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Numerical study on multi-order multi-dimensional fractional optimal control problem in general form

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Abstract

The aim of this work is application of Bernstein polynomials (BPs) for solving multi-order multi-dimensional fractional optimal control problem (MOMDFOCP). Firstly, by the Bernstein basis, we introduce operational matrices for Riemann-Liouville fractional integral and product in the arbitrary interval [a,b]. Then, via these matrices, we reduce the problem to the optimization problem. For solving this problem, we apply Lagrangian multipliers method. So, we can obtain approximate solution for MOMDFOCP. Results of some examples show that the obtained solutions are very accurate and in good agreement with exact solutions.

Keywords: Bernstein polynomials; Caputo fractional derivative; multi-order multi-dimensional fractional optimal control problem; operational matrices; Riemann-Liouville fractional integral.

1. Introduction

Recently, researchers in various fields of science are interested in using the fractional order differential equations for better interpretations of their results by incorporating more information in their models, for examples in biology, economics, polymer rheology, chemistry, mechanics, aerodynamics, control theory, regular variation in thermodynamics, biophysics, signal and image processing etc (Hilfer, 2000; Kilbas, *et al.*, 2006; Garg & Manohar, 2013; Ghany & Hyder, 2014). One of these models is fractional optimal control problem (FOCP). Indeed, optimal control problem for dynamic system with derivative of fractional order is called fractional optimal control problem. FOCP is very interesting for researchers and they have done many works in recent years, (Lotfi *et al.*, 2011; Agrawal, 2004; Agrawal & Baleanu, 2007; Baleanu *et al.*, 2009). In most of these papers, the authors worked on FOCP with one state, one control function and one order for fractional derivative. But in this paper, we deal FOCP with several states, control functions and multi-order for fractional derivatives that call multi-order multi-dimensional fractional optimal control problem (MOMDFOCP). Indeed, this problem covers the previous problems in the field of FOCP. So, solving MOMDFOCP will be very important. Earlier, we used Bernstein operational matrices of Caputo derivative, Riemann-Liouville fractional integral and product in interval [0,1] for solving fractional quadratic Riccati differential equations (Baleanu *et al.*, 2013), multi-order fractional differential equations (Rostamy *et al.*, 2013 and 2014), nonlinear system of fractional differential equations (Alipour & Baleanu, 2013), Abel's integral equation (Alipour & Rostamy, 2011) and time varying fractional optimal control problems (Alipour & Rostamy, 2013). Now, in this work, we introduce Bernstein operational matrices of Riemann-Liouville fractional integral and product in the arbitrary interval [a,b] then apply them for solving MOMDFOCP.

In this work, we consider the multi-order multi-dimensional fractional optimal control problem as follows:

$$\textit{Minimize} \quad J\left(t, X(t), U(t)\right) = \int_a^b f\left(t, X(t), U(t)\right) dt, \tag{1}$$

subject to the system of dynamic constrains

$$g_i\left(t, {}_a^C D_t^{\alpha_i} X(t), X(t), U(t)\right) = 0, \quad i = 1, \ldots, n, \; a < t \le b, \tag{2}$$

and system of inequality constrains

$$h_r\left(t, X(t), U(t)\right) \le 0, \quad r = 1, \ldots, l, \quad a < t \le b, \tag{3}$$

and the initial condition

$$X(a) = X_a, \tag{4}$$

where $X(t)=[x_1(t),\ldots,x_n(t)]^T$ and $U(t)=[u_1(t),\ldots,u_k(t)]^T$ are state and control functions, respectively. Also, $\alpha_i=\left[\alpha_{i1},\ldots,\alpha_{in}\right]^T,\left(0<\alpha_{in}<\ldots<\alpha_{i2}<\alpha_{i1}\le 1\right)$, $X_a=[x_{a,1},\ldots,x_{a,n}]^T$, ${}_a^CD_t^{\alpha_i}X(t)=[{}_a^CD_t^{\alpha_{i1}}x_1(t),\ldots,{}_a^CD_t^{\alpha_{in}}x_n(t)]^T$ and $f,h_r:[a,b]\times\Re^{n+k}\to\Re$, $g_i:[a,b]\times\Re^{2n+k}\to\Re$ are polynomial functions.

We organize the rest of this paper as follows: In Section 2, we present basic definitions and properties in fractional calculus. In Section 3, we introduce BPs and apply them to approximate functions. Also, some useful Lemmas and Corollaries of BPs are proposed in this section. We get the operational matrix for Riemann-Liouville fractional integral by BPs in Section 4. In Section 5, we use BPs for solving MOMDFOCPs. In section 6, some examples are applied to show applicability and accuracy of the proposed method. Conclusions of our works are in final section.

2. Basic definitions and properties in fractional calculus

In this section, we remark some basic definitions and properties of the fractional calculus.

Definition 2.1. (Caputo, 1967) The Riemann-Liouville fractional integral operator of order $\alpha\ge 0$ for function $f(t)$, is defined as

$$ {}_aI_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\quad \alpha>0,\,t>a, $$

$$ {}_aI_t^0f(t)=f(t), \tag{5} $$

and for $n-1<\alpha\le n$, $n\in\mathrm{N}$, $t>a$, the fractional derivative of $f(t)$ in the Caputo sense is defined as

$$ \begin{aligned} {}_a^CD_t^{\alpha}f(t)&={}_aI_t^{n-\alpha}\;{}_a^CD_t^{n}f(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)dx\,. \end{aligned} \tag{6} $$

Now, we can propose the following properties for $\alpha\ge\beta\ge 0$ and $n-1<\alpha\le n$, $n\in\mathrm{N}$ (Luchko & Gorneflo, 1998; Miller & Ross, 1993; Oldham & Spanier, 1974):

$$ i)\;{}_aI_t^{\alpha}(t-a)^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-a)^{\alpha+\gamma}, \tag{7} $$

$$ ii)\;{}_a^CD_t^{\alpha}\;{}_aI_t^{\alpha}f(t)=f(t), \tag{8} $$

$$ iii)\;{}_aI_t^{\alpha}\;{}_a^CD_t^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(a^+)\frac{(t-a)^k}{k!},\;t>a, \tag{9} $$

$$ iv)\;{}_a^CD_t^{\beta}f(t)={}_aI_t^{\alpha-\beta}\;{}_a^CD_t^{\alpha}f(t). \tag{10} $$

3. Bernstein polynomials and approximations

The Bernstein polynomials (BPs) of degree m on the interval $[a,b]$ are defined as follows (Kreyszig, 1978):

$$ B_{i,m}(t)=\binom{m}{i}\frac{(t-a)^i(b-t)^{m-i}}{(b-a)^m},\quad i=0,1,\cdots,m. \tag{11} $$

Since, set $\{B_{0,m}(t),B_{1,m}(t),\cdots,B_{m,m}(t)\}$ form a basis in the polynomials space of degree m on the interval $[a,b]$, so we can write the following express for every polynomial of degree m:

$$ P(t)=\sum_{i=0}^{m}c_i\,B_{i,m}(t). \tag{12} $$

Lemma 3.1. Let

$\overline{T}_m(t)=\left[1,(t-a),(t-a)^2,\ldots,(t-a)^m\right]^T$ and

$\Phi_m(t)=\left[B_{0,m}(t),B_{1,m}(t),B_{2,m}(t),\ldots,B_{m,m}(t)\right]^T$, then

$$ \Phi_m(t)=A\,\overline{T}_m(t), \tag{13} $$

where $A=\left(a_{i,j}\right)_{i,j=1}^{m+1}$ and

$$ a_{i+1,j+1}=\begin{cases}\dfrac{(-1)^{j-i}}{(b-a)^j}\dbinom{m}{i}\dbinom{m-i}{j-i} & i\le j,\\ 0 & i>j,\end{cases} $$

$i,j=0,1,\ldots,m.$

Proof. From expansion of $(b-t)^{m-i}$, we can get

$$ \begin{aligned} B_{i,m}(t)&=\binom{m}{i}\frac{(t-a)^i(b-t)^{m-i}}{(b-a)^m}\\ &=\binom{m}{i}\frac{(t-a)^i((b-a)+(a-t))^{m-i}}{(b-a)^m}\\ &=\binom{m}{i}\frac{(t-a)^i}{(b-a)^m}\\ &\sum_{k=0}^{m-i}\binom{m-i}{k}(b-a)^{m-i-k}(a-t)^k\\ &=\sum_{k=0}^{m-i}(-1)^k\binom{m}{i}\binom{m-i}{k}\left(\frac{t-a}{b-a}\right)^{k+i}\\ &=\sum_{j=i}^{m}(-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}\left(\frac{t-a}{b-a}\right)^{j},\\ &i=0,1\ldots,m. \end{aligned} $$

So, it is clear that we can write $\Phi_m(t)=A\,\overline{T}_m(t)$.□

Corollary 3.2. We can approximate a function $y(t)$ as follows

$$y(t) \approx \sum_{i=0}^{m} c_i B_{i,m}(t) = c^T \Phi_m(t), \tag{14}$$

where $c^T = [c_1, c_2, \cdots, c_m] = \left(\int_a^b y(t)\, \Phi_m(t)^T dt \right) Q^{-1}$, such that matrix dual $Q = \left(Q_{i,j}\right)_{i,j=1}^{m+1}$ and

$$Q_{i+1,j+1} = \int_a^b B_{i,m}(t) B_{j,m}(t) dt = \frac{(b-a)\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}},$$

$$i, j = 0,1, \cdots m. \tag{15}$$

Definition 3.3. We denote the operational matrix of product for vector c based on basis $\Phi_m(t)$ by $\hat{C}$ and define as:

$$c^T \Phi_m(t) \Phi_m(t)^T \approx \Phi_m(t)^T \hat{C}. \tag{16}$$

To get $\hat{C}$, we can follow the steps below.

By (13) we obtain

$$c^T \Phi_m(t) \Phi_m(t)^T = c^T \Phi_m(t) \left(\overline{T}_m(t)^T A^T \right)$$
$$= \Big[c^T \Phi_m(t), (t-a)\left(c^T \Phi_m(t)\right),$$
$$\ldots, (t-a)^m \left(c^T \Phi_m(t)\right) \Big] A^T$$
$$= \Big[\sum_{i=0}^{m} c_i B_{i,m}(t), \sum_{i=0}^{m} c_i (t-a) B_{i,m}(t),$$
$$\ldots, \sum_{i=0}^{m} c_i (t-a)^m B_{i,m}(t) \Big] A^T.$$

Then, we can apply the approximation $(t-a)^k B_{i,m}(t) \approx e_{k,i} \Phi_m(t)$, $(i,k = 0,1,\ldots,m)$, $e_{k,i} = \left[e_{k,i}^0 \;,\; e_{k,i}^1 \;,\; \ldots \;,\; e_{k,i}^m \right]^T$ where

$$e_{k,i} = \frac{Q^{-1}\binom{m}{i}(b-a)^{k+1}}{2m+k+1}$$
$$\left[\frac{\binom{m}{0}}{\binom{2m+k}{i+k}}, \frac{\binom{m}{1}}{\binom{2m+k}{i+k+1}}, \ldots, \frac{\binom{m}{m}}{\binom{2m+k}{i+k+m}} \right]^T,$$
$$i, k = 0,1,\ldots,m.$$

So, we get

$$\sum_{i=0}^{m} c_i (t-a)^k B_{i,m}(t) \approx \sum_{i=0}^{m} c_i \left(\sum_{j=0}^{m} e_{k,i}^j B_{j,m}(t) \right)$$
$$= \sum_{j=0}^{m} B_{j,m}(t) \left(\sum_{i=0}^{m} c_i e_{k,i}^j \right)$$
$$= \Phi_m(t)^T \left[\sum_{i=0}^{m} c_i e_{k,i}^0, \sum_{i=0}^{m} c_i e_{k,i}^1, \ldots, \sum_{i=0}^{m} c_i e_{k,i}^m \right]^T$$
$$= \Phi_m(t)^T \underbrace{\left[e_{k,0}, e_{k,1}, \ldots, e_{k,m} \right]}_{V_k} c = \Phi_m(t)^T V_k c,$$

Finally, by the above results, we can write

$$c^T \Phi_m(t) \Phi_m(t)^T$$
$$\approx \Phi_m(t)^T \underbrace{\left[V_0 c, V_1 c, \ldots, V_m c \right] A^T}_{\hat{C}},$$

and therefore $\hat{C}$ is obtained.

Corollary 3.4. Let $y(t) \approx c^T \Phi_m(t)$, $x(t) \approx d^T \Phi_m(t)$, then We can approximate the functions $x(t)\, y(t)$ and $y^k(t)$ as follows:

$$\begin{aligned} & y(t)\, x(t) \approx \Phi_m(t)^T \hat{C} d, \\ & y^k(t) \approx \Phi_m(t)^T \hat{C}^{k-1} c. \end{aligned} \tag{17}$$

Proof. Refer to Rostamy *et al.* (2014).

4. Bernstein operational matrix of Riemann-Liouville fractional integral

Now, we want to get the operational matrix for the Riemann-Liouville fractional integral by BPs. We can write:

$${}_aI_t^{\alpha} \Phi_m(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \Phi_m(\tau) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} A \overline{T}_m(\tau) d\tau$$
$$= A \left[{}_aI_t^{\alpha} 1, {}_aI_t^{\alpha} (t-a), \ldots, {}_aI_t^{\alpha} (t-a)^m \right]^T$$
$$= A \Big[\frac{0!}{\Gamma(\alpha+1)} (t-a)^{\alpha}, \frac{1!}{\Gamma(\alpha+2)} (t-a)^{\alpha+1},$$
$$\ldots, \frac{m!}{\Gamma(\alpha+m+1)} (t-a)^{\alpha+m} \Big]^T = A\, D T_{m,\alpha}(t),$$

where D and $T_{m,\alpha}$ are as follows:

$$D_{i,j}=\begin{cases}\dfrac{i!}{\Gamma(\alpha+i+1)} & i=j,\\ 0 & i\neq j,\end{cases}\quad i,j=0,1,\dots,m,$$

$$T_{m,\alpha}(t)=\left[(t-a)^{\alpha},(t-a)^{\alpha+1},\dots,(t-a)^{\alpha+m}\right]^T.$$

Now, we need to approximate $(t-a)^{\alpha+i}$ $(i=0,1,\dots,m)$ with respect to BPs by using (14). Therefore, we have

$(t-a)^{\alpha+i}\approx E_i^T\Phi_m(t)$,

where

$$E_i=\frac{(b-a)^{\alpha+i+1}m!}{\Gamma(i+m+\alpha+2)}Q^{-1}\left[\frac{\Gamma(i+\alpha+1)}{0!},\frac{\Gamma(i+1+\alpha+1)}{1!},\dots,\frac{\Gamma(i+m+\alpha+1)}{m!}\right]^T,$$

$i=0,\dots,m.$

So, we can write

$${}_aI_t^{\alpha}\Phi_m(t)\approx\underbrace{AD\left[E_0,E_1,\dots,E_m\right]^T}_{F_\alpha}\Phi_m(t),$$

Finally, we obtain

$${}_aI_t^{\alpha}\Phi_m(t)\approx F_\alpha\ \Phi_m(t). \tag{18}$$

We denote the Bernstein operational matrix of Riemann-Liouville fractional integral of order α by F_α.

5. Bps for solving MOMDFOCP

For converting the inequality constrains (3) into the equality constrains we can use unknown functions $w_r(t)\,(r=1,\dots,l)$ as follows :

$$h_r\left(t,X(t),U(t)\right)+w_r(t)^2=0,\ r=1,\dots l. \tag{19}$$

By (14), we can apply the following approximations:

$${}_a^CD_t^{\alpha_{i1}}x_i(t)\approx C_{i1}^T\Phi_m(t),\quad i=1,\dots,n, \tag{20}$$

$$u_j(t)\approx U_j^T\Phi_m(t),\quad j=1,\dots,k, \tag{21}$$

$$w_r(t)\approx W_r^T\Phi_m(t)\quad r=1,\dots,l. \tag{22}$$

From (9), (10), (18) and (20) we have

$$\begin{aligned}x_i(t)&={}_aI_t^{\alpha_{i1}}\,{}_a^CD_t^{\alpha_{i1}}x_i(t)+x_{a,i}\\&\approx{}_aI_t^{\alpha_{i1}}\left(C_i^T\Phi_m(t)\right)+x_{a,i}\,\Lambda_m^T\Phi_m(t)\\&=C_i^T\,{}_aI_t^{\alpha_{i1}}\Phi_m(t)+x_{a,i}\Lambda_m^T\Phi_m(t)\\&\approx\left(C_i^TF_{\alpha_{i1}}+x_{a,i}\Lambda_m^T\right)\Phi_m(t),\end{aligned} \tag{23}$$

and

$$\begin{aligned}{}_a^CD_t^{\alpha_{ij}}x_i(t)&={}_aI_t^{\alpha_{i1}-\alpha_{ij}}\,{}_a^CD_t^{\alpha_{i1}}x_i(t)\\&\approx{}_aI_t^{\alpha_{i1}-\alpha_{ij}}\left(C_i^T\Phi_m(t)\right)\\&=C_i^T\,{}_aI_t^{\alpha_{i1}-\alpha_{ij}}\Phi_m(t)\\&\approx C_i^TF_{\alpha_{i1}-\alpha_{ij}}\Phi_m(t),\end{aligned} \tag{24}$$

$j=2,\dots,n,$

where $\Lambda_m=[\underbrace{1,1,\dots,1}_{m+1}]^T$. Also, from Corollary 3.4 and (22) we have

$$w_r(t)^2\approx\Phi_m(t)^T\hat{W}_rW_r, \tag{25}$$

where $\hat{W}_r$ is the operational matrix of product for vector W_r. Therefore, the problem (1)-(4) is reduced as follows:

$$\begin{aligned}\text{Minimize}\quad&\int_a^b f(t,(C_1^TF_{\alpha_{11}}+x_{0,1}\Lambda_m^T)\Phi_m(t),\\&\dots,(C_n^TF_{\alpha_{n1}}+x_{0,n}\Lambda_m^T)\Phi_m(t),\\&U_1^T\Phi_m(t),\dots,U_k^T\Phi_m(t))dt,\end{aligned} \tag{26}$$

subject to the system of dynamic constrains

$$\begin{aligned}&g_i\left(t,\left(C_1^T\bar{F}_{\alpha_1}+x_{0,1}\bar{\Lambda}_m^T\right)\Phi_m(t),\dots,\right.\\&\left.\left(C_n^T\bar{F}_{\alpha_n}+x_{0,n}\bar{\Lambda}_m^T\right)\Phi_m(t),U_1^T\Phi_m(t),\dots,U_k^T\Phi_m(t)\right)\\&=0,\qquad i=1,\dots,n,\end{aligned} \tag{27}$$

$$\begin{aligned}&h_r\left(t,\left(C_1^TF_{\alpha_{11}}+x_{0,1}\bar{\Lambda}_m^T\right)\Phi_m(t),\dots,\right.\\&\left.\left(C_n^TF_{\alpha_{n1}}+x_{0,n}\bar{\Lambda}_m^T\right)\Phi_m(t),U_1^T\Phi_m(t),\dots,U_k^T\Phi_m(t)\right)\\&+\Phi_m(t)^T\hat{W}_rW_r=0,\qquad r=1,\dots,l,\end{aligned} \tag{28}$$

where

$$\bar{F}_{\alpha_i}=\left[I_m,F_{\alpha_{i1}},F_{\alpha_{i1}-\alpha_{i2}},\dots,F_{\alpha_{i1}-\alpha_{in}}\right],$$

$$\bar{\Lambda}_m=\underbrace{\left[0_m,\Lambda_m,0_m,\dots,0_m\right]}_{n+1}$$

and I_m, 0_m are identity matrix and zero vector from order $m+1$, respectively.

Now, by (14) we approximate all of the known functions in the problem (26)-(28) and , by using Corollaries 3.4 we get the following approximations:

$$\int_a^b f\left(t, X(t), U(t)\right) dt \approx \Psi\left(C_1, \dots, C_n, U_1, \dots, U_k\right), \tag{29}$$

$$g_i\left(t, {}_a^C D_t^{\alpha_i} X(t), X(t), U(t)\right) \approx G_i\left(C_1, \dots, C_n, U_1, \dots, U_k\right)\Phi_m(t), \quad i = 1, \dots, n, \tag{30}$$

and

$$h_r\left(t, X(t), U(t)\right) + w_r(t)^2 \approx \left(\underbrace{H_r\left(C_1, \dots, C_n, U_1, \dots, U_k\right) + W_r^T \hat{W}_r^T}_{\tilde{H}_r(C_1,\dots,C_n,U_1,\dots,U_k,W_r)}\right)\Phi_m(t), \quad r = 1, \dots, l, \tag{31}$$

where

$$\Psi, G_i, H_r : \Re^{(m+1)\times n} \times \Re^{(m+1)\times k} \to \Re^{1\times(m+1)}.$$

Now, by multiplying $\Phi_m(t)^T$ in right hand of (30) and (31), then integration in interval $[a, b]$ and by noting that matrix Q is invertible, we reach an optimization problem as follows:

$$\mathit{Minimize} \quad \Psi\left(C_1, \dots, C_n, U_1, \dots, U_k\right), \tag{32}$$

subject to the system of algebraic equations

$$G_i\left(C_1, \dots, C_n, U_1, \dots, U_k\right) = 0, \quad i = 1, \dots, n, \tag{33}$$

$$\tilde{H}_r\left(C_1, \dots, C_n, U_1, \dots, U_k, W_r\right) = 0, \quad r = 1, \dots, l. \tag{34}$$

We apply the Lagrange method for solving this optimization problem. Therefore, we introduce Lagrange function as:

$$\begin{aligned} L\big(&C_1, \dots, C_n, U_1, \dots, U_k, W_1, \dots, W_l, \lambda_1, \dots, \lambda_n, \tilde{\lambda}_1, \dots, \tilde{\lambda}_l\big) \\ &= \Psi\left(C_1, \dots, C_n, U_1, \dots, U_k\right) + \sum_{i=1}^{n} \lambda_i G_i\left(C_1, \dots, C_n, U_1, \dots, U_k\right) \\ &+ \sum_{r=1}^{l} \tilde{\lambda}_r \tilde{H}_r\left(C_1, \dots, C_n, U_1, \dots, U_k, W_r\right), \end{aligned} \tag{35}$$

where $\lambda_i\ (i = 1, \dots, n)$ and $\tilde{\lambda}_r\ (r = 1, \dots, l)$ call the Lagrange multipliers. Now, by considering the necessary conditions for the extremum we can get the following systems of algebraic equations

$$\frac{\partial L}{\partial C_i} = 0, \tag{36}$$

$$\frac{\partial L}{\partial \lambda_i} = 0, \quad i = 1, \dots, n, \tag{37}$$

$$\frac{\partial L}{\partial U_j} = 0, \quad j = 1, \dots, k, \tag{38}$$

$$\frac{\partial L}{\partial W_r} = 0, \tag{39}$$

$$\frac{\partial L}{\partial \tilde{\lambda}_r} = 0, \quad r = 1, \dots, l. \tag{40}$$

By solving system (36)-(40), we obtain C_i, U_j, W_r, λ_i and $\tilde{\lambda}_r$. So, from (23) and (21) we get the approximations of the state functions $x_i(t)$ and the control functions $u_j(t)$, respectively.

6. Numerical simulations

To demonstrate the applicability and to validate the numerical scheme, we apply the present method for the following examples.

Example 6.1

We consider the following multi-order two-dimensional FOCP

$$\mathit{Minimize} \quad J = \int_0^1 x_1^2(t) + x_2^2(t) + 0.005 u^2(t)\, dt,$$

subject to

$$ {}_0^C D^{\alpha_1} x_1(t) - x_2(t) = 0, $$

$$ {}_0^C D^{\alpha_2} x_2(t) + x_2(t) - u(t) = 0, $$

$$ x_2(t) - 8(t-0.5)^2 + 0.5 \leq 0, $$

$$ x_1(0) = 0, \quad x_2(0) = -1. $$

We can see the approximate solutions of the state and the control functions for $\alpha_1 = 1$ and $\alpha_2 = 0.8, 0.9, 1$ in Figures 1-3. These Figures show that for fixed $\alpha_1 = 1$, as α_2 approaches to 1, the approximate solutions approach to the solutions for $\alpha_1 = \alpha_2 = 1$. Moreover, in Table 1, we report the obtained results for cost function J by the present method and compare with the other methods for $\alpha_1 = \alpha_2 = 1$.

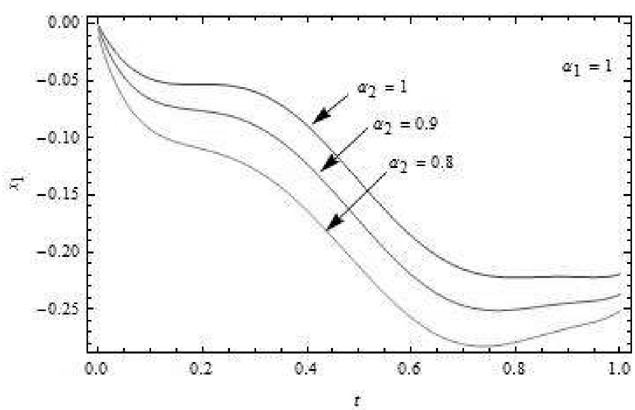

Fig. 1. Plot of $x_1(t)$ for $m = 8$, $\alpha_1 = 1$ and $\alpha_2 = 0.8, 0.9, 1$ in Example 6.1

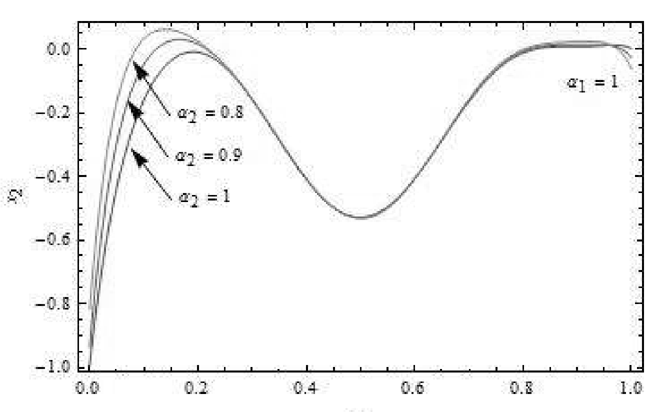

Fig. 2. Plot of $x_2(t)$ for $m = 8$, $\alpha_1 = 1$ and $\alpha_2 = 0.8, 0.9, 1$ in Example 6.1

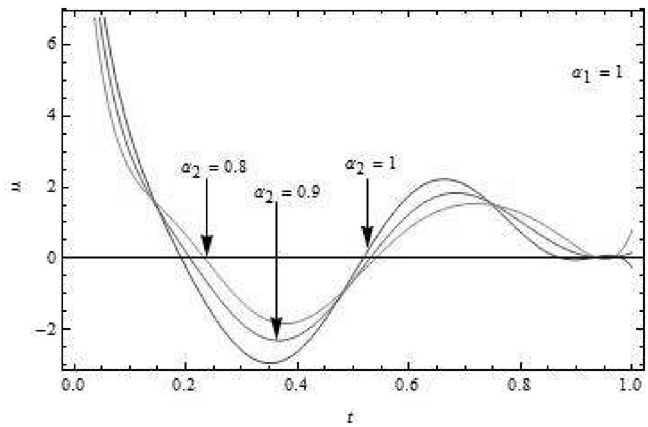

Fig. 3. Plot of $u(t)$ for $m = 8$, $\alpha_1 = 1$ and $\alpha_2 = 0.8, 0.9, 1$ in Example 6.1

Table 1. Obtained results of J for $\alpha_1 = \alpha_2 = 1$ in Example 6.1

Method	J
Classical Chebyshev (Vlassenbroak, 1998) $m = 13, k = 28$	0.171850
Fourier-based (Yen & Nagurka, 1991) $k = 9$	0.17013
Spectral Chebyshev (Jaddu, 2002) $k = 7$	0.170785
Hybrid functions (Marzban & Razzaghi, 2003) $w = 15, M = 4, N = 4$	0.17013640
Haar functions (Marzban & Razzaghi, 2010) $k = 128, w = 100$	0.170103
Chebyshev Finite Difference (Maleki, 2011) $N = 10$	0.170875
Present Method $m = 8$	0.169030

Example 6.2

Consider the following problem

$$ Maximize \quad J = \int_0^{\ln(2)} x(t)\, dt, $$

subject to

$$ {}_0^C D^{\alpha} x(t) - x(t) - u(t) = 0, $$

$$ -1 \leq u(t) \leq 1, $$

$$ x(t) + u(t) \leq 2, $$

$$ x(0) = 0. $$

This problem has exact solution $J = 0.30682$ for $\alpha = 1$ (Elnager, 1993). In Figure 4, we can observe the approximate solution of the state function by present method for $\alpha = 0.8, 0.9, 1$. In Table 2, we tabled the approximations of cost function J by present method with compare by Elnager (1993) for $\alpha = 1$.

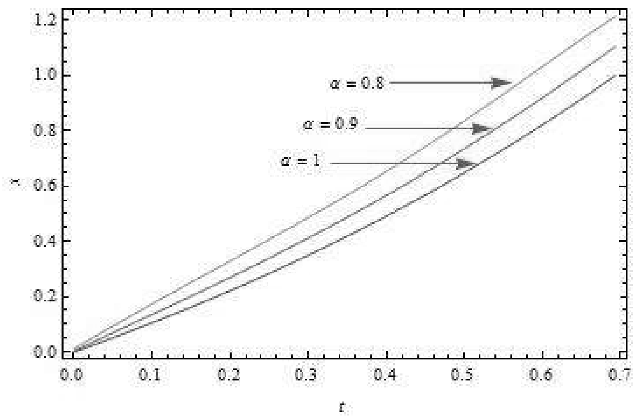

Fig. 4. Plot of $x\,(t)$ for $m=5$ and $\alpha=0.8,0.9,1$ in Example 6.2

Table 2. The approximations of cost function J for α = 1 in Example 6.2.

Method	J
Rationalized Haar Functions (Elnager, 1993) $K = 8$	0.30696
Present method $m = 5$	0.30685

Example 6.3

Consider the two-dimensional FOCP as follows

$$\textit{Minimize} \quad J=\frac{1}{2}\int_0^{\frac{\pi}{2}} 4x_2^2(t)+u^2(t)\,dt\,,$$

subject to the system dynamics

$${}_0^C D^{\alpha} x_1(t)-u(t)=0\,,$$

$${}_0^C D^{\alpha} x_2(t)-x_1(t)=0,$$

with initial conditions

$$x_1(0)=0\,,\ \ x_2(0)=10.$$

This problem for α = 1 have the exact solution (Datta & Mohan, 1995)

$$x_1(t)=-\frac{20\left(e^{\pi-t}+e^{t}\right)\sin(t)}{1+e^{\pi}}\,,$$

$$x_2(t)=\frac{10\left(e^{t}\left(\cos(t)-\sin(t)\right)+e^{\pi-t}\left(\cos(t)+\sin(t)\right)\right)}{1+e^{\pi}}\,,$$

$$u(t)=p(t)\,x_1(t)+q(t)\,x_2(t)\,,$$

where

$$p(t)=-\frac{\sinh(\pi-2t)-\sin(\pi-2t)}{\cosh^2\left(\frac{\pi}{2}-t\right)+\cos^2\left(\frac{\pi}{2}-t\right)}\,,$$

$$q(t)=-\frac{\cosh(\pi-2t)-\cos(\pi-2t)}{\cosh^2\left(\frac{\pi}{2}-t\right)+\cos^2\left(\frac{\pi}{2}-t\right)}\,.$$

We plot the obtained results of the state and the control variables by our method for m = 5 and $\alpha=0.8, 0.9, 1$ in Figures 5-7. These Figures show the approximate solutions for both the state and the control functions approach to the analytical solutions for α = 1 when α approaches to 1, as expected. Also, we report the absolute errors for our method with α = 1 and m = 5, 7 for different value t in Tables 3-5. So, we show the results will be more accurate as m be increased.

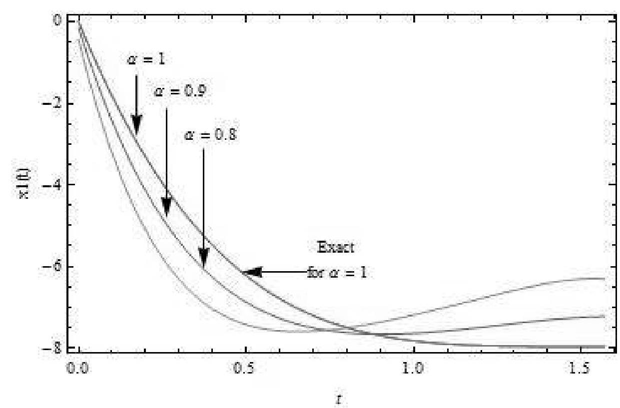

Fig. 5. Plot of $x_1(t)$ for m = 5 and $\alpha=0.8, 0.9, 1$ in Example 6.3

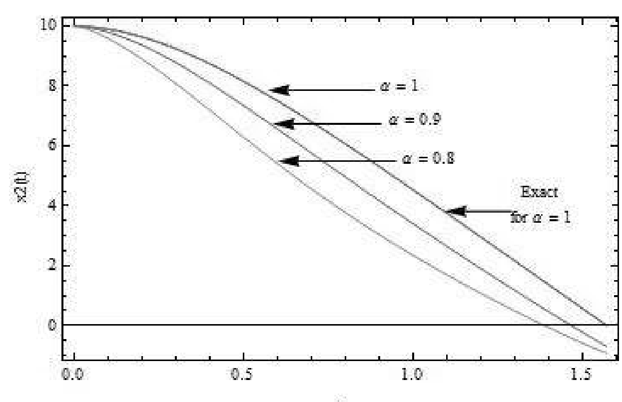

Fig. 6. Plot of $x_2(t)$ for m = 5 and $\alpha=0.8, 0.9, 1$ in Example 6.3

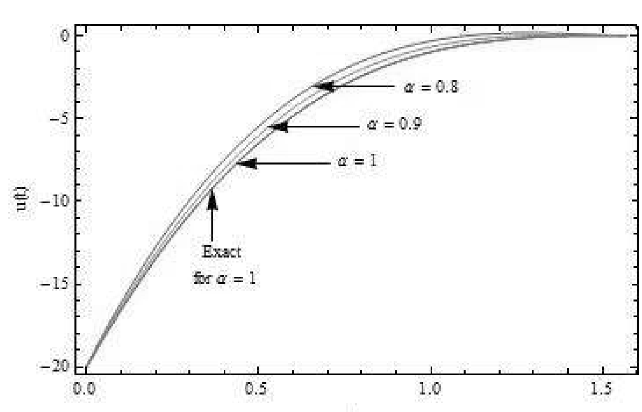

Fig. 7. Plot of $u\,(t)$ for m = 5 and $\alpha=0.8, 0.9, 1$ in Example 6.3

Table 3. Absolute error of $x_1(t)$ for $\alpha = 1$ and different values t in Example 6.3.

t	$m = 5$	$m = 7$
0.3	0.00018419	2.15677×10^{-6}
0.6	0.0000873692	1.10226×10^{-6}
0.9	0.0000982275	8.13366×10^{-7}
1.2	0.000244898	2.25033×10^{-8}
1.5	0.000123368	3.34895×10^{-6}

Table 4. Absolute error of $x_2(t)$ for $\alpha = 1$ and different values t in Example 6.3.

t	$m = 5$	$m = 7$
0.3	0.0000618499	1.65842×10^{-6}
0.6	0.0000350435	6.71922×10^{-7}
0.9	0.0000188201	4.14416×10^{-7}
1.2	0.0000659208	2.44523×10^{-7}
1.5	0.00002995	2.78407×10^{-6}

Table 5. Absolute error of $u(t)$ for $\alpha = 1$ and different values t in Example 6.3.

t	$m = 5$	$m = 7$
0.3	0.000361339	6.96315×10^{-7}
0.6	0.000111793	1.44511×10^{-7}
0.9	0.000321687	7.55399×10^{-9}
1.2	0.000625898	3.04542×10^{-7}
1.5	0.000345349	1.35264×10^{-6}

Example 6.4

We consider the following problem (Pooseh *et al.,* 2014)

$$Minimize \quad J = \int_0^1 \left(t\,u(t) - (\alpha+2)\,x(t)\right)^2 dt,$$

subject to the system dynamic

$${}_0^C D^1 x(t) + {}_0^C D^\alpha x(t) - u(t) - t^2 = 0,$$

with boundary conditions

$$x(0) = 0, \quad x(1) = \frac{2}{\Gamma(\alpha+3)}.$$

For this problem we have the exact solution $x(t) = \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}$, $u(t) = \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}$.

For $m = 7$, the approximate solutions of the states functions and the control functions are plotted in the Figures 8 and 9. Similar to previous examples, these Figures show that as α approaches to 1, the obtained solutions approach to the ones for $\alpha = 1$ as expected. Also, the errors in the norm L^2 $\left(\|y(t)\|_{L^2[a,b]} = \sqrt{\int_a^b |y(t)|^2\, dt}\right)$ for approximate solutions that have obtained by our method are reported in Tables 6 and 7. We can see that by increasing m, the results will be more accurate.

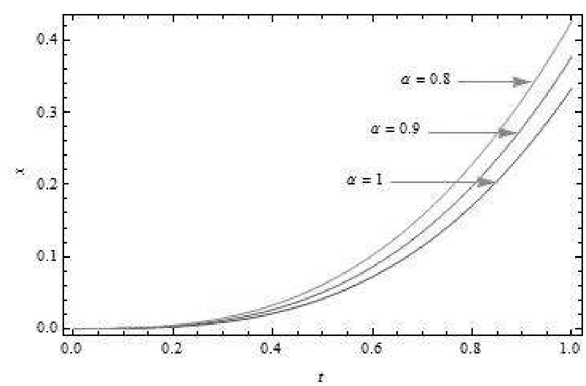

Fig. 8. Plot of $x(t)$ for $m = 7$ and $\alpha = 0.8, 0.9, 1$ in Example 6.4.

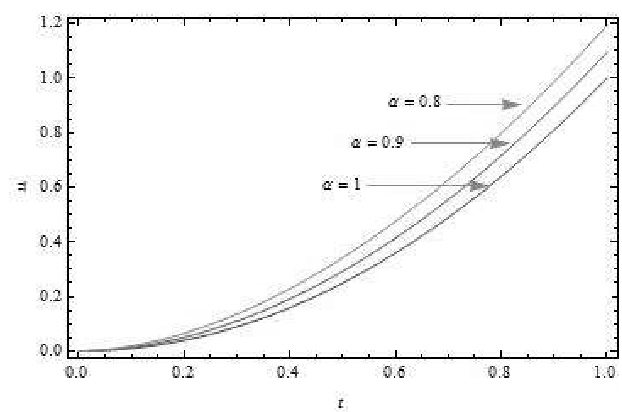

Fig. 9. Plot of $u(t)$ for $m = 7$ and $\alpha = 0.8, 0.9, 1$ in Example 6.4.

Table 6. $\|x(t) - x_m(t)\|_{L^2[0,1]}$ for different values m and α in Example 6.4.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
7	2.89933×10^{-17}	7.5999×10^{-7}	1.89168×10^{-6}
9	2.41324×10^{-17}	1.65651×10^{-7}	4.27512×10^{-7}
12	1.03578×10^{-17}	2.89815×10^{-8}	7.7282×10^{-8}

Table 7. $\|u(t) - u_m(t)\|_{L^2[0,1]}$ for different values m and α in Example 6.4.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
7	1.93584×10^{-16}	0.0000120377	0.0000314255
9	3.53488×10^{-16}	4.09026×10^{-6}	0.0000112193
12	8.26356×10^{-17}	1.16833×10^{-6}	3.38702×10^{-6}

Example 6.5

Consider the following multi-order FOCP

$$Minimize \quad J = \int_0^1 \left(u(t) - x(t)\right)^2 dt,$$

subject to the system dynamic

$${}_0^C D^{\alpha_1} x(t) + {}_0^C D^{\alpha_2} x(t) = u(t) - x(t) + t^3 + \frac{6t^{\alpha_1-\alpha_2+3}}{\Gamma(\alpha_1-\alpha_2+4)},$$

with boundary conditions

$$x(0)=0, \quad x(1)=\frac{6}{\Gamma(\alpha_1+4)}.$$

This problem have the exact solution $x(t)=u(t)=\frac{6t^{\alpha_1+3}}{\Gamma(\alpha_1+4)}$. In Figures 10 and 11, we can observe plots of the approximate solutions of the states function and the control function for $m = 4$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.3, 0.4, 0.5$. Moreover, in Tables 8 and 9, we can see the errors in the norm L^2 for the obtained solutions. Similar to previous examples, we conclude the results will be better as m be increased.

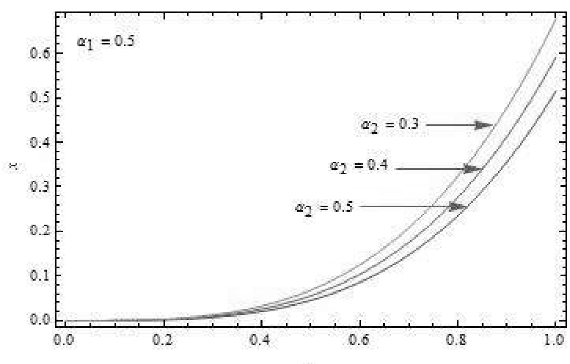

Fig. 10. Plot of $x(t)$ for $m = 4$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.3, 0.4, 0.5$ in Example 6.5.

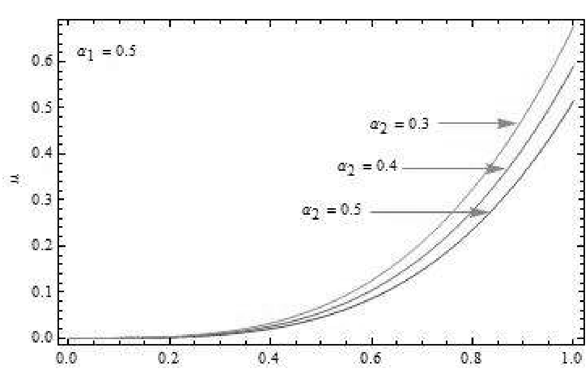

Fig. 11. Plot of $u(t)$ for $m = 4$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.3, 0.4, 0.5$ in Example 6.5.

Table 8. $\|x(t)-x_m(t)\|_{L^2[0,1]}$ for $\alpha_1 = 0.5$ and different values m and α_2 in Example 6.5.

m	$\alpha_2 = 0.5$	$\alpha_2 = 0.4$	$\alpha_2 = 0.3$
4	0.0000827126	0.0000855434	0.0000807791
8	3.11006×10^{-7}	3.99823×10^{-7}	4.68549×10^{-7}
12	1.36782×10^{-8}	1.93924×10^{-8}	2.51756×10^{-8}

Table 9. $\|u(t)-u_m(t)\|_{L^2[0,1]}$ for $\alpha_1 = 0.5$ and different values m and α_2 in Example 6.5.

m	$\alpha_2 = 0.5$	$\alpha_2 = 0.4$	$\alpha_2 = 0.3$
4	0.000287038	0.000302329	0.000291907
8	1.05323×10^{-6}	1.44989×10^{-6}	1.81476×10^{-6}
12	4.56923×10^{-8}	7.1658×10^{-8}	1.01974×10^{-7}

7. Conclusions

In this paper, we have gotten the Bernstein operational matrices of Riemann-Liouville fractional integral and product in the arbitrary interval [a,b]. Then by using these matrices, we have approximated MOMDFOCP by a parametric optimization. By Lagrangian method, we reduced the optimization problem to a system of algebraic equations that be solved easily. The examples as shown in the proposed method simply works and is very much applicable. The results show the approximate solutions for both the state and the control functions approach to the analytical solutions for $\alpha = 1$ when α approaches to 1 and m be fixed. Also, we see the results will be more accurate as m be increased.

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دراسة عددية عن مسألة التحكم الأمثل الكسري متعدد الرتب متعدد الأبعاد

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ملخص

الهدف من هذا العمل هو تطبيق متعددة حدود برنستاين (BPs) لحل مسألة التحكم الأمثل الكسري متعدد الرتب متعدد الأبعاد (MOMDFOCP). بدايةً، من خلال دوال برنستاين، نقدم مصفوفات تشغيلية للتكامل الكسري لريمان–ليوفيل (Riemann-Liouville) والمضروب في الفترة [أ، ب]. ومن ثَم، ومن خلال تلك المصفوفات، فإننا نقوم بتحويل المسألة إلى مسألة في الأمثلة. ولحل هذه المسألة، نطبق طريقة مضروب لاجرانج (Lagrangian multipliers). بذلك، يمكننا الحصول على حل تقريبي لمسألة التحكم الأمثل الكسري متعدد الرتب متعدد الأبعاد (MOMDFOCP). وتوضح نتائج بعض الأمثلة أن الحلول الناتجة دقيقة جدا ومتوافقة بشكل جيد مع الحلول الفعلية.