-symmetric manifold.

### $\xi$ BELONGS TO THE NULLITY DISTRIBUTION

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold for which $\xi$ belongs to the generalized $(k, \mu)$-nullity distribution, from (3) we have
\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y],
\]
where both $k$ and $\mu$ are smooth functions on $M^{2n+1}$. Throughout the paper, we denote by $\mathcal{D}$ the distribution which is defined by $\mathcal{D} = \ker(\eta) = \text{Im}(\phi)$. Replacing $Y$ by $\xi$ in equation (13) gives that $lX = k(X - \eta(X)\xi) + \mu h'X$, making using of equations (4) and (6) in this equation then we get $\phi l\phi X = -k(X - \eta(X)\xi) + \mu h'X$. Substituting the above equation into (8) we have
\[
h'^2 = (k + 1)\phi^2 \quad (\Leftrightarrow h'^2 = (k + 1)\phi^2).
\]

Let $X \in \mathcal{D}$ be an eigenvector field of $h'$ with the corresponding eigenvalue $\lambda$, from relation (14) we have that $\lambda^2 = -(k + 1)$. It follows that $k \leq -1$ and $\lambda = \pm \sqrt{-k - 1}$. In what follows, we denote by $[\lambda]'$ and $-\lambda'$ the eigenspaces associated with $h'$ corresponding to the eigenvalue $\lambda \neq 0$ and $-\lambda$ of $h'$ respectively. Thus we have the following two lemmas.

**Lemma 1** (Pastore & Saltarelli, 2011). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost
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Let $\xi$ be a $(k, \mu)'$-nullity condition satisfies $\mu = -2$. Thus, making use of the above Lemma 2 and Theorem 5.1 of Pastore & Saltarelli (2011), we obtain the following lemma.

**Lemma 3** (Wang & Liu). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost Kenmotsu manifold with $\xi$ belonging to the generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then we have

$$QX = -2nX + 2n(k + 1)\eta(X)\xi + [\mu - 2(n - 1)]h'X$$

for any $X \in \Gamma(TM)$. Moreover, if both $k$ and $\mu$ are constants, then we have

$$QX = -2nX + 2n(k + 1)\eta(X)\xi - 2nh'X$$

for any $X \in \Gamma(TM)$. In both cases, the scalar curvature of $M^{2n+1}$ is $2n(k - 2n)$.

Similarly, making use of Theorem 4.1 of Pastore & Saltarelli (2011), by a straightforward computation the present authors also obtained the following lemma.

**Lemma 4** (Wand & Liu). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost Kenmotsu manifold with $h \neq 0$. If the generalized $(k, \mu)'$-nullity condition holds, then

$$\xi(\lambda) = -\lambda(\mu + 2), \quad \xi(k) = -2(k + 1)(\mu + 2). \quad (15)$$

Moreover, if $2n + 1 \geq 5$, then we have

$$X(\lambda) = 0, \quad X(k) = 0, \quad X(\mu) = 0 \quad (16)$$

for any $X \in \mathcal{D}$.

**Lemma 2** (Dileo & Pastore, 2009). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h \neq 0$ and $\xi$ belongs to the $(k, \mu)'$-nullity distributions. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \quad R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0, \quad R(X_\lambda, Y_{-\lambda})Z_\lambda = (k + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \quad R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = (k - 2\lambda)[g(Y_{-\lambda}, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \quad R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k + 2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],$$

where $\lambda^2 = -(k + 1)$.
Kenmotsu manifold with \( \xi \) belonging to the generalized \((k, \mu)\)'-nullity distribution and \( h' \neq 0 \). If \( n > 1 \), then we have
\[
QX = -2nX + 2n(k + 1)\eta(X)\xi - 2(n - 1)h'X + \mu hX \tag{19}
\]
for any \( X \in \Gamma(TM) \). Moreover, the scalar curvature of of \( M^{2n+1} \) is \( 2n(k - 2n) \).

**\( \phi \)-RECURRENT ALMOST KENMOTSU MANIFOLDS**

We now give a classification result of a type of almost Kenmotsu manifolds.

**Theorem 1.** Let \( (M^{2n+1}, \phi, \xi, \eta, g) \) be a \((2n + 1)\)-dimensional \( \phi \)-recurrent almost Kenmotsu manifold with \( h' \neq 0 \). Suppose that the characteristic vector field \( \xi \) belongs to the \((k, \mu)\)'-nullity distribution, then \( k' = -2 \) and hence \( M^{2n+1} \) is locally isometric to the product \( \mathbb{H}^{n+1}(-4) \times \mathbb{R}^n \).

**Proof.** Assume that \( M^{2n+1} \) is a \( \phi \)-recurrent almost Kenmotsu manifold, by virtue of equations (4) and (11) we obtain
\[
-(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z \tag{20}
\]
for any vector fields \( X, Y, Z, W \in \Gamma(TM) \). Taking the inner product of relation (20) with arbitrary vector field \( U \in \Gamma(TM) \) we get
\[
-g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) = A(W)g(R(X, Y)Z, U) \tag{21}
\]
for any vector fields \( X, Y, Z, W \in \Gamma(TM) \). Considering a local orthonormal basis \( \{E_i : i = 1, 2, \cdots, 2n + 1\} \) of tangent space at each point of the manifold \( M^{2n+1} \). By setting \( X = U = E_i \) in equation (21) and taking summation over \( i : 1 \leq i \leq 2n + 1 \), we obtain
\[
-(\nabla_W S)(Y, Z) + \eta((\nabla_W R)(\xi, Y)Z) = A(W)S(Y, Z) \tag{22}
\]
for any \( Y, Z, W \in \Gamma(TM) \). In view of the skew-symmetry property of the curvature tensor \( R \) we conclude that \( \eta((\nabla_W R)(\xi, Y)\xi) = 0 \) for any \( Y, W \in \Gamma(TM) \). Thus it follows from (22) that
\[
-(\nabla_W S)(Y, \xi) = A(W)S(Y, \xi) \tag{23}
\]
for any \( Y, W \in \Gamma(TM) \).

Applying Lemma 3 in this context, we obtain from (18) that \( Q\xi = 2nk\xi \). Replacing \( Y \) by \( \xi \) in relation (23) yields that \( 2nkA(W) = -g(\nabla_W S)(\xi, \xi) = 0 \) for any \( W \in \Gamma(TM) \), then we get \( A = 0 \) and hence \( M^{2n+1} \) is \( \phi \)-symmetric. Also, it follows from (18) that
\[(\nabla_Y Q)X + 2n(\nabla_Y h')X = 2n(k + 1)[\eta(X) Y - 2\eta(X) \eta(Y) \xi + \eta(X) h'Y + g(X, Y) \xi + g(h'X, Y) \xi]\]

for any \(X, Y \in \Gamma(TM)\). Noticing that \(g((\nabla_X Q)Y, Z) = (\nabla_X S)(Y, Z)\) for any \(X, Y, Z \in \Gamma(TM)\) and making use of (23) we obtain

\[\langle (\nabla_X S)(Y, \xi) \rangle = g((\nabla_X Q)Y, \xi) = -2n \{ g(\nabla_X h'Y, \xi) - (k + 1)[g(Y, W) + g(h'Y, W) - \eta(Y) \eta(W)] \}\]

\[= -2n \{ g(h^2Y, W) + (k + 1)[g(Y, W) - \eta(Y) \eta(W)] + (k + 2)g(h'Y, W) \}\]

for any vector fields \(Y, W \in \Gamma(TM)\). Taking into account \(A = 0\) and comparing the above equation with (23) we get

\[g(h^2Y, W) + (k + 1)[g(Y, W) - \eta(Y) \eta(W)] + (k + 2)g(h'Y, W) = 0\]

(24)

for any \(Y, W \in \Gamma(TM)\). Letting \(Y \in [\lambda]'\) in relation (24) and applying Lemma 2 we obtain

\[\lambda^2 + (k + 2)\lambda + k + 1 = 0.\]

(25)

In view of the fact that \(\lambda^2 = -(k + 1)\) and the assumption that \(h\) is non-vanishing, then we see from (25) that \(k = -2\) and hence \(\lambda = \pm 1\). The remaining proof is easy to check. For the sake of completeness, we give the details of the remaining proof. Without losing the generality, we now choose \(\lambda = 1\), then by Lemma 2 we get:

\[R(X, Y)Z_L = -4[g(Y, Z_L)X - g(X, Z_L)Y_L] \text{ and } R(X, Y)L Z_L = 0\]

(26)

for any \(X, Y, Z_L \in [\lambda]'\) and \(X_L, Y_L, Z_L \in [-\lambda]'\). Moreover, noticing \(\mu = -2\) then it follows from (13) that \(K(X, \xi) = -4\) for any \(X \in [\lambda]'\) and \(K(X, \xi) = 0\) for any \(X \in [-\lambda]'\). As shown in Dileo & Pastore (2009) that the distribution \([\xi] \oplus [\lambda]'\) is integrable with totally geodesic leaves and the distribution \([-\lambda]'\) is integrable with totally umbilical leaves by \(H = -(1 - \lambda)\xi\), where \(H\) is the mean curvature vector field for the leaves of \([-\lambda]'\) immersed in \(M^{2n+1}\). Being \(\lambda = 1\), we know that two orthogonal distribution \([\xi] \oplus [\lambda]'\) and \([-\lambda]\) are both integrable with totally geodesic leaves. This completes the proof.

**Corollary 1.** A locally symmetric almost Kenmotsu manifold with the characteristic vector field belonging to the \((k, \mu)'\)-nullity distribution and the non-vanishing tensor \(h\) is locally isometric to the product \(\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n\).

The above result was proved by Dileo & Pastore (2009).
Theorem 2. Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional \(\phi\)-recurrent almost Kenmotsu manifold with \(n > 1\) and \(h' \neq 0\). If the characteristic vector field \(\xi\) belongs to the generalized \((k, \mu)\)\(^{-}\)-nullity distribution, then the 1-form \(A\) is given by

\[
A = -\frac{1}{k}\,dk,
\]

where \(k\) is a non-zero function on \(M^{2n+1}\).

Proof. Proceeding similarly to proof of Theorem 1, we see that the relation (23) holds in this context. Since \(\xi\) belongs to the generalized \((k, \mu)\)\(^{-}\)-nullity distribution, then by applying Lemma 3 we may obtain

\[
(\nabla_W Q)Y = 2nW(k)\eta(Y)\xi + W(\mu)h'Y + [\mu - 2(n - 1)](\nabla_W h')Y
\]

\[
+ 2n(k + 1)|\eta(Y)|W - 2\eta(Y)\eta(W)\xi + \eta(Y)h'W + g(W, Y)\xi + g(h'Y, W)\xi
\]

for any \(Y, W \in \Gamma(TM)\). Taking the inner product of the above equation with \(\xi\) and making use of \(g((\nabla_W Q)Y, Z) = (\nabla_W S)(Y, Z)\) we obtain

\[
(\nabla_W S)(Y, \xi) = 2nW(k)\eta(Y) + 2n(k + 1)|g(Y, W) - \eta(W)\eta(Y)|
\]

\[
+ (2n - 2 - \mu)g(h'^2Y, W) + (2nk + 4n - \mu - 2)g(h'W, Y)
\]

for any \(Y, W \in \Gamma(TM)\). Replacing \(Y\) by \(\xi\) in the above equation and using the first term of (6) we see that

\[
(\nabla_W S)(\xi, \xi) = 2nW(k)
\]

for any \(W \in \Gamma(TM)\). On the other hand, using relation (17) we get \(Q\xi = 2nk\xi\) then it follows from equation (23) that

\[
(\nabla_W S)(\xi, \xi) = -A(W)S(\xi, \xi) = -2nkA(W)
\]

for any \(W \in \Gamma(TM)\), this means that

\[
kA(W) = -W(k)
\]

(28)

for any \(W \in \Gamma(TM)\). Moreover, from (14) we see that the smooth function \(k\) satisfies \(k \leq -1\), hence, (27) follows from (28). This completes the proof.

Corollary 2. Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional \(\phi\)-recurrent almost Kenmotsu manifold with the characteristic vector field \(\xi\) belonging to the generalized \((k, \mu)\)\(^{-}\)-nullity distribution and \(h' \neq 0\). Then the following statements are equivalent:

(i): \(k\) is a constant;

(ii): \(M^{2n+1}\) is \(\phi\)-symmetric;

(iii): \(\xi\) belongs to the \((k, \mu)\)\(^{-}\)-nullity distribution.

Moreover, if the dimension of \(M^{2n+1}\) is assumed to be greater than 5, then the above three assertions are equivalent to the following statement:

(iv): the vector field associated to the 1-form \(A\) is orthogonal to \(\xi\).

Proof. The equivalence between (i) and (ii) follows from relation (27). If \(k\): is a
constant, then by (14) we know that the eigenvalues $\lambda$ and $-\lambda$ of $h'$ are also non-zero constants. Using the first term of equation (15) we conclude that $\mu = -2$, that is, $\xi$ belongs to the $(k, -2)$-nullity distribution. Conversely, (iii) $\Rightarrow$ (i) is obviously.

Next, we assume that $n > 1$ and prove (iv) $\Rightarrow$ (i). If the vector field associated to the 1-form $A$ is orthogonal to $\xi$, in view of (27) we obtain $\xi(k) = 0$. As the dimension of $M^{2n+1}$ is greater than 5, then the second term of equation (16) and $\xi(k) = 0$ assure that $k$ is a constant. Also, (i) $\Rightarrow$ (iv) follows from (27). This completes the proof.

We now present some classification theorems of almost Kenmotsu manifolds for which $\xi$ belongs to the $(k, \mu)$-nullity distribution.

**Theorem 3.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional $\phi$-recurrent almost Kenmotsu manifold with the characteristic vector field $\xi$ belonging to the $(k, \mu)$-nullity distribution. Then $M^{2n+1}$ is of constant sectional curvature $-1$, provided that the vector field associated to the 1-form $A$ is not orthogonal to $\xi$.

**Proof.** If $\xi$ belongs to the $(k, \mu)$-nullity distribution, it follows from Dileo & Pastore (2009) that $k = -1$ and hence $h = 0$. Thus, equation (7) becomes $\nabla_X \xi = X - \eta(X)\xi$ for any $X \in \Gamma(TM)$, and from (10) we see that

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X$$ (29)

for any $X, Y \in \Gamma(TM)$. Taking the inner product of relation (29) with arbitrary vector field $Z \in \Gamma(TM)$ we get

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X)$$ (30)

for any $X, Y, Z \in \Gamma(TM)$. Moreover, taking the covariant differentiation along arbitrary vector field $W \in \Gamma(TM)$ on (29) and making use of (29) we get

$$\langle \nabla_W R(X, Y) \xi, Z \rangle = \nabla_W R(X, Y) \xi - R(\nabla_W X, Y) \xi - R(X, \nabla_W Y) \xi - R(X, Y) \nabla_W \xi$$

$$= -R(X, Y)W - g(Y, W)X + g(X, W)Y$$ (31)

for any $X, Y, W \in \Gamma(TM)$. Replacing $Z$ by $\xi$ in relation (30) gives

$$\eta(R(X, Y)\xi) = \eta(X)\eta(Y) - \eta(Y)\eta(X) = 0$$

for any $X, Y \in \Gamma(TM)$, thus, using this equation and taking the inner product of relation (31) with $\xi$ we obtain

$$\eta(\langle \nabla_W R(X, Y) \xi, Z \rangle) = 0$$ (32)

for any $X, Y, W \in \Gamma(TM)$. In view of (31) and (32) we obtain from (11) that

$$\langle \nabla_W R(X, Y) \xi, \xi \rangle = -A(W)R(X, Y)\xi$$ (33)
for any $X, Y, W \in \Gamma(TM)$. Using (31) in the left hand side of equation (33) gives that
\[
R(X, Y)W - g(X, W)Y + g(Y, W)X = A(W)R(X, Y)\xi
\]
(34)
for any $X, Y, W \in \Gamma(TM)$. Replacing $W$ by $\xi$ in (34) and using (30) we obtain
\[
A(\xi)R(X, Y)\xi = 0
\]
for any $X, Y \in \Gamma(TM)$. Suppose that the vector field associated to the 1-form $A$ is not orthogonal to $\xi$, we see from the above relation that
\[
R(X, Y)\xi = 0
\]
for any $X, Y \in \Gamma(TM)$. Substituting the above relation into (34) we get
\[
R(X, Y)W = -|g(Y, W)X - g(X, W)Y|
\]
for any $X, Y, W \in \Gamma(TM)$. Thus we complete the proof.

**Corollary 3.** A $\phi$-symmetric almost Kenmotsu manifold with the characteristic vector field belonging to the $(k, \mu)$-nullity distribution is of constant sectional curvature $-1$.

Obviously, the above result extends Corollary 6 of Kenmotsu (1972).

**Theorem 4.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional $\phi$-recurrent almost Kenmotsu manifold with $h \neq 0$. If the characteristic vector field $\xi$ belongs to the generalized $(k, \mu)$-nullity distribution and $n > 1$, then the 1-form $A$ is given by
\[
A = -\frac{1}{k} dk,
\]
(35)
where $k$ is a non-zero function on $M^{2n+1}$.

**Proof.** Similarly as in the proof of Theorem 2, we obtain that relation (28) holds for any vector field $W \in \Gamma(TM)$. On the other hand, from a result of Pastore & Saltarelli (2011), we know that the smooth function $k$ satisfies $k \leq -1$. Therefore, (35) follows from (28).

**Corollary 4.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional $\phi$-recurrent almost Kenmotsu manifold with $h \neq 0$. If the characteristic vector field $\xi$ belongs to the generalized $(k, \mu)$-nullity distribution and $n > 1$, then the vector field associated to the 1-form $A$ is never orthogonal to the characteristic vector field $\xi$.

**Proof.** It is easy to see from (35) that $kA(\xi) = -\xi(k')$, using this equation in the second term of relation (15) we obtain
\[
A(\xi) = \frac{4(k + 1)}{k}.
\]
(36)
Suppose that the vector field associated to the 1-form $A$ is orthogonal to $\xi$, that is, $A(\xi) = \{0\}$. We observe from Proposition 3.1 of Pastore & Saltarelli (2011) that relation (14) holds in this context. Thus, it follows from (36) that $k = -1$ and hence by relation (14) we may get $h = \{0\}$, we arrive at a contradiction. This completes the proof.

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 حول منطويات قرب - الكنموتوسوم المعاداة

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خلاصة

الغرض من هذا البحث هو دراسة المنطويات القرب كنموتوسوم المعاداة و المتناظرة التي لها حقل متجهي مميز و ينتمي إلى توزيع صغير.