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New characterizations of k-normal and k-EP matrices

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Abstract

In this paper, some new characterizations of k-normal and k-EP matrices are obtained using the core-EP decomposition. We obtain several equivalent conditions for a matrix A to be k-normal and k-EP in terms of certain generalized inverses.

Keywords: Core-EP decomposition; generalized inverses; k-core EP matrices; k-EP matrices; k-normal matrices.

1. Introduction

The concepts of the classes of k-normal matrices and k-EP matrices were introduced by Malik *et al.* in (Malik *et al.*, 2016) where the authors studied characterizations and properties of both k-normal and k-EP matrices using the Hartwing-Spindelböck decomposition. More properties of these two types of matrices have been given in (Ferreyra *et al.*, 2018; Wang *et al.*, 2019). Inspired by the previous work, the intention of this paper is to discuss both classes and their further properties and characterizations using some generalized inverses.

The classical Moore-Penrose inverse(Penrose *et al.*, 1955) and Drazin inverse(Drazin *et al.*, 1958) were defined in the fifties and have been thoroughly studied since then. On the other hand, some generalized inverses such as core inverse(Baksalary *et al.*, 2010), core EP inverse(**?**), DMP inverse(Malik *et al.*, 2014), WG inverse(Wang *et al.*, 2018), etc., were introduced in the last decade. Nowadays, they attract the attention of many researchers.

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices and $\mathbb{Z}^+$ denotes the set of all positive integers. The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^*, $r(A)$ and I_n will denote the range space, null space, conjugate transpose, rank of $A \in \mathbb{C}^{m\times n}$ and the identity matrix of order n. Ind(A) means the index of $A \in \mathbb{C}^{n\times n}$. Let $\mathbb{C}_k^{n\times n}$ be the set consisting of $n \times n$ complex matrices with index k.

For convenience, throughout the paper we will use the following notations: $\mathbb{C}_n^{k-\mathrm{N}}$, $\mathbb{C}_n^{k,\textcircled{\dagger}}$ and $\mathbb{C}_n^{k-\mathrm{EP}}$ will denote the subsets of $\mathbb{C}^{n\times n}$ consisting of k-normal, k-core EP and k-EP matrices, respectively, i.e.,

$$\mathbb{C}_n^{k-\mathrm{N}} = \{A \mid A \in \mathbb{C}^{n\times n}, A^kA^* = A^*A^k\};$$

$$\begin{aligned}\mathbb{C}_n^{k,\textcircled{\dagger}} &= \{A \mid A \in \mathbb{C}_k^{n\times n}, A^kA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^k\} \\ &= \{A \mid A \in \mathbb{C}_k^{n\times n}, A^k \in \mathbb{C}_n^{\mathrm{EP}}\};\end{aligned}$$

$$\mathbb{C}_n^{k-\mathrm{EP}} = \{A \mid A \in \mathbb{C}_k^{n\times n}, A^kA^\dagger = A^\dagger A^k\}.$$

The structure of this article is as follows: In the Section 2, we discuss several sufficient and necessary conditions for the class of k-normal matrices in terms of generalized inverses. The Section 3 is devoted to the characterizations of the sets $\mathbb{C}_n^{k,\textcircled{\dagger}}$ and $\mathbb{C}_n^{k-\mathrm{EP}}$.

2. Properties of the k-normal matrices

In this section, we will consider the class of k-normal matrices in terms of some generalized inverses.

In the following lemma, we will present the core-EP decomposition, which was given by Wang in (Wang *et al.*, 2016).

Lemma 2.1 (Wang *et al.*, 2016)(*core-EP decomposition*) *Let* $A \in \mathbb{C}_k^{n\times n}$. *Then* A *can be represented as*

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{1}$$

where $T \in \mathbb{C}^{t\times t}$ *is nonsingular and* $t = r(T) = r(A^k)$, N *is nilpotent with index* k, *and* $U \in \mathbb{C}^{n\times n}$ *is unitary.*

Moreover, the representation of A *given by* (1) *is unique (Wang et al., 2016, Theorem 2.4). Furthermore, in that case the core-EP inverse of* A *is given by*

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2}$$

Next we will introduce the following notations, that will be used throughout this paper.

Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1). Then

$$\triangle = [TT^* + S(I_{n-t} - N^\dagger N)S^*]^{-1};$$

$$\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j};$$

$$T_q = \sum_{j=0}^{q-1} T^j S N^{q-1-j} (q \in \mathbb{Z}^+).$$

Lemma 2.2 *Let* $A \in \mathbb{C}_k^{n\times n}$ *be given by* (1). *Then (Ferreyra et al., 2018; Wang et al., 2018):*

$$A^\dagger = U\begin{bmatrix} T^*\triangle & -T^*\triangle S N^\dagger \\ (I_{n-t} - N^\dagger N)S^*\triangle & N^\dagger - (I_{n-t} - N^\dagger N)S^*\triangle S N^\dagger \end{bmatrix} U^*; \tag{3}$$

$$A^D = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{bmatrix} U^*; \tag{4}$$

$$A^{D,\dagger} = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} N N^\dagger \\ 0 & 0 \end{bmatrix} U^*; \tag{5}$$

$$A^{\dagger,D} = U\begin{bmatrix} T^*\triangle & T^*\triangle T^{-k}\widetilde{T} \\ (I_{n-t} - N^\dagger N)S^*\triangle & (I_{n-t} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T} \end{bmatrix} U^*; \tag{6}$$

$$A^{\textcircled{w}} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*. \tag{7}$$

According to (1) and (3), we have

$$AA^\dagger = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix} U^*, \tag{8}$$

$$A^\dagger A = U\begin{bmatrix} T^*\triangle T & T^*\triangle S(I_{n-t} - N^\dagger N) \\ (I_{n-t} - N^\dagger N)S^*\triangle T & N^\dagger N + (I_{n-t} - N^\dagger N)S^*\triangle S(I_{n-t} - N^\dagger N) \end{bmatrix} U^*. \tag{9}$$

Lemma 2.3 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1). Then $A \in \mathbb{C}_n^{k-N}$ if and only if $T^kT^* = T^*T^k$ and $S = 0$.*

Proof. *Using (1), we have*

$$A^k = U \begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix} U^*. \tag{10}$$

Now it is easy to check that $A \in \mathbb{C}_n^{k-N}$ if and only if $T^kT^* = T^*T^k$ and $S = 0$.

Lemma 2.4 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $q \in \mathbb{Z}^+$. Then $T_q = 0$ if and only if $S = 0$. In particular, $\widetilde{T} = 0$ if and only if $S = 0$.*

Proof. The proof is similar to (Wang *et al.*, 2019), Theorem 2.3.

Lemma 2.5 (Sylvester *et al.*, 1884) *Let $A \in \mathbb{C}^{p\times p}$ and $B \in \mathbb{C}^{q\times q}$ have no common eigenvalues. Then $AY - YB = 0$ has a unique solution $Y = 0$, where $Y \in \mathbb{C}^{p\times q}$.*

Theorem 2.6 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $q \in \mathbb{Z}^+$. Then the following conditions are equivalent:*

(a) $A \in \mathbb{C}_n^{k-N}$;

(b) $A^{k+1}A^\dagger A^* = A^*A^{k+1}A^\dagger$;

(c) $A^*A^\dagger A^{k+1} = A^\dagger A^{k+1}A^*$;

(d) $A^*A^{k+q} = A^kA^*A^q$;

(e) $A^{k+q}A^* = A^qA^*A^k$.

Proof. That (a) implies all other items (b), (c), (d) and (e) follow directly by Lemma 2.3.

$(b) \Rightarrow (a)$. It follows from (1), (3) and (10) that $T^kT^* = T^*T^k$ and $S = 0$. Hence by Lemma 2.3, we have that (a) holds.

$(c) \Rightarrow (a)$. By taking the conjugate transpose of $A^*A^\dagger A^{k+1} = A^\dagger A^{k+1}A^*$ and applying item (b) we get that $A \in \mathbb{C}_n^{k-\mathrm{N}}$.

$(d) \Rightarrow (a)$. Since $A^*A^{k+q} = A^kA^*A^q$, it follows from (1) and (10) that $T^kT^* = T^*T^k$ and $S = 0$. Hence by Lemma 2.3, we have that (a) holds.

$(e) \Rightarrow (a)$. By taking the conjugate transpose of $A^{k+q}A^* = A^qA^*A^k$ and applying point $(d) \Rightarrow (a)$, we can deduce $A \in \mathbb{C}_n^{k-\mathrm{N}}$.

Next we will present 10 conditions involving A^*, X, A and their powers to assure that $A \in \mathbb{C}_n^{k-\mathrm{N}}$, where $X \in \{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\dagger,D}, A^{\textcircled{w}}\}$.

Theorem 2.7 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $X \in \{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\dagger,D}, A^{\textcircled{w}}\}$. The following assertions are equivalent:*

(a) $A \in \mathbb{C}_n^{k-N}$;

(b) $A(A^*)^kX = (A^*)^k$;

(c) $X(A^*)^kA = (A^*)^k$;

(d) $(A^*)^kAX = A(A^*)^kX$;

(e) $(A^*)^kXA = A(A^*)^kX$;

(f) $A^*A^kX = A^kA^*X$;

(g) $A^*A^{k+1}X = XA^{k+1}A^*$;

(h) $A^*A^{k+1}X = A^{k+1}XA^*$;

(i) $A^*XA^{k+1} = XA^{k+1}A^*$;

(j) $A^*XA^{k+1} = A^{k+1}XA^*$;

(k) $A^*AX^{k+1} = AX^{k+1}A^*$.

Proof. From (2), (4), (5), (6) and (7), that (a) implies all items $(b)-(k)$ can be directly verified by Lemma 2.3.

On the converse, we have to prove that each of conditions $(b)-(k)$ implies that $T^kT^* = T^*T^k$ and $S=0$.

$(b)\Rightarrow(a)$. Assume that $A(A^*)^kA^{\dagger,D} = (A^*)^k$. From (6) and (10), we obtain that

(i) $(T(T^k)^* + S\widetilde{T}^*)T^*\triangle = (T^k)^*$;

(ii) $(T(T^k)^* + S\widetilde{T}^*)T^*\triangle T^{-k}\widetilde{T} = 0$;

(iii) $N\widetilde{T}^*T^*\triangle = \widetilde{T}^*$;

(iv) $N\widetilde{T}^*T^*\triangle T^{-k}\widetilde{T} = 0$.

From (iii) we have that $H\widetilde{T} - \widetilde{T}N^* = 0$, where $H = (\triangle^*T)^{-1}$. Notice that H is invertible and N^* is nilpotent, hence H and N^* have no common eigenvalues. By Lemma 2.5, we get that $\widetilde{T}=0$, which implies $S=0$. Now we obtain $T^kT^* = T^*T^k$ by (i). The other cases follow similarly.

$(c)\Rightarrow(a)$. Let $\widetilde{X}\in\{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\textcircled{W}}\}$. From $\widetilde{X}(A^*)^kA = (A^*)^k$, using (2), (4), (5), (7) and (10), it can be easily verified that $T^*T^k = T^*T^k$ and $S=0$. If $A^{\dagger,D}(A^*)^kA = (A^*)^k$, then it follows from (6), (10) and Lemma 2.4 that $T^*T^k = T^*T^k$ and $S=0$.

$(d)\Rightarrow(a)$ and $(e)\Rightarrow(a)$. These proofs are similar to the proof of the part $(b)\Rightarrow(a)$.

$(f)\Rightarrow(a)$. By (2), (4), (5), (6), (7) and (10), it follows from $A^*A^kX = A^kA^*X$ that $T^*T^k = T^*T^k$ and $S=0$.

$(g)\Rightarrow(a)$. Let $\widetilde{X}\in\{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\textcircled{W}}\}$. If $A^*A^{k+1}\widetilde{X} = \widetilde{X}A^{k+1}A^*$, then we get $T^*T^k = T^*T^k$ and $S=0$ by (2), (4), (5), (7) and (10). In the case when $A^*A^{k+1}A^{\dagger,D} = A^{\dagger,D}A^{k+1}A^*$, the proof is similar to that for $(b)\Rightarrow(a)$.

$(h)\Rightarrow(a)$. If $A^*A^{k+1}X = A^{k+1}XA^*$, then it follows from (2), (4), (5), (6), (7) and (10) that $T^*T^k = T^*T^k$ and $S=0$.

$(i)\Rightarrow(a)$, $(j)\Rightarrow(a)$ and $(k)\Rightarrow(a)$. These are all similar to the proof of $(g)\Rightarrow(a)$.

3. More properties of the k-core EP and k-EP matrices

In the section, we discuss the necessary and sufficient conditions to satisfy a matrix A such that $A\in\mathbb{C}_n^{k,\textcircled{\dagger}}$ and $A\in\mathbb{C}_n^{k-\text{EP}}$ using some generalized inverses.

Lemma 3.1 (Ferreyra *et al.*, 2018) *Let* $A\in\mathbb{C}_k^{n\times n}$ *be given by* (1). *Then* $A\in\mathbb{C}_n^{k,\textcircled{\dagger}}$ *if and only if* $\widetilde{T}=0$.

We recall that the class of k-core EP matrices are defined by satisfying $A^kA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^k$(Ferreyra *et al.*, 2018), which is equivalent with $A^k(A^k)^\dagger = (A^k)^\dagger A^k(in\ short,\ A^k\in\mathbb{C}_n^{EP})$. Next, we will give a new sufficient and necessary condition for A such that $A\in\mathbb{C}_n^{k,\textcircled{\dagger}}$.

Theorem 3.2 *Let* $A\in\mathbb{C}_k^{n\times n}$ *be given by* (1) *and* $q\in\mathbb{Z}^+$. *Then* $A\in\mathbb{C}_n^{k,\textcircled{\dagger}}$ *if and only if* $A^q(A^k)^\dagger = (A^k)^\dagger A^q$.

Proof. It follows from (Ferreyra *et al.*, 2018) that

$$(A^k)^\dagger = U\begin{bmatrix} (T^k)^*(T^k(T^k)^* + \widetilde{T}\widetilde{T}^*)^{-1} & 0 \\ \widetilde{T}^*(T^k(T^k)^* + \widetilde{T}\widetilde{T}^*)^{-1} & 0 \end{bmatrix}U^*. \tag{11}$$

From $A^q(A^k)^\dagger = (A^k)^\dagger A^q$ and $(T^k)^*(T^k(T^k)^* + \widetilde{T}\widetilde{T}^*)^{-1}$ is invertible, we obtain $\widetilde{T}=0$, we now have $A\in\mathbb{C}_n^{k,\textcircled{\dagger}}$ by Lemma 3.1.

Conversely, if $A\in\mathbb{C}_n^{k,\textcircled{\dagger}}$, it is simple to show that $A^q(A^k)^\dagger = (A^k)^\dagger A^q$ by Lemma 3.1.

Next we will consider different characterizations of $A \in \mathbb{C}_n^{k,\textcircled{\dagger}}$ using several generalized inverses.

Theorem 3.3 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $X \in \{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\dagger,D}, A^{\textcircled{w}}\}$. The following are equivalent:*

(a) $A \in \mathbb{C}_n^{k,\textcircled{\dagger}}$;

(b) $(A^*)^k XA = (A^*)^k$;

(c) $AX(A^*)^k = (A^*)^k$;

(d) $(A^*)^k AX = AX(A^*)^k$;

(e) $(A^*)^k XA = XA(A^*)^k$;

(f) $(A^*)^k AX = XA(A^*)^k$;

(g) $(A^*)^k XA = AX(A^*)^k$.

Proof. The proofs of $(a) \Leftrightarrow (b)$ and $(a) \Leftrightarrow (c)$ follow directly by (2), (4), (5), (6) and (7).

$(a) \Rightarrow (d)$. If $A \in \mathbb{C}_n^{k,\textcircled{\dagger}}$, it is not difficult to verify that $(A^*)^k AX = AX(A^*)^k$ by Lemma 3.1.

$(d) \Rightarrow (a)$. By (2), (4), (5), (6), (7) and Lemma 3.1, we can deduce that $A \in \mathbb{C}_n^{k,\textcircled{\dagger}}$.

The proofs of $(a) \Rightarrow (e)$, $(a) \Rightarrow (f)$ and $(a) \Rightarrow (g)$ are similar to the proof of $(a) \Rightarrow (d)$.

The proofs of $(e) \Rightarrow (a)$, $(f) \Rightarrow (g)$ and $(g) \Rightarrow (a)$ are similar to the proof of $(d) \Rightarrow (a)$.

In (Ferreyra *et al.*, 2018), the authors presented some equivalent conditions for $A^kA^\dagger = A^\dagger A^k$. Inspired by this work, we will present several new characterizations of the class of k-EP matrices.

Lemma 3.4 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $p \geq k$. The following are equivalent:*

(a) $A \in \mathbb{C}_n^{k-EP}$;

(b) (Ferreyra *et al.*, 2018) $S = SN^\dagger N$ *and* $\widetilde{T} = \widetilde{T}NN^\dagger$;

(c) $A^{p+1}A^\dagger = A^\dagger A^{p+1}$.

Proof. The proof follows directly by (1), (3) and (10).

Theorem 3.5 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $p, q \geq k$. The following are equivalent:*

(a) $A \in \mathbb{C}_n^{k-EP}$;

(b) $AA^\dagger(A^*)^p = (A^*)^pA^\dagger A$;

(c) $A^{p+1}A^\dagger = A^p$ *and* $A^\dagger A^{q+1} = A^q$;

(d) $A^{p+1}A^\dagger + A^\dagger A^{q+1} = A^p + A^q$.

Proof. $(a) \Rightarrow (b)$. It follows from Lemma 3.4 that $A^{p+1}A^\dagger = A^\dagger A^{p+1}$, then by taking the conjugate transpose of $A^{p+1}A^\dagger = A^\dagger A^{p+1}$, we obtain $AA^\dagger(A^*)^p = (A^*)^pA^\dagger A$.

$(b) \Rightarrow (c)$. By taking the conjugate transpose of $AA^\dagger(A^*)^p = (A^*)^pA^\dagger A$, we get that $A^{p+1}A^\dagger = A^\dagger A^{p+1}$. Then, by Lemma 3.4, we obtain that $A \in \mathbb{C}_n^{k-\mathrm{EP}}$. Furthermore, we can directly check $A^{p+1}A^\dagger = A^p$ and $A^\dagger A^{q+1} = A^q$ by condition (b) of Lemma 3.4.

$(c) \Rightarrow (d)$. It is evident.

$(d) \Rightarrow (a)$. Premultplying and postmultiplying the condition $A^{p+1}A^\dagger + A^\dagger A^{q+1} = A^p + A^q$ by A, we get $A^{p+2}A^\dagger = A^{p+1}$ and $A^\dagger A^{q+2} = A^{q+1}$, respectively, which imply $A \in \mathbb{C}_n^{k-\mathrm{EP}}$.

In the next result, we will show certain necessary and sufficient conditions for a matrix A such that $A \in \mathbb{C}_n^{k-\mathrm{EP}}$ using MP and Drazin inverse.

Theorem 3.6 *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (1) and $q \in \mathbb{Z}^+$. The following are equivalent:*

(a) $A \in \mathbb{C}_n^{k-EP}$;

(b) $A^q(A^D)^kA^\dagger = A^\dagger A^q(A^D)^k$;

(c) $A^\dagger A^2(A^D)^kA^\dagger = (A^D)^k$;

(d) $(A^D)^kA^\dagger = A^\dagger(A^D)^k$.

Proof. $(b) \Leftrightarrow (a)$. By (1) and (4), we get that $A^q(A^D)^kA^\dagger = A^\dagger A^q(A^D)^k$ is equivalent with $S = SN^\dagger N$ and $\widetilde{T} = \widetilde{T}NN^\dagger$, i.e., $A \in \mathbb{C}_n^{k-\mathrm{EP}}$.

The proofs of $(c) \Leftrightarrow (a)$ and $(d) \Leftrightarrow (a)$ follow as the proof of the part $(b) \Leftrightarrow (a)$.

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