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Generalized roughness in $(\in, \in\vee q)$-fuzzy ideals of hemirings

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Abstract

Generalized roughness for fuzzy ideals in hemirings is studied. Approximations for fuzzy prime ideals are discussed. It is shown that generalized lower approximation as well as generalized upper approximation of $(\in, \in\vee q)$–fuzzy prime (semiprime, respectively) ideals of hemirings are $(\in, \in\vee q)$–fuzzy prime (semiprime, respectively) ideals.

Keywords: Fuzzy ideals; fuzzy sets; ideals; rough sets; semirings.

1. Introduction

Hemirings/semirings are algebraic structures, weaker than rings. Due to applications of hemirings in computer science, coding theory and algebra, scholars working in these areas have keen interest in them, Golan (1992). Ideals play a fundamental role in the study of structure of rings. However in hemirings they have limited applications. This limitation does not make their study uninteresting. Fuzzy semiring theory is a generalization of classical semiring theory. Many authors have studied different types of fuzzy ideals. Fuzzy semirings were introduced and first investigated by Ahsan *et al*. (1993).

Ming & Ming (1980) introduced the idea of fuzzy point. Notion of its belongingness and quasi-coincidence with a fuzzy set is very important in the study of fuzzy algebraic structures. This concept of fuzzy point played a seminal role for the introduction of (α, β)-fuzzy subgroups by Bhakat & Das (1992). Therefore (α, β)-fuzzy ideals of hemirings are very nice generalization of fuzzy ideals. Among these $(\in, \in\vee q)$–fuzzy ideals are most significant. Many authors have studied these, for details (Dudek *et al.,* 2009; Zulfiqar & Shabir, 2015).

Theory of rough sets was introduced by Pawlak (1982). In this theory, equivalence relation among the elements of a set is the key notion to discuss uncertainty. But in daily life, due to our limited knowledge about the elements of a set, it is often difficult to find an equivalence relation among the elements of these sets. Therefore authors sought more general rough sets models with less restrictions. Covering based rough sets and generalized rough sets are among these models. Davvaz (2008) initiated the study of generalized rough sets. In generalized rough set theory, set-valued maps are employed to define approximations of a set instead of equivalence relations. These maps give rise to relations, more general than equivalences. As a result we have a more flexible rough set model.

Theory of fuzzy sets proposes a very nice approach to study vagueness. As fuzzy set theory and rough set theory are two different approaches to handle uncertainty, these two can be combined in a very fruitful manner. Concepts of fuzzy rough and rough fuzzy sets are introduced in Dubois & Prade (1990).

Roughness in algebraic and fuzzy algebraic structures have been investigated by many scholars. Roughness in groups and subgroups is investigated in Biswas & Nanda (1994). Roughness in various other algebraic structures is investigated by many authors, for details (Biswas & Nanda, 1994; Davvaz & Mahdavipour, 2006; Jun, 2003; Kuroki, 1997; Kuroki & Wang, 1996). Generalized roughness or T-roughness in fuzzy algebraic structures has been discussed in Hosseini *et al*. (2012). However, in case of $(\in, \in\vee q)$-fuzzy algebraic structures much attention has yet not been paid. Therefore it is important to study roughness in generalized fuzzy algebraic structures such as $(\in, \in\vee q)$-fuzzy ideals of hemirings.

The arrangement of this paper is as stated next. In Section 2, few basic concepts having connection with hemirings, fuzzy sets, rough sets and $(\in, \in\vee q)$-fuzzy ideals are introduced. In Section 3, lower approximations and upper approximations of fuzzy subhemirings are studied. Concept of generalized roughness for fuzzy ideals is introduced. Then roughness for fuzzy semi-prime ideals and fuzzy prime ideals is studied. Notions of approximations of $(\in, \in\vee q)$-fuzzy subhemirings are investigated in Section 4. Then generalized roughness in $(\in, \in\vee q)$-fuzzy ideals is studied in the same section. In Section 5, it is seen that approximations of $(\in, \in\vee q)$-fuzzy semiprime ideals are $(\in, \in\vee q)$-fuzzy semiprime ideals. Further lower approximations and upper approximations of $(\in, \in\vee q)$-fuzzy prime ideals are studied. In Section 6, conclusions are stated.

2. Preliminaries

Some basic notions about hemirings, fuzzy hemirings and rough sets are introduced here. These concepts will be useful in later sections. A semiring $\langle H, +, \cdot\rangle$ is an algebraic structure over a non-empty crisp set H with two binary operations denoted by "+" and "·", such that $\langle H, +\rangle$ and $\langle H, \cdot\rangle$ are semigroups and "·" distributes over "+" from both sides. If $0 \in H$ such that $a + 0 = a = 0 + a$ and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in H$, then we call it a zero element of H. A semiring with commutative "+" and having a zero element is called a hemiring. If I is a non-empty subset of a hemiring H, then it is called left (right respectively) ideal, if $I + I \subseteq I$ and $H \cdot I \subseteq I$ ($I \cdot H \subseteq I$ respectively). A non-empty subset I is called an ideal if it is a left ideal as well as a right ideal of H. If the ideal I of H is such that $J^2 \subseteq I$ implies $J \subseteq I$, for all ideals J of H, then I is called semiprime. An ideal I is called a prime ideal of H if $JK \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$ for all ideals J and K of H. For undefined terms related to hemirings in this paper, see Golan (1992). Moreover in this paper H stands for a hemiring, unless stated otherwise.

Example 1. Let $H = \{0, a, d\}$ be a set. Addition and multiplication Tables 1 make H a hemiring.

Table 1. Addition and multiplication tables for H.

$+$	0	a	d
0	0	a	d
a	a	a	a
d	d	a	d

$\cdot$	0	a	d
0	0	0	0
a	0	a	a
d	0	a	d

Here $\{0\}$, $\{0, a\}$ and $\{0, a, d\}$ are ideals of the hemiring H.

A fuzzy subset of H is any mapping $\mu: H \longrightarrow [0,1]$, Zadeh (1965).

Definition 2. (Ming & Ming, 1980) A fuzzy subset of H is called a fuzzy point if

$$\mu(y) = \begin{cases} t & if \quad y = x \\ 0 & otherwise \end{cases}$$

Here x is the support of μ and $t \in [0,1]$ is its value. A fuzzy point is expressed by x_t.

Definition 3. (Ming & Ming, 1980) Let μ be a fuzzy subset and x_t be a fuzzy point.

1. When $\mu(x) \geq t$, then it means that x_t belongs to μ and is written as $x_t \in \mu$.
2. When $\mu(x) + t > 1$, then x_t is said to be quasi-coincident with μ and is denoted as $x_t q\, \mu$.
3. When $\mu(x) \geq t$ or $\mu(x) + t > 1$, then x_t belongs to μ or x_t is quasi-coincident with μ and is denoted as $x_t \in \vee q\, \mu$.

When any of $x_t \in \mu$, $x_t q\, \mu$ or $x_t \in \vee q\, \mu$ does not hold, then we write $x_t \overline{\in}\, \mu$, $x_t \overline{q}\, \mu$ or $x_t \overline{\in \vee q}\, \mu$ respectively.

Definition 4. (Dudek *et al.*, 2009) Any fuzzy subset μ of H is called a fuzzy subhemiring of H, if $\forall\ x, y \in H$, the following hold

$$\mu(x + y) \geq \min\{\mu(x), \mu(y)\}. \tag{1}$$

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}. \tag{2}$$

Definition 5. (Dudek *et al.*, 2009) A fuzzy subset μ of H is called a fuzzy left (right, respectively) ideal of H, if $\forall\, x, y \in H$, it satisfies (1) and

$$\mu(xy) \geq \mu(y) \tag{3}$$

(respectively $\mu(xy) \geq \mu(x)$)

and the inequality $\mu(0) \geq \mu(x)$ is satisfied for every $x \in H$.

Definition 6. (Dudek *et al.*, 2009) Any fuzzy ideal μ of H is called semi-prime if $\mu(x^2) = \mu(x)$ and prime if $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$, $\forall\, x, y \in H$.

Definition 7. (Dudek *et al.*, 2009) Any fuzzy subset of H, is called an $(\in, \in\vee q)$-fuzzy subhemiring of H, if

$$x_t, y_r \in \mu \longrightarrow (x + y)_{min(t,r)} \in \vee q\mu\,. \tag{4}$$

and $x_t\,, y_r \in \mu \longrightarrow (xy)_{min(t,r)} \in \vee q\mu$. (5)

Definition 8. (Dudek *et al*., 2009) Any fuzzy subset of H is called an $(\in, \in\vee q)$–fuzzy left (right, respectively) ideal of H, if (4) is satisfied and

$x_t \in \mu\,, y \in H \longrightarrow (yx)_t \in \vee q\mu$ (respectively $(xy)_t \in \vee q\mu$).

Definition 9. (Dudek *et al*., 2009) If μ is an $(\in, \in\vee q)$–fuzzy ideal of H, then it is called semiprime if $\forall\, x \in H$ and $t \in (0,1]$, $(x^2)_t \in \mu$ implies $x_t \in \vee q\mu$.

Definition 10. (Dudek *et al*., 2009) If μ is an $(\in, \in\vee q)$–fuzzy ideal of H, then it is called prime if $\forall\; x, y \in H$ and $t \in (0,1]$, $(xy)_t \in \mu$ implies $x_t \in \vee q\mu$ or $y_t \in \vee q\mu$.

Theorem 11**.** (Dudek *et al*., 2009) For any fuzzy subset μ of H, condition (4) is equivalent to

$\mu(x + y) \geq min\{\mu(x), \mu(y), 0.5\} \forall\, x, y \in H.$

Theorem 12**.** (Dudek *et al*., 2009) For any fuzzy subset μ of H, condition (5) is equivalent to

$\mu(xy) \geq min\{\mu(x), \mu(y), 0.5\}\;\; \forall\, x, y \in H.$

Theorem 13. (Dudek *et al*., 2009) For any fuzzy subset μ of H, following conditions are equivalent

$$x_t \in \mu\,, y \in H \longrightarrow (yx)_t \in \vee q\mu\,,$$
$$\mu(yx) \geq min\{\mu(x), 0.5\},\;\; \forall\, x, y \in H.$$

Theorem 14. (Dudek *et al*., 2009) If μ is an $(\in, \in\vee q)$–fuzzy ideal of H, then it is called semiprime if and only if

$\mu(x) \geq min\{\mu(x^2), 0.5\},\;\; \forall\, x \in H.$

Theorem 15. (Dudek *et al*., 2009) If μ is an $(\in, \in\vee q)$–fuzzy ideal of H, then it is called prime if and only if $\forall\, x, y \in H$

$max\{\mu(x), \mu(y)\} \geq min\{\mu(xy), 0.5\}.$

Throughout this paper we shall employ m for min and M for max, unless stated otherwise. Theory of rough sets has ability to handle uncertainty in a very nice way. Pawlak (1982) initiated this theory. In the following, some basic ideas of it are given. Let W be a non-empty crisp finite set with an equivalence relation ρ, moreover equivalence class containing some element $x \in W$ is denoted by $[x]_\rho$. We call (W, ρ) an approximation space. Consider a crisp subset B of W. If we are able to write B as union of some classes obtained by ρ, then B is definable, otherwise it is not definable. If B is not definable, then we can define two approximations of B, which are definable subsets of W. These approximations are defined as follows:

$$\underline{app}(B) = \{x \in W :\; [x]_\rho \subseteq B\}$$
$$\overline{app}(B) = \{x \in W :\; [x]_\rho \cap B \neq \phi\}$$

A rough set is the pair $\left(\underline{app}(B), \overline{app}(B)\right)$. If $\underline{app}(B) = \overline{app(B)}$, then B is definable set.

Definition 16. (Kazanci & Davvaz, 2008) Let (W, ρ) be an approximation space and μ be a fuzzy subset. If $x \in W$, then

$$\underline{app}(\mu)(x) = \bigwedge_{b \in [x]_\rho} (\mu)(b)$$

and $\overline{app}(\mu)(x) = \bigvee_{b \in [x]_\rho} (\mu)(b),$

where $\underline{app}(\mu)(x)$ is lower approximation and $\overline{app}(\mu)(x)$ is upper approximation of the fuzzy set μ.

The pair $\left(\underline{app}(\mu)(x), \overline{app}(\mu)(x)\right)$ is called the rough fuzzy set if $\underline{app}(\mu) = \overline{app}(\mu)$.

In Definition 16, ρ is an equivalence relation on W, but in daily life situations, apparently for the elements of the set W, there does not exist any equivalence. Therefore to handle situations like this, we have to define approximations of a fuzzy set in a more general context.

Definition 17. (Ali *et al*., 2012) Let H and $H^/$ be two hemirings. A set-valued map $F: H \longrightarrow P^*(H^/)$ is called a set-valued homomorphism, if $\forall\;\, y\,, z \in H$

(1) $F(y) + F(z) \subseteq F(y + z)$,

(2) $F(y)F(z) \subseteq F(y\,z)$

and it is called strong set-valued homomorphism if

(3) $F(y) + F(z) = F(y + z)$,

(4) $F(y)F(z) = F(y\,z)$.

Here $P^*(H^/)$ means the collection of all non-empty crisp subsets of $H^/$.

It must be noted that for each element $x \in H$, the image $F(x)$ is a non-empty subset of H. Such maps exist naturally. For example in case of groups, canonical maps, that map an element to a coset, are set-valued maps. From here onward by SVH we will mean set-valued homomorphism, whereas SSVH stands for strong set-valued homomorphism. Moreover F will denote the map $F: H \longrightarrow P^*(H)$, unless stated otherwise. Now concept introduced in Definition 16 can be generalized in the following.

Definition 18. Let μ be any fuzzy subset of H and

$F: H \longrightarrow P^*(H)$ be SVH. Then for every $x \in H$, we define fuzzy subsets

$$\underline{F}(\mu)(x) = \bigwedge_{z \in F(x)} (\mu)(z)$$

And $\overline{F}(\mu)(x) = \bigvee_{z \in F(x)} (\mu)(z)$.

$\underline{F}(\mu)$ is the lower approximation and $\overline{F}(\mu)$ is the upper approximation of the fuzzy set μ with respect to the mapping F. The pair $\left(\underline{F}(\mu), \overline{F}(\mu)\right)$ is called a rough fuzzy set if $\underline{F}(\mu) \neq \overline{F}(\mu)$.

3. Lower and upper approximations of fuzzy ideals

It has been seen that set-valued maps are very helpful to study roughness in hemirings (Ali *et al*., 2012). This concept is being extended to study roughness in fuzzy hemirings. In this section, initially the approximations of fuzzy subhemirings are studied. Then it is established that approximations of fuzzy semiprime ideals of hemirings are fuzzy semiprime ideals. Further, it will be shown that lower approximations and upper approximations of fuzzy prime ideals of hemirings are fuzzy prime ideals. Therefore we begin with:

Theorem 19. Let F be a SSVH and μ be a fuzzy subhemiring of H. Then $\underline{F}(\mu)$ is a fuzzy subhemiring of H.

Proof. As μ is fuzzy subhemiring of H, so by Definition 4, $\mu(y+z) \geq \mu(y) \wedge \mu(z)$ and $\mu(yz) \geq \mu(y) \wedge \mu(z)$, $\forall\ y, z \in H$. Consider

$\underline{F}(\mu)(y+z)$

$$= \bigwedge_{a \in F(y+z)} \mu(a)$$

$$= \bigwedge_{a \in [F(y)+F(z)]} \mu(a)$$

Now $a = c + d$ such that $c \in F(y)$ and $d \in F(z)$

$$= \bigwedge_{(c+d) \in [F(y)+F(z)]} \mu(c+d)$$

$$\geq \bigwedge_{\substack{c \in F(y) \\ d \in F(z)}} \left(\mu(c) \wedge \mu(d)\right)$$

$$= \left(\bigwedge_{c \in F(y)} \mu(c)\right) \wedge \left(\bigwedge_{d \in F(z)} \mu(z)\right)$$

$= \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z)$

$$\underline{F}(\mu)(y+z) \geq \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z) \qquad (6)$$

Similarly, one can also show that

$$\underline{F}(\mu)(yz) \geq \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z) \qquad (7)$$

It is clear from (6) and (7) that, $\underline{F}(\mu)$ is a fuzzy subhemiring of H.

Now, we show that the lower approximation of a fuzzy subhemiring is not a fuzzy subhemiring for SVH in general.

Example 20. Let $H = \{0, a, b, c, d\}$ be hemiring with multiplication defined as $y \cdot z = 0, \forall\ y, z \in H$ and

Table 2. Addition table for *H*.

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	c	a	b	a
b	b	a	b	c	b
c	c	b	c	a	c
d	d	a	b	c	d

Let $F: H \longrightarrow P^*(H)$ be defined by $F(0) = \{0\}$, $F(a) = \{c\}$, $F(b) = F(d) = \{b, d\}$ and $F(c) = \{a\}$. Clearly F is a SVH.

Let μ be a fuzzy subset of H given by $\mu(0) = 1$, $\mu(a) = \mu(c) = 0.7$, $\mu(b) = 0.8$ and $\mu(d) = 0.6$. Then μ is a fuzzy subhemiring of H. Then by using Definition 18, $\underline{F}(\mu)(0) = 1$, $\underline{F}(\mu)(a) = 0.7$, $\underline{F}(\mu)(b) = \underline{F}(\mu)(d) = 0.6$ and $\underline{F}(\mu)(c) = 0.7$.

As $\underline{F}(\mu)(yz) \geq \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z)$ holds for $\underline{F}(\mu)$.

But $\underline{F}(\mu)(y+z) \geq \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z)$ is not satisfied in this case, because $\underline{F}(\mu)(a+c) = \underline{F}(\mu)(b) = 0.6$ and $\underline{F}(\mu)(a) \wedge \underline{F}(\mu)(c) = 0.7 \wedge 0.7 = 0.7$.

Hence, lower approximation of a fuzzy subhemiring may not be a fuzzy subhemiring by employing SVH.

In the next result, it is seen that upper approximation of a fuzzy subhemiring is a fuzzy subhemiring.

Theorem 21. Let F be a SVH and μ be fuzzy subhemiring of H. Then $\overline{F}(\mu)$ is fuzzy subhemiring of H.

Proof. As μ is fuzzy subhemiring of H, so by Definition 4, $\mu(y+z) \geq \mu(y) \wedge \mu(z)$ and $\mu(yz) \geq \mu(y) \wedge \mu(z)$, $\forall\ y, z \in H$. Consider

$\overline{F}(\mu)(y) \wedge \overline{F}(\mu)(z)$

$$= \left(\bigvee_{c \in F(y)} \mu(c) \right) \wedge \left(\bigvee_{d \in F(z)} \mu(z) \right)$$

$$= \bigvee_{\substack{c \in F(y) \\ d \in F(z)}} \big(\mu(c) \wedge \mu(d)\big)$$

$$= \bigvee_{(c+d) \in [F(y)+F(z)]} \big(\mu(c) \wedge \mu(d)\big)$$

$$\leq \bigvee_{(c+d) \in F(y+z)} \big(\mu(c) \wedge \mu(d)\big)$$

$$\leq \bigvee_{(c+d) \in F(y+z)} \mu(c+d)$$

$$= \overline{F}(\mu)(y+z)$$

$$\overline{F}(\mu)(y+z) \geq \overline{F}(\mu)(y) \wedge \overline{F}(\mu)(z) \tag{8}$$

Similarly, one can also show that

$$\overline{F}(\mu)(yz) \geq \overline{F}(\mu)(y) \wedge \overline{F}(\mu)(z) \tag{9}$$

It is clear from (8) and (9) that, $\overline{F}(\mu)$ is a fuzzy subhemiring of H.

Now, lower approximations and upper approximations of fuzzy ideals are being studied in the following.

Theorem 22. Let F be a SSVH and μ be a fuzzy left (right, respectively) ideal of H. Then $\underline{F}(\mu)$ is a fuzzy left (right, respectively) ideal of H.

Proof. As μ is fuzzy left ideal of H, therefore by Definition 5, $\mu(y+z) \geq \mu(y) \wedge \mu(z)$ and $\mu(yz) \geq \mu(z)$, $\forall\ y, z \in H$. Consider

$\underline{F}(\mu)(y+z)$

$$= \bigwedge_{a \in F(y+z)} \mu(a)$$

$$= \bigwedge_{a \in [F(y)+F(z)]} \mu(a)$$

Now $a = c + d$ such that $c \in F(y)$ and $d \in F(z)$

$$= \bigwedge_{(c+d) \in [F(y)+F(z)]} \mu(c+d)$$

$$= \bigwedge_{\substack{c \in F(y) \\ d \in F(z)}} \mu(c+d)$$

$$\geq \bigwedge_{\substack{c \in F(y) \\ d \in F(z)}} \big(\mu(c) \wedge \mu(d)\big)$$

$$= \left(\bigwedge_{c \in F(y)} \mu(c) \right) \wedge \left(\bigwedge_{d \in F(z)} \mu(z) \right)$$

$$= \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z)$$

$$\underline{F}(\mu)(y+z) \geq \underline{F}(\mu)(y) \wedge \underline{F}(\mu)(z) \tag{10}$$

Similarly, one can also show that

$$\underline{F}(\mu)(yz) \geq \underline{F}(\mu)(z) \tag{11}$$

It is clear from (10) and (11) that, $\underline{F}(\mu)$ is fuzzy left ideal of H.

Example 23. Consider the hemiring as given in Example 1. Let $F: H \longrightarrow P^*(H)$ be defined by $F(0) = \{0, a\}$, $F(d) = \{a\}$ and $F(a) = \{d, a\}$. Then F is a SVH. Let μ be a fuzzy subset of H given by $\mu(0)1$, $\mu(a) = 0.7$ and $\mu(d) = 0.5$. Clearly μ is fuzzy left ideal of H. Then by using Definition 18, $\underline{F}(\mu)(0) = 0.7$, $\underline{F}(\mu)(a) = 0.5$ and $\underline{F}(\mu)(d) = 0.7$. Now $\underline{F}(\mu)(ad) \geq \underline{F}(\mu)(d)$ is not satisfied in this case because $\underline{F}(\mu)(ad) = \underline{F}(\mu)(a) = 0.5$ and $\underline{F}(\mu)(d) = 0.7$, so $\underline{F}(\mu)(ad) \ngeq \underline{F}(\mu)(d)$.

Hence, lower approximation of fuzzy left ideal is not fuzzy left ideal when using SVH in general.

Theorem 24. Let F be a SVH and μ be a fuzzy left (right, respectively) ideal of H. Then $\overline{F}(\mu)$ is fuzzy left (right, respectively) ideal of H.

Proof. As μ is fuzzy left ideal of H, therefore by Definition 5, $\mu(y+z) \geq \mu(y) \wedge \mu(z)$ and $\mu(yz) \geq \mu(z)$, $\forall\ y, z \in H$. Consider

$\overline{F}(\mu)(y) \wedge \overline{F}(\mu)(z)$

$$= \left(\bigvee_{u \in F(y)} \mu(u) \right) \wedge \left(\bigvee_{v \in F(z)} \mu(v) \right)$$

$$= \bigvee_{\substack{u \in F(y) \\ v \in F(z)}} \big(\mu(u) \wedge \mu(v)\big)$$

$$= \bigvee_{(u+v) \in [F(y)+F(z)]} \big(\mu(u) \wedge \mu(v)\big)$$

$$\leq \bigvee_{(u+v) \in F(y+z)} \big(\mu(u+v)\big)$$

$$\leq \bigvee_{(u+v) \in F(y+z)} \mu(u+v)$$

$$= \overline{F}(\mu)(y+z)$$

$$\overline{F}(\mu)(y+z) \geq \overline{F}(\mu)(y) \wedge \overline{F}(\mu)(z) \quad (12)$$

Similarly, it can be shown that

$$\overline{F}(\mu)(yz) \geq \overline{F}(\mu)(z) \quad (13)$$

It is clear from (12) and (13) that, $\overline{F}(\mu)$ is fuzzy left ideal of H.

Next, study of roughness in fuzzy semiprime ideals is being initiated and it is established that lower approximations and upper approximations of fuzzy semiprime ideals are fuzzy semiprime.

Theorem 25. Let F be a SSVH. If μ is fuzzy semiprime ideal of H, then $\underline{F}(\mu)$ is fuzzy semiprime ideal of H.

Proof. As μ is fuzzy semiprime ideal of H, therefore $\mu(y^2) = \mu(y)$, $\forall\ y \in H$ and μ is a fuzzy ideal of H, so by Theorem 22, $\underline{F}(\mu)$ is fuzzy ideal of H. To show that $\underline{F}(\mu)$ is fuzzy semiprime ideal of H, we need to show that $\underline{F}(\mu)(y^2) = \underline{F}(\mu)(y)$, $\forall\ y \in H$. For this consider

$$\underline{F}(\mu)(y)$$

$$= \bigwedge_{d \in F(y)} \mu(d)$$

$$= \bigwedge_{d \in F(y)} \mu(d^2)$$

$$= \bigwedge_{dd \in F(y)F(y)} \mu(d^2)$$

$$= \bigwedge_{d^2 \in F(yy)} \mu(d^2)$$

$$= \bigwedge_{d^2 \in F(y^2)} \mu(d^2)$$

$$= \underline{F}(\mu)(y^2)$$

Therefore $\underline{F}(\mu)$ is a fuzzy semiprime ideal of H.

Theorem 26. Let F be a SSVH. If μ is fuzzy semiprime ideal of H, then $\overline{F}(\mu)$ is fuzzy semiprime ideal of H.

Proof. Similar to the proof of Theorem 25.

Now, roughness in fuzzy prime ideals is being studied. First it is seen that lower approximation of fuzzy prime ideal is fuzzy prime ideal.

Theorem 27. Let F be a SSVH. If μ is a fuzzy prime ideal of H, then $\underline{F}(\mu)$ is fuzzy prime ideal of H.

Proof. As μ is fuzzy prime ideal of H, so $\mu(yz) = \mu(y)$ or $\mu(yz) = \mu(z)$, $\forall\ y, z \in H$. Since μ is fuzzy ideal of H, therefore by Theorem 24, $\underline{F}(\mu)$ is fuzzy ideal of H. We require $\underline{F}(\mu)(yz) = \underline{F}(\mu)(y)$ or $\underline{F}(\mu)(yz) = \underline{F}(\mu)(z)$. For this consider

$$\underline{F}(\mu)(yz)$$

$$= \bigwedge_{a \in F(yz)} \mu(a)$$

$$= \bigwedge_{a \in F(y)F(z)} \mu(a)$$

Now $a = bc$ where $b \in F(y)$ and $c \in F(z)$.

$$= \bigwedge_{bc \in F(y)F(z)} \mu(bc)$$

$$= \bigwedge_{\substack{b \in F(y) \\ c \in F(z)}} \mu(bc)$$

As μ is prime ideal of H, so either $\mu(bc) = \mu(b)$ or $\mu(bc) = \mu(c)$, therefore

$$\underline{F}(\mu)(yz) = \bigwedge_{b \in F(y)} \mu(b) = \underline{F}(\mu)(y)$$

or $\underline{F}(\mu)(yz) = \bigwedge_{c \in F(z)} \mu(z) = \underline{F}(\mu)(z)$

Therefore $\underline{F}(\mu)$ is fuzzy prime ideal of H.

Theorem 28. Let F be a SSVH. If μ is a fuzzy prime ideal of H, then $\overline{F}(\mu)$ is fuzzy prime ideal of H.

Proof. Similar to the proof of Theorem 27.

4. Approximations of $(\in, \in\vee q)$-fuzzy ideals

Notion of $(\in, \in\vee q)$-fuzzy structures was initiated by Bhakat & Das (1992). (α, β)-fuzzy algebraic structures are generalization of fuzzy algebraic structures. Various types of (α, β)-fuzzy ideals of hemirings are studied by Dudek *et al.* (2009).

Among these $(\in, \in\vee q)$-fuzzy ideals have great importance. $(\in, \in\vee q)$-fuzzy ideals are actually generalization of fuzzy ideals. In fuzzy algebraic structure roughness has been studied extensively, but no such study has been made for $(\in, \in\vee q)$-fuzzy algebraic structures. Therefore in this section, study of roughness in $(\in, \in\vee q)$-fuzzy subhemirings is initiated.

Theorem 29. Let F be a SSVH. If μ is an $(\in, \in\vee q)$-fuzzy subhemiring of H, then $\underline{F}(\mu)$ is $(\in, \in\vee q)$-fuzzy subhemiring of H.

Proof. Let $y_r, z_s \in \underline{F}(\mu)$, where $y, z \in H$ and $r, s \in (0,1]$, then $\underline{F}(\mu)(y) \geq r$ and $\underline{F}(\mu)(z) \geq s$. Consider

$\underline{F}(\mu)(y+z) = \bigwedge_{u \in F(y+z)} \mu(u)$

$= \bigwedge_{u \in [F(y)+F(z)]} \mu(u)$

Now $u = b + c$ where $b \in F(y)$ and $c \in F(z)$.

$= \bigwedge_{b+c \in [F(y)+F(z)]} \mu(b+c)$

$\geq \bigwedge_{b \in F(y),\, c \in F(z)} m(\mu(b), \mu(c), 0.5)$

$= m\left(\left(\bigwedge_{b \in F(y)} \mu(b)\right), \left(\bigwedge_{c \in F(z)} \mu(c)\right), 0.5\right)$

$= m(\underline{F}(\mu)(y), \underline{F}(\mu)(z), 0.5)$

$\geq m(r, s, 0.5)$, hence

$\underline{F}(\mu)(y+z) \geq m(r, s, 0.5)$

When $m(r,s) \leq 0.5$

We have $\underline{F}(\mu)(y+z) \geq m(r,s)$. So

$$(y+z)_{m(r,s)} \in \underline{F}(\mu). \quad (14)$$

When $m(r,s) > 0.5$

We have $\underline{F}(\mu)(y+z) \geq 0.5$, so $\underline{F}(\mu)(y+z) + m(r,s) > 1$. So

$$(y+z)_{m(r,s)} q\, \underline{F}(\mu). \quad (15)$$

From (14) and (15)

$$(y+z)_{m(r,s)} \in\vee\, q\underline{F}(\mu). \quad (16)$$

Similarly, it can be shown that

$$(yz)_{m(r,s)} q\, \underline{F}(\mu). \quad (17)$$

From (16) and (17), it is clear that $\underline{F}(\mu)$ is $(\in, \in\vee\, q)$-fuzzy subhemiring of H.

Theorem 30. Let F be a SVH. If μ is an $(\in, \in\vee\, q)$-fuzzy subhemiring of H, then $\overline{F}(\mu)$ is an $(\in, \in\vee\, q)$-fuzzy subhemiring of H.

Proof. Let $y_u, z_v \in \overline{F}(\mu)$, where $y, z \in H$ and $u, v \in (0,1]$, then $\overline{F}(\mu)(y) \geq u$ and $\overline{F}(\mu)(z) \geq v$. Consider

$m(u, v, 0.5) \leq m(\overline{F}(\mu)(y), \overline{F}(\mu), 0.5)$

$= m\left(\left(\bigvee_{b \in F(y)} \mu(b)\right), \left(\bigvee_{c \in F(z)} \mu(c)\right), 0.5\right)$

$= \bigvee_{b \in F(y),\, c \in F(z)} m(\mu(b), \mu(c), 0.5)$

$= \bigvee_{b+c \in [F(y)+F(z)]} m(\mu(b), \mu(c), 0.5)$

$\leq \bigvee_{b+c \in F(y+z)} m(\mu(b), \mu(c), 0.5)$

$\leq \bigvee_{b+c \in F(y+z)} m(b+c)$

$= \overline{F}(\mu)(y+z)$, hence

$\overline{F}(\mu)(y+z) \geq m(u, v, 0.5)$

When $m(u,v) \leq 0.5$

We have $\overline{F}(\mu)(y+z) \geq m(u,v)$, which implies

$$(y+z)_{m(u,v)} \in \overline{F}(\mu). \quad (18)$$

When $m(u,v) > 0.5$

We have $\overline{F}(\mu)(y+z) \geq 0.5$, so

$\overline{F}(\mu)(y+z) + m(u,v) > 1$. So

$$(y+z)_{m(u,v)} q\, \overline{F}(\mu). \quad (19)$$

From (18) and (19), it is clear that

$$(y+z)_{m(u,v)} \in\vee\, q\overline{F}(\mu). \quad (20)$$

Similarly, it can be shown that

$$(yz)_{m(u,v)} \in\vee\, q\overline{F}(\mu). \quad (21)$$

From (20) and (21), it is clear that $\overline{F}(\mu)$ is $(\in, \in\vee\, q)$-fuzzy subhemiring of H.

Theorem 31. Let F be a SSVH. If μ is an $(\in, \in\vee\, q)$-fuzzy left ideal of H, then $\overline{F}(\mu)$ is an $(\in, \in\vee\, q)$-fuzzy left ideal of H.

Proof. As μ is an $(\in, \in\vee\, q)$-fuzzy left ideal of H, so it is an $(\in, \in\vee\, q)$-fuzzy subhemiring of H. Therefore

$$x_t, y_r \in \overline{F}(\mu) \longrightarrow (x+y)_{m(t,r)} \in\vee\, q\overline{F}(\mu). \quad (22)$$

Next, let $x_t \in \overline{F}(\mu)$ and $y \in H$, then $\overline{F}(\mu)(x) \geq t$, consider

$\overline{F}(\mu)(yx)$

$= \bigvee_{x' \in F(yx)} \mu(x')$

$= \bigvee_{x' \in F(y)F(x)} \mu(x')$

Now $x' = ab$ such that $a \in F(y)$ and $b \in F(x)$.

$$= \bigvee_{ab \in F(y)F(x)} \mu(ab)$$

$$= \bigvee_{\substack{a \in F(y) \\ b \in F(x)}} \mu(ab)$$

$$\geq \bigvee_{b \in F(x)} m(\mu(b), 0.5)$$

$$= m\left(\bigvee_{b \in F(x)} \mu(b), 0.5\right)$$

$$= m(\overline{F}(\mu)(x), 0.5)$$

$$\geq m(t, 0.5)$$

When $t \leq 0.5$, we have $\overline{F}(\mu)(yx) \geq t$, that implies

$$(yx)_t \in \overline{F}(\mu) \quad (23)$$

When $t > 0.5$, we have $\overline{F}(\mu)(yx) \geq 0.5$, which implies $\overline{F}(\mu)(yx) + t > 1$, so

$$(yx)_t q\, \overline{F}(\mu) \quad (24)$$

From (23) and (24), we have $(yx)_t \in\vee\, q\overline{F}(\mu)$.

From (22) and (25), it is clear that $\overline{F}(\mu)$ is an $(\in, \in\vee q)$-fuzzy left ideal of H.

Theorem 32. Let F be a SSVH. If μ is an $(\in, \in\vee q)$-fuzzy left ideal of H, then $\underline{F}(\mu)$ is an $(\in, \in\vee q)$-fuzzy left ideal of H.

Proof. Similar to the proof of Theorem 31.

5. Approximations of $(\in, \in\vee q)$-fuzzy prime ideals

Notion of roughness is being extended to $(\in, \in\vee q)$-fuzzy semiprime ideals and $(\in, \in\vee q)$-fuzzy prime ideals. Therefore we discuss lower approximations and upper approximations of $(\in, \in\vee q)$-fuzzy semiprime ideals and then approximations of $(\in, \in\vee q)$-fuzzy prime ideals are being studied.

Theorem 33. Let F be a SSVH. If μ is an $(\in, \in\vee q)$-fuzzy semiprime ideal of H, then $\underline{F}(\mu)$ is an $(\in, \in\vee q)$-fuzzy semiprime ideal of H.

Proof. Let $y^2_p \in \underline{F}(\mu)$, then $\underline{F}(\mu)(y^2) \geq p$, where $y \in H$ and $p \in (0, 1]$. Consider

$$\underline{F}(\mu)(y)$$

$$= \bigwedge_{d \in F(y)} \mu(d)$$

$$\geq \bigwedge_{dd \in F(y)F(y)} m(\mu(d^2), 0.5)$$

$$= \bigwedge_{d^2 \in F(yy)} m(\mu(d^2), 0.5)$$

$$= m\left(\bigwedge_{d^2 \in F(y^2)} \mu(d^2), 0.5\right)$$

$$= m\left(\bigwedge_{d^2 \in F(y^2)} \mu(d^2), 0.5\right)$$

$$= m(\underline{F}(\mu)(y^2), 0.5)$$

$$\geq m(p, 0.5)$$

When $p \leq 0.5$, we have $\underline{F}(\mu)(y) \geq p$ and this implies $y_p \in \underline{F}(\mu)$. When $p > 0.5$, we have $\underline{F}(\mu)(y) \geq 0.5$, so $\underline{F}(\mu)(y) + p > 1$, which implies that $y_p\, q\underline{F}(\mu)$. Therefore, we get $y_p \in\vee\, q\underline{F}(\mu)$.

Thus $\underline{F}(\mu)$ is an $(\in, \in\vee q)$-fuzzy semiprime ideal of H.

Theorem 34. Let F be a SSVH. If μ is an $(\in, \in\vee q)$-fuzzy semiprime ideal of H, then $\overline{F}(\mu)$ is an $(\in, \in\vee q)$-fuzzy semiprime ideal of H.

Proof. Similar to the proof of Theorem 33.

Theorem 35. Let F be a SSVH. If μ is an $(\in, \in\vee q)$-fuzzy prime ideal of H, then $\underline{F}(\mu)$ is an $(\in, \in\vee q)$-fuzzy prime ideal of H.

Proof. As μ is an $(\in, \in\vee q)$-fuzzy prime ideal of H, therefore μ is an $(\in, \in\vee q)$-fuzzy ideal of H, so by Theorem 32, $\underline{F}(\mu)$ is $(\in, \in\vee q)$-fuzzy ideal of H. Further by Theorem 15

$$max(\mu(y), \mu(z)) \geq min(\mu(yz), 0.5).$$

Now let $(yz)_s \in \underline{F}(\mu)$, which implies $\underline{F}(\mu)(yz) \geq s$, where $y, z \in H$ and $s \in (0, 1]$. Consider

$$\underline{F}(\mu)(y) \vee \underline{F}(\mu)(z)$$

$$= \left(\bigwedge_{b \in F(y)} \mu(b)\right) \vee \left(\bigwedge_{c \in F(z)} \mu(c)\right)$$

$$= \bigwedge_{\substack{b \in F(y) \\ c \in F(z)}} (\mu(b) \vee \mu(c))$$

$$\geq \bigwedge_{\substack{b \in F(y) \\ c \in F(z)}} (\mu(bc) \wedge 0.5)$$

$= \bigwedge_{bc \in F(y)F(z)} (\mu(bc) \wedge 0.5)$

$= \bigwedge_{bc \in F(yz)} (\mu(bc) \wedge 0.5)$

$= \underline{F}(\mu)(yz) \wedge 0.5$

$\geq m(s, 0.5)$

When $s \leq 0.5$, we have $\underline{F}(\mu)(y) \vee \underline{F}(\mu)(z) \geq s$, then $\underline{F}(\mu)(y) \geq s$ or $\underline{F}(\mu)(z) \geq s$, therefore

$$y_s \in \underline{F}(\mu) \text{ or } z_s \in \underline{F}(\mu). \qquad (26)$$

When $s > 0.5$, we have $\underline{F}(\mu)(y) \vee \underline{F}(\mu)(z) \geq 0.5$, so $\underline{F}(\mu)(y) \geq 0.5$ or $\underline{F}(\mu)(z) \geq 0.5$. $\underline{F}(\mu)(y) + s > 1$ or $\underline{F}(\mu)(z) + s > 1$, hence

$$y_s q\underline{F}(\mu) \text{ or } z_s q\underline{F}(\mu). \qquad (27)$$

From (26) and (27), it is clear that $y_s \in\vee\, q\underline{F}(\mu)$ or $z_s \in\vee\, q\underline{F}(\mu)$. Therefore $\underline{F}(\mu)$ is an $(\in, \in\vee\, q)$–fuzzy prime ideal of H.

Theorem 36. Let F be a SSVH. If μ is an $(\in, \in\vee\, q)$-fuzzy prime ideal of H, then $\overline{F}(\mu)$ is an $(\in, \in\vee\, q)$-fuzzy prime ideal of H.

Proof. Similar to the proof of Theorem 35.

6. Conclusion

In the present paper, we see that the lower approximations of fuzzy subhemirings (fuzzy ideals, respectively) using SVH are fuzzy subhemirings (fuzzy ideals, respectively). The upper approximations of fuzzy subhemirings (fuzzy ideals, respectively) using SSVH are fuzzy subhemirings (fuzzy ideals, respectively). It is also seen that the approximations of fuzzy prime (semiprime, respectively) ideals using SSVH are fuzzy prime (semiprime, respectively) ideals.

We see that the lower approximation of an $(\in, \in\vee\, q)$-fuzzy subhemiring using SSVH is an $(\in, \in\vee\, q)$-fuzzy subhemiring and the upper approximation of an $(\in, \in\vee\, q)$-fuzzy subhemiring using SVH is an $(\in, \in\vee\, q)$-fuzzy subhemiring. It is also seen that the approximations of $(\in, \in\vee\, q)$-fuzzy ideals using SSVH are $(\in, \in\vee\, q)$-fuzzy ideals.

We believe that in the near future the idea of roughness using set-valued maps will be extended to other algebraic structures.

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الخشونة المُعَممة في النماذج الضبابية – $(\in, \in\vee q)$ لهيميرينغس (Hemirings)

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ملخص

تمت مناقشة الخشونة المُعَممة لنماذج ضبابية في هيميرينغس (Hemirings). وتمت مناقشة القيم التقريبية لنماذج رئيسية ضبابية. ويتضح أن القيم التقريبية الدنيا وكذلك القيم التقريبية العليا المُعَممة من هيميرينغس (Hemirings) للأمثلة الرئيسية الضبابية –$(\in, \in\vee q)$ (شبه رئيسية، على التوالي) عبارة عن نماذج رئيسية ضبابية –$(\in, \in\vee q)$ (شبه رئيسية، على التوالي).