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On pointwise convergence of bivariate nonlinear singular integral operators

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Abstract

In this paper, we present some theorems on pointwise convergence and the rate of pointwise convergence for the family of nonlinear bivariate singular integral operators of the following form:

$$T_{\lambda}(f;x,y)=\iint_{D} K_{\lambda}(t-x,s-y,f(t,s))dsdt, \quad (x,y)\in D,\ \lambda\in\Lambda,$$

where f is a real valued and integrable function on a bounded arbitrary closed, semi-closed or open region $D=\langle a,b\rangle\times\langle c,d\rangle$ in $\mathbb{R}^2$ or $D=\mathbb{R}^2$ and Λ is the set of non-negative indices with accumulation point λ_0.

Keywords: Lipschitz condition; pointwise convergence; rate of convergence; nonlinear bivariate integral operator; generalized Lebesgue point.

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1. Introduction

One of the main concerns of the approximation theory is to approximate the functions by using the functions having comparatively better properties. Some properties, which make the functions well-behaved, may be continuity, differentiability and integrability. Typically, those properties indicate polynomial functions. However, in some cases, for instance, the original function is integrable, using integral type operators is more appropriate than using polynomial type operators. Indeed, the non-integrability of the polynomials on the whole Euclidean space is a strong example for these kind of situations. Therefore, the integral type operators have been one of the main tools for the researchers of approximation theory, representation theory and singular integral theory.

Some studies, which have come to the fore in literature, can be summarized as follows:

Taberski (1962) studied the pointwise approximation of periodic and integrable functions by handling a two parameter family of convolution type singular integral operators of the form:

$$L_{\lambda}(f;x)=\int_{-\pi}^{\pi} f(t)K_{\lambda}(t-x)dt,\ \ x\in\langle-\pi,\pi\rangle,\ \lambda\in\Lambda, \tag{1}$$

where $\langle-\pi,\pi\rangle$ is an arbitrary closed, semi-closed or open interval in $\mathbb{R}$, $K_{\lambda}:\mathbb{R}\to\mathbb{R}_0^+$ denotes a family of periodic kernels satisfying the suitable conditions and Λ is a given set of non-negative numbers with accumulation point λ_0. The operator of equation (1) has great importance in the areas of generalized Fourier series, orthogonal series, theory of differential equations and harmonic analysis.

Following Taberski's (1962) line, Gadjiev (1968) proved the pointwise convergence of operators of equation (1) at a generalized Lebesgue point and established the convergence order. Rydzewska (1973) extended these results by obtaining pointwise approximation theorems for the functions $f \in L_1(-\pi,\pi)$ at a μ-generalized Lebesgue point. Further, Karsli & Ibikli (2007) considered the indicated operator in more general function spaces.

Taberski (1964), who gave rise to this theory, analyzed the convergence of bivariate singular integral operators depending on three parameters of the form:

$$L_\lambda\left(f;x,y\right)=\iint_Q f\left(t,s\right)K_\lambda\left(t-x,s-y\right)dsdt,\ \left(x,y\right)\in Q, \tag{2}$$

where $Q=\langle -\pi,\pi\rangle\times\langle -\pi,\pi\rangle$ is an arbitrary closed, semi-closed or open region, $K_\lambda:\mathbb{R}^2\to\mathbb{R}_0^+$ stands for a family of kernels comprising appropriate properties and $\lambda\in\Lambda$ is a given set of non-negative numbers with accumulation point λ_0. Later on, Rydzewska (1974) and Siudut (1988) improved the results of Taberski (1964) by changing the domain of integration. Recently, Uysal *et al.* (2015) and Yilmaz *et al.* (2014) studied the operators of equation (2) under the assumption of the kernel has a radial character. Also, Karsli (2015) and Yilmaz *et al.* (2017)* presented some pointwise convergence results on the approximation by convolution type double singular integral operators in different settings.

* This reference appeared after the paper was accepted for publication.

Musielak (1983) made a great contribution to the theory by presenting the problem concerning convergence of the nonlinear integral operators of the form:

$$T_w(f;s)=\int_a^b K_w(x-s,f(x))dx,\ s\in[a,b],\ w\in\Lambda, \tag{3}$$

and by assuming that the kernel K_w was Lipschitz with respect to second variable. After this famous paper, Swiderski & Washnicki (2000) studied the operators of equation (3) in some function spaces. Later on, Musielak (2000) studied some specific properties of the two dimensional integral operators analogous to operators of equation (3) in different function spaces. Also, Uysal (2016)** presented some weighted approximation results for two dimensional nonlinear singular integral operators. For deep and comprehensive analysis of several types of nonlinear integral operators and sampling type operators, which are considered in, for example, modular function spaces, the monograph (Bardaro *et al.,* 2003) is recommended by the authors.

The convergence of various operators were studied at different types of Lebesgue points: a family of singular integral operators in different settings (Karsli, 2006; Bardaro *et al.,* 2008; Bardaro *et al.,* 2011; Vinti & Zampogni, 2011), a family of nonlinear *m*-singular integral operators (Karsli, 2014), a family of nonlinear Mellin type convolution operators (Bardaro & Mantellini, 2006; Bardaro *et al.,* 2013).

The main concern of this paper is to investigate the pointwise convergence of nonlinear bivariate singular integral operators at a $\mu-$generalized Lebesgue point of the function $f\in L_1(D)$, where $L_1(D)$ consisting of the functions such that norm

** This reference appeared after the paper was accepted for publication.

of f, $\iint_D |f(t,s)|\,dsdt$, has a finite value and the rate of pointwise convergence of these operators in the following form:

$$T_\lambda(f;x,y)=\iint_D K_\lambda(t-x,s-y,f(t,s))\,dsdt,$$
$$(x,y)\in D,\ \lambda\in\Lambda, \qquad (4)$$

where $D=\langle a,b\rangle\times\langle c,d\rangle$ is an arbitrary bounded closed, semi-closed or open region in $\mathbb{R}^2$ or $D=\mathbb{R}^2$ and Λ is the set of non-negative indices with accumulation point λ_0. Here, $K_\lambda:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is a family of kernels which are integrable on $\mathbb{R}^2$.

The rest of the paper is structured as follows: In Preliminaries, we introduce the main definitions. In the next section, we give two theorems and establish the pointwise convergence of the operators of equation (4) and we estimate the rate of the pointwise convergence. The paper ends with a conclusion, which contains brief notes describing the contributions.

2. Preliminaries

Now, we give main definitions and remarks which are used in this manuscript.

The following definitions give the definitions of a $\mu-$generalized Lebesgue point of the function $f\in L_1(D)$ for $D=\langle a,b\rangle\times\langle c,d\rangle$ is an arbitrary bounded closed, semi-closed or open region in $\mathbb{R}^2$ and $D=\mathbb{R}^2$, respectively.

Definition 2.1. Assume that the function $\mu(t,s):\mathbb{R}^2\to\mathbb{R}$ is absolutely continuous in the sense of Carathéodory on $[0,b-a]\times[0,d-c]$, increasing with respect to t on $[0,b-a]$ and increasing with respect to s on $[0,d-c]$. Let $\mu(t,s)=0$ whenever $ts=0$. A point $(x_0,y_0)\in D$ is called a $\mu-$generalized Lebesgue point of the function $f\in L_1(D)$ if

$$\lim_{(h,k)\to(0,0)}\frac{1}{\mu(h,k)}\int_0^h\int_0^k |f(t+x_0,s+y_0)-f(x_0,y_0)|\,dsdt=0,$$

where $0<h<b-a$ and $0<k<d-c$.

Definition 2.2. Let δ_1 be an arbitrary positive real number. Assume that the function $\mu(t,s):\mathbb{R}^2\to\mathbb{R}$ is absolutely continuous in the sense of Carathéodory on $[0,\delta_1]\times[0,\delta_1]$ and increasing with respect to each variable seperately on $[0,\delta_1]$. Let $\mu(t,s)=0$ whenever $ts=0$.

A point $(x_0,y_0)\in\mathbb{R}^2$ is called a $\mu-$generalized Lebesgue point of the function $f\in L_1(\mathbb{R}^2)$ if

$$\lim_{(h,k)\to(0,0)}\frac{1}{\mu(h,k)}\int_0^h\int_0^k |f(t+x_0,s+y_0)-f(x_0,y_0)|\,dsdt=0,$$

where $0<h,k<\delta_1$.

For a deep analysis of the concept Carathéodory type absolute continuity, we refer to see Sremr (2010). Also, for some analogous and equivalent definitions, we refer the reader to see Rydzewska (1974). Note that the classical definition of Lebesgue point of $f\in L_1(D)$ is obtained by taking $\mu(t,s)=ts$.

Example 2.1. Let $f\in L_1(\mathbb{R}^2)$ be given by

$$f(t,s)=\begin{cases}1, & \text{if } ts=0,\\ \dfrac{1}{\sqrt{|t||s|(1+|t|)(1+|s|)}}, & \text{otherwise.}\end{cases}$$

Using definition of $\mu-$generalized Lebesgue point and taking $\mu(t,s)=\sqrt[4]{ts}e^{ts}$, we see that origin is a μ –generalized Lebesgue point of $f\in L_1(\mathbb{R}^2)$. For one dimensional analogue of the above function, we refer the reader to see Almali (2016).

Example 2.2. Let $h\in L_1(\mathbb{R}^2)$ be given by

$$h(t,s)=\begin{cases} tse^{-(t+s)}, & \text{if } (t,s)\in[0,1]\times[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

If we take $\mu(t,s)=t^{(\frac{1}{4}+1)}s^{(\frac{1}{4}+1)}$, then the origin is a μ –generalized Lebesgue point of $h\in L_1(\mathbb{R}^2)$. On the other hand, if we take $\alpha=\frac{1}{4}$, then the origin is also a generalized Lebesgue point.

Now, we present the class of kernels which will be used in the main theorems. In the construction stage of the following class, Rydzewska (1974); Siudut (1988); Bardaro *et al.* (2003) and Karsli (2006) are used as main reference works.

Definition 2.3. (*Class A*) Let $\lambda\in\Lambda$, where Λ is a set of non-negative indices with accumulation point λ_0 (or λ_0 denotes ∞). Further, let $K_\lambda:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ be a family of kernels which are integrable on $\mathbb{R}^2$ and the following conditions hold there:

a. $K_\lambda(t,s,0)=0$ for $(t,s)\in\mathbb{R}^2$, $\lambda\in\Lambda$.

b. Let $L_\lambda:\mathbb{R}^2\to\mathbb{R}$ be an integrable function on $\mathbb{R}^2$ such that

$$|K_\lambda(t,s,u)-K_\lambda(t,s,v)|\le L_\lambda(t,s)|u-v|,$$

holds for every $(t,s)\in\mathbb{R}^2$, for every u, $v\in\mathbb{R}$ and for each fixed $\lambda\in\Lambda$.

c. $\lim\limits_{\lambda\to\lambda_0}\iint\limits_{\xi\le\sqrt{t^2+s^2}} L_\lambda(t,s)\,ds\,dt=0,\ \forall\xi>0.$

d. $\lim\limits_{\lambda\to\lambda_0}\left[\sup\limits_{\xi\le\sqrt{t^2+s^2}} L_\lambda(t,s)\right]=0,\ \forall\xi>0.$

e.

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}\left|\iint\limits_{\mathbb{R}^2}K_\lambda(t-x,s-y,u)\,ds\,dt-u\right|=0,\ \forall u\in\mathbb{R}\cdot$$

f. $\|L_\lambda\|_{L_1(\mathbb{R}^2)}\le M<\infty.$

g. For a given positive real number δ_0 such that $\delta_0\ge\delta_1>0$, L_λ is monotonically increasing on $(-\delta_0,0]$ and monotonically decreasing on $[0,\delta_0)$ with respect to t and similarly L_λ is monotonically increasing on $(-\delta_0,0]$ and monotonically decreasing on $[0,\delta_0)$ with respect to s, for any $\lambda\in\Lambda$. Analogously, L_λ is bimonotonically increasing with respect to (t,s) on $[0,\delta_0)\times[0,\delta_0)$ and $(-\delta_0,0]\times(-\delta_0,0]$ and similarly L_λ is bimonotonically decreasing with respect to (t,s) on $[0,\delta_0)\times(-\delta_0,0]$ and $(-\delta_0,0]\times[0,\delta_0)$ for any $\lambda\in\Lambda$.

Finally, we need a singularity assumption on L_λ. The function L_λ is singular if it satisfies the property $\lim\limits_{\lambda\to\lambda_0}L_\lambda(t_0,s_0)=\infty$ at some points $(t_0,s_0)\in\mathbb{R}^2$.

Example 2.3. The first example of a kernel satisfying the above conditions is the linear kernel with respect to the third variable, i.e.,

$K_\lambda(t,s,u)=L_\lambda(t,s)u,$

where L_λ satisfies the conditions (c), (d), (f) and (g) of class A. Furthermore,

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}\iint\limits_{\mathbb{R}^2}L_\lambda(t-x,s-y)\,ds\,dt=1.$$

This case leads to well-known singular integral operators of convolution type. For details one may refer to Taberski (1964), Rydzewsk (1974), Siudut (1988), Swiderski & Washnicki (2000) and Yilmaz *et al*. (2014).

Example 2.4. Another kernel K_λ satisfying conditions of class A is the kernel given by

$$K_\lambda(t,s,u)=\begin{cases}\lambda^2\sin\frac{u}{2\lambda}, & \text{if } (t,s)\in[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}],\\ 0, & \text{if } (t,s)\in\mathbb{R}^2\setminus[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}],\end{cases}$$

where $\lambda\in\Lambda=\mathbb{N}$ and $\lambda_0=+\infty$. One dimensional analogue of this kernel function can be found in Swiderski & Washnicki (2000).

Remark 2.1. If the function $g:\mathbb{R}^2\to\mathbb{R}$ is bimonotonic on $[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]\subset\mathbb{R}^2$, then the following equality

$$V(g;[\alpha_1,\alpha_2]\times[\beta_1,\beta_2])=\bigvee_{\alpha_1}^{\alpha_2}\bigvee_{\beta_1}^{\beta_2}\left(g(t,s)\right)$$
$$=\left|g(\alpha_1,\beta_1)-g(\alpha_1,\beta_2)-g(\alpha_2,\beta_1)+g(\alpha_2,\beta_2)\right|$$

holds (Taberski, 1964; Ghorpade & Limaye, 2010).

3. Main results

This section starts with the following lemma which gives the existence of the operators of equation (4). For this kind of existence theorem see Karsli (2008).

Lemma 3.1. Assume that K_λ belongs to class A. If $f\in L_1(D)$, then $T_\lambda(f;x,y)\in L_1(D)$ and the following inequality

$$\left\|T_\lambda(f;x,y)\right\|_{L_1(D)}\le\left\|L_\lambda\right\|_{L_1(\mathbb{R}^2)}\left\|f\right\|_{L_1(D)}$$

holds for all $\lambda\in\Lambda$.

Proof. We will prove the lemma in two cases.

Case I: $D=\langle a,b\rangle\times\langle c,d\rangle$ is an arbitrary bounded closed, semi-closed or open region in $\mathbb{R}^2$. Suppose that $f\in L_1(D)$. To prove $T_\lambda(f;x,y)\in L_1(D)$, we have to show that the following expression

$$\left\|T_\lambda(f;x,y)\right\|_{L_1(D)}=\iint_D\left|\iint_D K_\lambda\left(t-x,s-y,f(t,s)\right)dsdt\right|dydx$$

remains finite. The extension of $f:D\to\mathbb{R}$, which is denoted by g, is as follows:

$$g(t,s)=\begin{cases}f(t,s), & \text{if } (t,s)\in D,\\ 0, & \text{if } (t,s)\in\mathbb{R}^2\setminus D.\end{cases}$$

Obviously, the following equality:

$$\left\|T_\lambda(f;x,y)\right\|_{L_1(D)}=\iint_D\left|\iint_D K_\lambda\left(t-x,s-y,f(t,s)\right)dsdt\right|dydx$$
$$=\iint_D\left|\iint_{\mathbb{R}^2} K_\lambda\left(t-x,s-y,g(t,s)\right)dsdt\right|dydx$$

holds. In view of condition (a) and (Lipschitz) condition (b) of class A, Fubini's Theorem e. g. Butzer & Nessel (1971) and condition (f) of class A, we have

$$\left\|T_\lambda(f;x,y)\right\|_{L_1(D)}\le\iint_D\left(\iint_{\mathbb{R}^2}\left|g(t+x,s+y)\right|L_\lambda(t,s)dsdt\right)dydx$$
$$\le\iint_{\mathbb{R}^2}L_\lambda(t,s)\left(\iint_{\mathbb{R}^2}\left|g(t+x,s+y)\right|dydx\right)dsdt$$
$$=\left\|L_\lambda\right\|_{L_1(\mathbb{R}^2)}\left\|f\right\|_{L_1(D)}.$$

Case II: $D=\mathbb{R}^2$. This follows directly from case I; we have the following relations

$$\left\|T_\lambda(f;x,y)\right\|_{L_1(\mathbb{R}^2)}\le\left\|L_\lambda\right\|_{L_1(\mathbb{R}^2)}\left\|f\right\|_{L_1(\mathbb{R}^2)}.$$

Hence the lemma is proved.

The following theorems give the pointwise approximation of the operators of equations (4) to the function f at a $\mu-$generalized Lebesgue point of $f\in L_1(D)$ whenever D is an arbitrary bounded region in $\mathbb{R}^2$ such that closed, semi-closed or open and $D=\mathbb{R}^2$, respectively.

Theorem 3.1. Assume that K_λ belongs to class A. If $(x_0,y_0)\in D$ is a $\mu-$generalized Lebesgue point of the function $f \in L_1(D)$, then

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} \left|T_\lambda(f;x,y) - f(x_0,y_0)\right| = 0$$

on any set Z on which the function $\Omega_\delta(x,y;\lambda)$ defined by

$$\Omega_\delta(x,y;\lambda) = \int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} L_\lambda(t-x,s-y)\left|d_t d_s \mu(|t-x_0|,|s-y_0|)\right|$$
$$+2\int_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x,0)\left|d_t \mu(|t-x_0|,|y-y_0|)\right|$$
$$+2\int_{y_0-\delta}^{y_0+\delta} L_\lambda(0,s-y)\left|d_s \mu(|x-x_0|,|s-y_0|)\right|$$
$$+4L_\lambda(0,0)\mu(|x-x_0|,|y-y_0|),$$

where $0 < 2\delta < \min\{b-a, d-c\}$, remains bounded as (x,y,λ) tends to (x_0,y_0,λ_0).

Here, $\left|d_t d_s \mu(|t-x_0|,|s-y_0|)\right|$, $\left|d_t \mu(|t-x_0|,|y-y_0|)\right|$ and $\left|d_s \mu(|x-x_0|,|s-y_0|)\right|$ denote Riemann-Stieltjes measure and for any constant $C>0$, $Z = \{(x,y,\lambda) \in D\times\Lambda : \Omega_\delta(x,y;\lambda) < C\}$.

Proof. Suppose that $(x_0,y_0) \in D$ is a $\mu-$generalized Lebesgue point of function $f \in L_1(D)$. Let $x_0 \le 0$, $y_0 \le 0$, $0 < x_0 - x < \delta/2$, for all $\delta > 0$ satisfying $x_0+\delta < b$ and $x_0 - \delta > a$ and $0 < y_0 - y < \delta/2$ for all $\delta>0$ satisfying and $y_0 - \delta > c$. Set $B_\delta = \left\{(t,s)\in D : (t-x_0)^2 + (s-y_0)^2 < \delta^2 : (x_0,y_0)\in D\right\}$.

Now, from Definition 2.1, for all given $\varepsilon > 0$ there exists $\delta > 0$ such that for all h and k satisfying $0 < h,k \le \delta$ the inequality

$$\int_{x_0}^{x_0+h}\int_{y_0-k}^{y_0} \left|f(t,s) - f(x_0,y_0)\right| ds dt < \varepsilon\mu(h,k) \qquad (5)$$

holds. Set $I(x,y,\lambda) = \left|T_\lambda(f;x,y) - f(x_0,y_0)\right|$. By the aid of conditions (b) and (e) of class A and extension $g(t,s)$ of $f(t,s)$ in $\mathbb{R}$ (Lemma 3.1), we get

$$I(x,y,\lambda) \le \iint_D \left|f(t,s) - f(x_0,y_0)\right| L_\lambda(t-x,s-y) ds dt$$
$$+\left|\iint_{\mathbb{R}^2} K_\lambda(t-x,s-y,f(x_0,y_0)) ds dt - f(x_0,y_0)\right|$$
$$+\left|f(x_0,y_0)\right| \iint_{\mathbb{R}^2\setminus D} L_\lambda(t-x,s-y) ds dt$$
$$= I_1 + I_2 + I_3.$$

From conditions (e) and (c), $I_2 \to 0$ and $I_3 \to 0$ as (x,y,λ) tends to (x_0,y_0,λ_0), respectively. Set $Q = \langle x_0-\delta, x_0+\delta\rangle \times \langle y_0-\delta, y_0+\delta\rangle$. Therefore, the integral I_1 may be written in the form:

$$I_1 = \left\{\iint_{D\setminus Q} + \iint_Q\right\} \left|f(t,s) - f(x_0,y_0)\right| L_\lambda(t-x,s-y) ds dt$$
$$= I_{11} + I_{12}.$$

By initial assumptions which we have supposed at the beginning of the proof and condition (d), $I_{11} \to 0$ as (x,y,λ) tends to (x_0,y_0,λ_0). Obviously, the following equality holds for I_{12}

$$I_{12} = \left\{\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} + \int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0}\right\} \left|f(t,s) - f(x_0,y_0)\right| L_\lambda(t-x,s-y) ds dt$$
$$+\left\{\int_{x_0-\delta}^{x_0}\int_{y_0}^{y_0+\delta} + \int_{x_0}^{x_0+\delta}\int_{y_0}^{y_0+\delta}\right\} \left|f(t,s) - f(x_0,y_0)\right| L_\lambda(t-x,s-y) ds dt$$
$$= I_{121} + I_{122} + I_{123} + I_{124}.$$

In view of the inequality (5) and the new function:

$$F(t,s) = \int_{x_0}^{t}\int_{s}^{y_0} \left|f(u,v) - f(x_0,y_0)\right| dv du,$$

for all t and s satisfying $0 < t - x_0 \le \delta$ and $0 < y_0 - s \le \delta$, the relation

$$F(t,s)\le \varepsilon\mu(t-x_0,y_0-s) \qquad (6)$$

is obtained. The following Riemann-Stieltjes integral representation for the integral I_{121}

$$\left|I_{121}\right| = \left|\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} L_\lambda(t-x,s-y)\,d_t d_s\left[-F(t,s)\right]\right|$$

holds. Applying bivariate integration by parts to the Riemann-Stieltjes integral $\left|I_{121}\right|$, (Taberski, 1964; Jawarneh & Noorani, 2011), we have

$$\begin{aligned}\left|I_{121}\right| = \Bigg| &-\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} F(t,s)\,d_t d_s\left[L_\lambda(t-x,s-y)\right]\\ &-\int_{x_0}^{x_0+\delta} F(t,y_0-\delta)\,d_t\left[L_\lambda(t-x,y_0-\delta-y)\right]\\ &+\int_{y_0-\delta}^{y_0} F(x_0+\delta,s)\,d_s\left[L_\lambda(x_0+\delta-x,s-y)\right]\\ &+F(x_0+\delta,y_0-\delta)L_\lambda(x_0+\delta-x,y_0-\delta-y)\Bigg|.\end{aligned}$$

From inequality (6), we can write

$$\begin{aligned}\left|I_{121}\right| \le\, &\varepsilon\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} \mu(t-x_0,y_0-s)\left|d_t d_s\left[L_\lambda(t-x,s-y)\right]\right|\\ &+\varepsilon\int_{x_0}^{x_0+\delta} \mu(t-x_0,\delta)\left|d_t\left[L_\lambda(t-x,y_0-\delta-y)\right]\right|\\ &+\varepsilon\int_{y_0-\delta}^{y} \mu(\delta,y_0-s)\left|d_s\left[L_\lambda(x_0+\delta-x,s-y)\right]\right|\\ &+\varepsilon\mu(\delta,\delta)L_\lambda(x_0+\delta-x,y_0-\delta-y).\end{aligned}$$

Now, bivariation and single variations are given as follows:

$$A_1(t,s)=\begin{cases}\bigvee\limits_{t}^{x_0+\delta-x}\ \bigvee\limits_{y_0-\delta-y}^{s} L_\lambda(u,v), & x_0-x\le t<x_0+\delta-x,\\ & y_0-\delta-y<s\le y_0-y\ ,\\ 0, & \text{otherwise,}\end{cases}$$

$$A_2(t)=\begin{cases}\bigvee\limits_{t}^{x_0+\delta-x} L_\lambda(u,y_0-\delta-y), & x_0-x\le t<x_0+\delta-x,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$A_3(s)=\begin{cases}\bigvee\limits_{y_0-\delta-y}^{s} L_\lambda(x_0+\delta-x,v), & y_0-\delta-y<s\le y_0-y\\ 0, & \text{otherwise.}\end{cases}$$

Since $\mu(t,s)$ is absolutely continuous in the sense of Carathéodory on $[0,\delta]\times[0,\delta]$, it has both mixed second order partial derivatives almost everywhere on the indicated rectangle and they are equivalent; we refer the reader to Sremr (2010). Therefore, taking above variations into account and applying bivariate integration by parts to the last inequality for I_{121}, we have

$$\left|I_{121}\right|\le -\varepsilon\int_{x_0-x}^{x_0-x+\delta}\int_{y_0-y-\delta}^{y_0-y}\begin{bmatrix}A_1(t,s)+A_2(t)+A_3(s)+\\ L_\lambda(x_0+\delta-x,y_0-\delta-y)\end{bmatrix}d_t d_s$$
$$\left[\mu(t-x_0+x,y_0-s-y)\right].$$

For the similar situations, we refer the reader to Taberski (1964) and Rydzewska (1974). The rest of the operations are performed quite similar to that of Theorem 1 in Uysal *et al*. (2015). Therefore, we skip this part. In view of Remark 2.1 and condition (g) the following inequality:

$$\begin{aligned}\left|I_{121}\right|\le\, &\varepsilon\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} L_\lambda(t-x,s-y)\left|d_t d_s\mu(|t-x_0|,|s-y_0|)\right|\\ &+2\varepsilon\int_{x_0}^{x_0+\delta}\int_{0}^{y_0-y} L_\lambda(t-x,0)\left|d_t d_s\mu(|t-x_0|,|y_0-s-y|)\right|\end{aligned}$$

holds. Since $\mu(t,s)$ also has both integrable first order partial derivatives on $[0,\delta]\times[0,\delta]$ we refer the reader to Sremr (2010), we have

$$\begin{aligned}\left|I_{121}\right|\le\, &\varepsilon\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} L_\lambda(t-x,s-y)\left|d_t d_s\mu(|t-x_0|,|s-y_0|)\right|\\ &+2\varepsilon\int_{x_0}^{x_0+\delta} L_\lambda(t-x,0)\left|d_t\mu(|t-x_0|,|y_0-y|)\right|.\end{aligned}$$

Computing the integrals I_{122}, I_{123} and I_{124} with the same idea and combining the respective inequalities we obtain the inequality: $I_{12}\le\varepsilon\Omega_\delta(x,y;\lambda)$ for $0<2\delta<\min\{b-a,d-c\}$.

Therefore, if the points $(x,y,\lambda)\in Z$ are sufficiently close to (x_0,y_0,λ_0), then we have $I_{12}<\varepsilon C$. Hence $I_{12}\to 0$ as (x,y,λ) tends to (x_0,y_0,λ_0). Note that, if we reverse the initial inequalities which we have supposed at the beginning of the proof, then we arrive at the same conclusion. Thus the proof is completed.

Theorem 3.2. Assume that K_λ belongs to class A. If $(x_0,y_0)\in\mathbb{R}^2$ is a $\mu-$generalized Lebesgue point of the function $f\in L_1(\mathbb{R}^2)$, then

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}\left|T_\lambda(f;x,y)-f(x_0,y_0)\right|=0$$

on any set Z on which the function $\Omega_\delta(x,y;\lambda)$, which is defined in Theorem 3.1, remains bounded as (x,y,λ) tends to (x_0,y_0,λ_0). Here, for an arbitrary real number $\delta_1>0$, $0<\delta<\delta_1$ and for any constant $C>0$, $Z=\{(x,y,\lambda)\in\mathbb{R}^2\times\Lambda:\Omega_\delta(x,y;\lambda)<C\}$.

Proof. Suppose that $(x_0,y_0)\in\mathbb{R}^2$ is a $\mu-$generalized Lebesgue point of function $f\in L_1(\mathbb{R}^2)$ and the equivalent initial assumptions given in Theorem 3.1 hold. From Definition 2.2, for all given $\varepsilon>0$ there exists $\delta>0$ such that for all h and k satisfying $0<h,k\le\delta$ the inequality:

$$\int_{x_0}^{x_0+h}\int_{y_0-k}^{y_0}\left|f(t,s)-f(x_0,y_0)\right|dsdt<\varepsilon\mu(h,k)$$

holds. Set $I(x,y,\lambda)=\left|T_\lambda(f;x,y)-f(x_0,y_0)\right|$. By conditions (b) and (e) of class A, we get

$$I(x,y,\lambda)\le\iint_{\mathbb{R}^2}\left|f(t,s)-f(x_0,y_0)\right|L_\lambda(t-x,s-y)\,dsdt$$
$$+\left|\iint_{\mathbb{R}^2}K_\lambda(t-x,s-y,f(x_0,y_0))\,dsdt-f(x_0,y_0)\right|$$
$$=I_1+I_2.$$

From condition (e), $I_2\to 0$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$. Therefore, I_1 may be written in the form:

$$I_1=\left\{\iint_{\mathbb{R}^2\setminus Q}+\iint_Q\right\}\left|f(t,s)-f(x_0,y_0)\right|L_\lambda(t-x,s-y)\,dsdt$$
$$=I_{11}+I_{12},$$

where $Q=\langle x_0-\delta,x_0+\delta\rangle\times\langle y_0-\delta,y_0+\delta\rangle$.

Since

$$I_{11}=\iint_{\mathbb{R}^2\setminus Q}\left|f(t,s)-f(x_0,y_0)\right|L_\lambda(t-x,s-y)\,dsdt$$
$$\le\sup_{\xi\le\sqrt{u^2+v^2}}L_\lambda(u,v)\|f\|_{L_1(\mathbb{R}^2)}$$
$$+\left|f(x_0,y_0)\right|\iint_{\xi\le\sqrt{u^2+v^2}}L_\lambda(u,v)\,dvdu,\ \text{for all } \xi>0,$$

by conditions (d) and (c) $I_{11}\to 0$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

The remaning part of the proof is almost the same as the proof of Theorem 3.1.

Thus the proof is completed.

4. Rate of convergence

The next theorem gives the rate of pointwise convergence of the operators of equation (4) to the function f at a $\mu-$generalized Lebesgue point of $f\in L_1(D)$ whenever D is an arbitrary bounded region in $\mathbb{R}^2$ such that closed, semi-closed or open or $D=\mathbb{R}^2$.

Theorem 4.1. Suppose that the hypothesis of Theorem 3.1 (Theorem 3.2) is satisfied. Further, $\Omega_\delta(x,y;\lambda)\to 0$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ for some $\delta>0$ given in Theorem 3.1 (Theorem 3.2) and the following conditions:

(*i*) $\sup_{\xi\le\sqrt{t^2+s^2}}L_\lambda(t,s)=o(\Omega_\delta(x,y;\lambda)),\ \forall\xi>0,$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$

(ii) $\iint\limits_{\xi\le\sqrt{t^2+s^2}} L_\lambda(t,s)\,dsdt = o(\Omega_\delta(x,y;\lambda)),\ \forall\xi>0,$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$

(iii) $\left|\iint\limits_{\mathbb{R}^2} K_\lambda(t,s,u)\,dsdt - u\right| = o(\Omega_\delta(x,y;\lambda))$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$

are satisfied. Then, at each $\mu-$generalized Lebesgue point of $f\in L_1(D)$ we have as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$

$$\left|T_\lambda(f;x,y)-f(x_0,y_0)\right| = o(\Omega_\delta(x,y;\lambda)).$$

Proof. This follows immediately from the hypothesis of Theorem 3.1 (Theorem 3.2).

Example 4.1. Let the kernel function K_λ be given as

$$K_\lambda(t,s,u)=\begin{cases}\frac{\lambda u}{2}+\sin\frac{\lambda u}{2},\ \text{if } (t,s)\in[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}],\\ 0,\ \text{if } (t,s)\in\mathbb{R}^2\setminus[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}],\end{cases}$$

where $\lambda\in\Lambda=\mathbb{N}$ and $\lambda_0=+\infty$. Let $u,v\in\mathbb{R}$.

One dimensional analogue of this kernel function can be found in Swiderski & Washnicki (2000).

Observe that

$\left|K_\lambda(t,s,u)-K_\lambda(t,s,v)\right|\le\lambda|u-v|$ for $(t,s)\in[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]$

and

$\left|K_\lambda(t,s,u)-K_\lambda(t,s,v)\right|=0$ for $(t,s)\in\mathbb{R}^2\setminus[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]$.

Since $K_\lambda(t,s,0)=0$, condition (a) is satisfied. Hence we have

$$L_\lambda(t,s)=\begin{cases}\lambda,\ \text{if } (t,s)\in[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}],\\ 0,\ \text{if } (t,s)\in\mathbb{R}^2\setminus[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}].\end{cases}$$

Verifications of the conditions (c) and (d) of class A follow from definition of $L_\lambda(t,s)$. Moreover,

$$\lim_{\lambda\to\infty}\iint\limits_{\mathbb{R}^2}K_\lambda(t,s,u)dsdt=\lim_{\lambda\to\infty}\left(\frac{2}{\lambda}\sin\frac{\lambda u}{2}+u\right)=u$$

and $\|L_\lambda\|_{L_1(\mathbb{R}^2)}=2$. Since L_λ takes constant values for any $\lambda\in\Lambda$, monotonicity conditions are clearly satisfied. Since $\lim\limits_{\lambda\to\infty}L_\lambda(0,0)=\infty$, singularity condition is fulfilled.

Let $(x_0,y_0)=(0,0)$, $\mu(t,s)=ts$ and $[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\times[\frac{-1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2\lambda}}]\subseteq[-\delta,\delta]\times[-\delta,\delta]$.

Hence, we get

$$\Omega_\delta(x,y;\lambda)=4\lambda\delta^2+4\lambda\delta|y|+4\lambda\delta|x|+4\lambda|x||y|.$$

To find the rate of convergence, suppose that the following equality

$$\lim_{(x,y,\lambda)\to(0,0,\infty)}\Omega_\delta(x,y;\lambda)=\lim_{(x,y,\lambda)\to(0,0,\infty)}\left(4\lambda\delta^2+4\lambda\delta|y|+4\lambda\delta|x|+4\lambda|x||y|\right)=0$$

holds. Consequently the following expressions:

$$\lim_{(x,y,\lambda)\to(0,0,\infty)}\Omega_\delta(x,y;\lambda)=0\Leftrightarrow\delta=|x|=|y|=O(\tfrac{1}{\lambda^\alpha}),$$

$\alpha>1/2$

and

$$\Omega_\delta(x,y;\lambda)=O(\frac{1}{\lambda^{2\alpha-1}}),\ \text{for } \alpha>1/2$$

are obtained. Since $\frac{2}{\lambda}\sin\frac{\lambda u}{2}=o(\frac{1}{\lambda^{2\alpha-1}})$, $1/2<\alpha<1$, we have

$$\left|\iint\limits_{\mathbb{R}^2}K_\lambda(t,s,f(x_0,y_0))dsdt-f(x_0,y_0)\right|=o(\frac{1}{\lambda^{2\alpha-1}}),\ \ 1/2<\alpha<1$$

By taking $\xi=\frac{1}{\sqrt{2\lambda}}$, we see that $\sup\limits_{\frac{1}{\sqrt{2\lambda}}\le\sqrt{t^2+s^2}}L_\lambda(t,s)=O(\frac{1}{\sqrt{\lambda}})$. Therefore, we get

$$\sup_{\frac{1}{\sqrt{2\lambda}}\le\sqrt{t^2+s^2}}L_\lambda(t,s)=o(\tfrac{1}{\lambda^\alpha}),\ 0<\alpha<1/2.$$

Similarly, the following equality:

$$\iint\limits_{\frac{1}{\sqrt{2\lambda}}\le\sqrt{t^2+s^2}}L_\lambda(t,s)dsdt=o(\tfrac{1}{\lambda^\alpha}),\ \ 0<\alpha<1/2$$

holds. Therefore, verifications of the hypotheses (i) and (ii) are completed. We finally have

$\left|T_\lambda(f;x,y)-f(x_0,y_0)\right|=o(\frac{1}{\lambda^\alpha}),\ \ 0<\alpha<1/2$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

5. Conclusion

In this paper, we investigated the pointwise convergence and the rate of pointwise convergence for the family of nonlinear bivariate singular integral operators of the equation (4). This study may be seen as a continuation and further generalization of the previous studies, such as Rydzewska (1974) and Karsli (2006). For this purpose, we used two dimensional counterparts of some concepts given in one dimensional case, such as absolute continuity, monotonicity, integration by parts method. By using these notions a special class of kernel functions, called class A, has been presented. Also, for this definition the previous studies are used, such as Karsli (2006). Therefore, main results are presented as Theorem 3.1 and Theorem 3.2. By using these theorems, we obtained the rate of pointwise convergence and gave an example.

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حول تقارب نقطة بنقطة لمشغلات تكاملية أحادية لا خطية ذات متغيرين

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خلاصة

نعرض في هذا البحث بعض النظريات عن تقارب نقطة بنقطة ومعدل التقارب نقطة بنقطة لعائلة مشغلات تكاملية أحادية لا خطية ذات متغيرين للصيغة التالية:

$$T_\lambda(f;x,y) = \iint_D K_\lambda(t-x, s-y, f(t,s))\,ds\,dt, \qquad (x,y) \in D,\ \lambda \in \Lambda,$$

حيث أن f عبارة عن دالة ذات قيمة حقيقية لها تكامل في منطقة اختيارية محدودة مغلقة أو شبه مغلقة أو مفتوحة $D = \langle a,b \rangle \times \langle c,d \rangle$ في $\mathbb{R}^2$ أو $D = \mathbb{R}^2$ و Λ هي مجموعة مؤشرات مع نقطة تجمع λ_0 .