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On the powers of the Kummer distribution

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Abstract

The Kummer distribution is a probability distribution, whose density is given by
$$f(x) = Cx^{\alpha-1}(1+\delta x)^{-\gamma}e^{-\beta x}, \quad x > 0,$$
where $\alpha, \beta, \delta > 0, \gamma \in \mathbb{R}$, and C is a normalizing constant. In this paper, the distributions of random variable X^p, $p > 0$, where X has the Kummer distribution, are considered with the conditions being IFR/DFR, some properties of moments depending on the parameters and the moment-(in) determinacy. In the case of moment-indeterminacy, exemplary Stieltjes classes are constructed.

Keywords: Failure rate; Kummer distribution; moment-(in) determinacy; Stieltjes class.
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1. Introduction

A parametric family of distributions, known as Kummer distributions, was first introduced by Armero & Bayarri (1997) to perform a statistical analysis of $M/M/\infty$ queuing systems. They proposed the following definition.

Definition 1. A random variable X is said to have *Kummer distribution* with parameters α, β, γ, δ if its probability density equals:

$$f(x|\alpha,\beta,\gamma,\delta) = \begin{cases} Cx^{\alpha-1}(1+\delta x)^{-\gamma}e^{-\beta x}, & x > 0 \\ 0, & x \le 0 \end{cases} \qquad (1)$$

where $\alpha, \beta, \delta > 0$, $\gamma \in \mathbb{R}$, and $C = C(\alpha,\beta,\gamma,\delta)$ is a normalizing constant.

Notice that the constant C in (1) can be obtained from

$$C^{-1} = \int_0^\infty x^{\alpha-1}(1+\delta x)^{-\gamma}e^{-\beta x}dx = \delta^{-\alpha}\Gamma(\alpha)U(\alpha, \alpha-\gamma+1, \beta/\delta)$$

where $U(a,b,z)$ is the Kummer's function of the second kind defined by:

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}\, t^{a-1}(1+t)^{b-a-1}dt, \quad \mathrm{Re}(a) > 0, \mathrm{Re}(z) > 0.$$

See Abramowitz & Stegun (1972) formula 13.2.5. The involvement of Kummer's function in the density lent the name to the distribution. When $\delta = 1$, the distribution can be viewed as scale-standard, and without loss of generality only $\delta = 1$ will be studied here, although all the results can be easily extended to any $\delta > 0$. The notation $X \sim \mathrm{Kum}(\alpha,\beta,\gamma)$ means that

$$f_X(x) := f(x|\alpha,\beta,\gamma) = Cx^{\alpha-1}(1+x)^{-\gamma}e^{-\beta x}, \quad x > 0 \qquad (2)$$

where

$$C = C(\alpha,\beta,\gamma) = \left(\int_0^\infty x^{\alpha-1}(1+x)^{-\gamma}e^{-\beta x}dx\right)^{-1}$$

is a normalizing constant.

When $\gamma = 0$, the density in (1) is that of the classical Gamma(α,β)-distribution. Therefore, the focus of the current work will be on $\gamma \neq 0$. Generalization with $\gamma \neq 0$ widens the class of distributions and provides additional possibilities for its usage in theoretical research as well as in practical applications. The importance of the Kummer distribution for probability theory has been demonstrated in Koudou & Vallois (2012), where this distribution is shown to play a major role in the investigation of the Matsumoto-Yor type independence property. To be specific, the Kummer distribution provides solution to some characterization problems. Among other results, the following elegant relation has been obtained: If independent random variables X and Y have $\mathrm{Kum}(\alpha,\gamma-\alpha,\beta)$ and Gamma$(\gamma-\alpha,\beta)$ distributions, respectively, then $X+Y$ follows $\mathrm{Kum}(\alpha,\alpha-\gamma,\beta)$ distribution. Their results were extended to random matrices in Koudou (2012).

On the other hand, the Kummer distribution in the form (1) naturally emerges in the study of the queuing systems with infinite service channels, in which both inter-arrival and service times are exponential. The reader is referred to Armero & Bayarri (1997). As the queuing theory has a wide range of applications in engineering and service industries, such studies make

room for further interest in the Kummer distribution and serve as additional motivation to examine its various properties. Furthermore, in the case $\alpha < 1$ and $\gamma < 0$,
when $\sqrt{-\gamma} > \sqrt{\beta} + \sqrt{1-\alpha}$, the density decreases from $+\infty$, reaches the minimum (anti-mode), then increases to the maximum (mode), and finally decreases to zero as $x \to +\infty$. For example, when an energetic particle stream is injected into the plasma, such "bimodal" distributions reflect the bump-in-tail instability phenomenon well-known in the plasma physics. Chen (1984) and Krall & Trivelpiece (1973), Sect 9.4.

In the sequel, we consider an extended family of distributions related to $\mathrm{Kum}(\alpha,\beta,\gamma)$. To be specific, given $X \sim \mathrm{Kum}(\alpha,\beta,\gamma)$, we consider the distributions of the powers $X^p, p > 0$, whose densities can be derived using the well-known formula

$$f_{X^p}(x) = f_X(x^{1/p})\frac{d}{dx}(x^{1/p}), \qquad x > 0.$$

The powers of random variables being involved in the Box-Cox transformation are commonly used in the statistical practice Box & Cox (1964). Bearing in mind the importance of the powers of random variables, the powers of Kummer distribution will also be under scrutiny in the current work. At this stage, the definition below is introduced.

Definition 2. A random variable X is said to have p*-Kummer distribution* ($p > 0$) with parameters α, $\beta > 0$, $\gamma \in \mathbb{R}$, and is written $X \sim \mathrm{Kum}_p(\alpha,\beta,\gamma)$, if its probability density equals

$$f_p(x) := f_p(x|\alpha,\beta,\gamma) = C_p x^{\frac{\alpha}{p}-1}\left(1 + x^{\frac{1}{p}}\right)^{-\gamma} e^{-\beta x^{\frac{1}{p}}},\ x > 0, \qquad (3)$$

where $C_p = C_p(\alpha,\beta,\gamma) = [p\Gamma(\alpha)U(\alpha,\alpha-\gamma+1,\beta)]^{-1}$ is a normalizing constant.

The preceding discussion implies that $X \sim \mathrm{Kum}(\alpha,\beta,\gamma)$ if and only if $X^p \sim \mathrm{Kum}_p(\alpha,\beta,\gamma)$. Clearly, when $\gamma = 0$, the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ coincides with a generalized Gamma distribution. The family of such generalized Gamma distributions is a rich and important one, since it contains such well-known distributions as exponential, Gamma, Weibull, and others. Stoyanov & Tolmatz (2004), formula (1) and Stoyanov (2013), Section 11.4.

It should be mentioned here that probability densities associated with various generalizations of Γ-function, including Kummer's confluent hypergeometric function, have been researched during the last decade in different ligths. The reader is referred to the works Al-Saqabi *et al.* (2003), Al-Zamel (2001), Fereira & Salinas (2010), Joarder & Omar (2013), Nagar & Gupta (2002) and Saxena *et al.* (2007), together with the references therein to follow the progress in the area. This paper is aimed to contribute to the ongoing research by exhibiting new results related to the Kummer distribution.

On a note of history, the Kummer distribution was studied by Ng & Kotz (1995). They considered two classes of Kummer distributions: Kummer-Gamma, which coincides with the one given by Definition 1, and Kummer-Beta which generalizes the classical Beta distribution. Not only did they analyze those distributions, they also outlined a series of problems, some of which - relations between the moments of the Gamma and Kummer distributions, and the behavior of the failure rate - are addressed here. It should be pointed out that, in Ng & Kotz (1995), the density function of the Kummer-Gamma distribution is written in a slightly different form than in (2), namely

$$f(x) = Cx^{\alpha-1}(1+\delta x)^{\gamma}e^{-\beta x}, \quad x > 0.$$

In this paper, we employ the notation of Armero & Bayarri (1997) and write the density in the form (2) using the name 'Kummer distribution'. Additional information on this can also be found in Johnson *et al.* (1995).

This work establishes new results related to the Kummer and p-Kummer distributions. First, the failure rate of the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ depending on the parameters is examined. The similarities and distinctions with the classical case $\gamma = 0$, $p = 1$ are put forth. Further, for $X \sim \mathrm{Kum}_p(\alpha,\beta,\gamma)$, the monotonocity of the moments $m_k = \boldsymbol{E}[X^k]$ with respect to γ is shown. Finally, the moment-determinacy and moment-indeterminacy for the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ are investigated and, in the event of moment-indeterminacy, illustrative Stieltjes classes are set up.

Before we start, let us recall some necessary notations and definitions introduced in Stoyanov (2004). Consider a probability distribution P for which moments of all orders are finite. If the moment problem for P possesses a unique solution, then P is referred to as *moment-determinate* or *M-determinate*; otherwise, it is referred to as *moment-indeterminate* or *M-indeterminate*. There are quite a number of conditions to investigate M-determinacy and M-indeterminacy. The most widely used are the Cramér, Carleman, and Krein conditions. See, for example, Stoyanov (2013), Section 11. Even though, these results allow us to establish M-(in)determinacy, they do not provide any practical tool to construct different distributions with the same moments in the case of M-indeterminacy. Once a distribution is absolutely continuous, the problem can be handled by means of *Stieltjes classes*. The term 'Stieltjes class' has been proposed by Stoyanov (2013) and reflects the contributions of Stieltjes to the moment problem.

Definition 3. Let f be a probability density having finite moments of all orders, and h be an integrable function on $(-\infty,\infty)$ such that $\text{vraisup}|h(x)| = 1$. If, for all $n \in \mathbb{N}_0$,

$$\int_{\mathbb{R}} x^n\, h(x) f(x)\, dx = 0,$$

then h is called a *perturbation* of f. Equivalently, one can also say that the product hf has its *all moments vanishing*.

Definition 4. Given a probability density f and its perturbation h, the set

$$\boldsymbol{S} = \boldsymbol{S}(f,h) := \{\omega_\varepsilon(x): \omega_\varepsilon(x) = f(x)[1+\varepsilon h(x)],\ x \in \mathbb{R}, \varepsilon \epsilon [-1,1]\}, \quad (4)$$

is said to be a *Stieltjes class* for density f based on perturbation h.

It can be readily seen that $\boldsymbol{S}$ is an infinite family of probability densities all having the same moments as f. Using different perturbation functions h, one may construct various Stieltjes classes related to a given density f. During the last decade, the Stieltjes classes have attracted the attention of many researchers, who have proposed a variety of methods to produce such classes (Kleiber, 2013; Ostrovska & Stoyanov, 2005; Pakes, 2007; Stoyanov & Tolmatz, 2005). Towards the end, this paper deals with the moment-determinacy for $\text{Kum}_p(\alpha,\beta,\gamma)$, and present the Stieltjes classes in the situation of moment-indeterminacy when $p > 2$.

2. Failure rate

Let X be a random variable with density f and distribution function F. The *failure rate* - or the hazard function - of F is defined by:

$$H(x) = \frac{f(x)}{1-F(x)}. \quad (5)$$

The distribution is *increasing failure rate* (IFR) if $H(x)$ is increasing in x, and *decreasing failure rate* (DFR) if $H(x)$ is decreasing in x.

In this section, the restrictions on the parameters stipulating that the failure rate of $\text{Kum}_p(\alpha,\beta,\gamma)$ is increasing/decreasing are established. The next theorems provide the conditions for the Kummer distribution, whose density is defined by (3), to be IFR/DFR for all $x > 0$.

Theorem 1. *Assume that the distribution is not exponential; that is* $\alpha = 1$ *and* $\gamma = 0$ *cannot occur simultaneously. The Kummer distribution is:*
(i) *IFR for all* $x > 0$ *if and only if both* $\alpha \geq 1$ *and* $\gamma \leq \alpha - 1$;
(ii) *DFR for all* $x > 0$ *if and only if both* $0 < \alpha < 1$ *and* $\gamma \geq \alpha - 1$.

Proof. Let $X \sim \text{Kum}(\alpha,\beta,\gamma)$. Then, one can write:

$$H(x) = \frac{x^{\alpha-1} e^{-\beta x}(1+x)^{-\gamma}}{\int_x^\infty t^{\alpha-1}\, e^{-\beta t}(1+t)^{-\gamma} dt}, \quad (6)$$

whence

$$\frac{1}{H(x)} = \int_x^\infty \left(\frac{t}{x}\right)^{\alpha-1} \left(\frac{1+t}{1+x}\right)^{-\gamma} e^{-\beta(t-x)} dt$$

$$= \int_0^\infty \left(1+\frac{t}{x}\right)^{\alpha-1} \left(1+\frac{t}{1+x}\right)^{-\gamma} e^{-\beta t} dt. \quad (7)$$

Set

$$V(x) = \left(1+\frac{t}{x}\right)^{\alpha-1} \left(1+\frac{t}{1+x}\right)^{-\gamma},$$

and obtain

$$V'(x) = -\frac{tV(x)}{x(x+t)(1+x)(1+x+t)} \times [(\alpha-1-\gamma)x^2 + (2(\alpha-1) + t(\alpha-1-\gamma))x + (\alpha-1)(1+t)] \quad (8)$$

(i) Obviously, the sign of $V'(x)$ depends solely on the expression in the brackets and the latter is positive on the whole interval $(0,+\infty)$ if and only if $\alpha \geq 1$ and also $\gamma \leq \alpha - 1$ so that both equalities do not hold simultaneously. Consequently, in this case, $1/H(x)$ is decreasing, implying that $H(x)$ is monotone increasing for all $x > 0$.

(ii) Clearly, under the stated conditions regarding α and γ, the derivative $V'(x)$ is positive and, correspondingly, the distribution is DFR for all $x > 0$. □

In the case $\gamma = 0$, the results on the failure rate of $\Gamma(\alpha,\beta)$-distribution are recovered. After more careful examination of (8), the next statement can be easily reached.

Theorem 2. (i) *If* $\gamma > \alpha - 1$, *then there exists a point* x_0 *depending on the parameters such that the Kummer distribution is IFR on* $(0,x_0)$ *and DFR on* (x_0,∞).
(ii) *If* $\gamma < \alpha - 1$, *then there exists a point* x_0 *depending on the parameters such that the Kummer distribution is DFR on* $(0,x_0)$ *and IFR on* (x_0,∞).

It is worth pointing out that as $x \to \infty$, $H(x)$ approaches the failure rate of the exponential distribution with parameter β. Namely,

Corollary 3. *For all values of* α, β, γ, *one has:*

$$\lim_{x\to+\infty} H(x) = \beta.$$

Proof. This statement follows from (7) after passing to limit as $x \to \infty$, which can be justified by the Lebesgue Dominated Convergence Theorem.

The last statement demonstrates that the graph of the failure rate of Kummer distribution can not have the bathtub shape.

Having $p \neq 1$, the modification of the proof of the Theorem 1 can be used. It should be emphasized that the asymptotic behavior of the failure rate for $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ with $p \neq 1$ depends solely on p, rather than α, β and γ. More precisely, the statement below is valid:

Theorem 4. *Let* $X \sim \mathrm{Kum}_p(\alpha,\beta,\gamma)$ *and* $1 \neq p > 0$. *Then,*

(i) *when* $p > 1$, $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ *is DFR for* x *large enough and all* $\alpha > 0$, $\beta > 0$, $\gamma \in \mathbb{R}$.

(ii) *when* $0 < p < 1$, $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ *is IFR for* x *large enough and all* $\alpha > 0, \beta > 0, \gamma \in \mathbb{R}$.

Proof. Writing as in (6), one obtains with the help of (3), that

$$H_p(x) = \frac{x^{\alpha/p-1}e^{-\beta x^{1/p}}\left(1+x^{1/p}\right)^{-\gamma}}{\int_x^\infty t^{\alpha/p-1}\, e^{-\beta t^{1/p}}(1+t^{1/p})^{-\gamma}dt}, \quad (9)$$

and hence,

$$\frac{1}{H_p(x)} = \int_x^\infty \left(\frac{t}{x}\right)^{\frac{\alpha}{p}-1}\left(\frac{1+t^{\frac{1}{p}}}{1+x^{\frac{1}{p}}}\right)^{-\gamma} e^{-\beta\left(t^{\frac{1}{p}}-x^{\frac{1}{p}}\right)}dt$$

$$= \int_0^\infty pV\left(x^{1/p}\right)e^{-\beta t}dt$$

where

$$V(x) = x^{p-\alpha}(t+x)^{\alpha-1}\left(1+\frac{t}{1+x}\right)^{-\gamma}.$$

Thence,

$$\frac{V'(x)}{V(x)} = \frac{p-\alpha}{x} + \frac{\alpha-1}{t+x} + \frac{\gamma t}{(1+x)(1+t+x)}$$

$$\sim \frac{p-1}{x}, \quad \text{as } x \to +\infty, \quad (10)$$

which shows that, for x large enough, $V'(x)(p-1) > 0$ provided that $1 \neq p > 0$, and $V'(x) < 0$ provided $0 < p < 1$. This observation completes the proof.

3. Properties of moments

Since the Kummer distribution is a generalization of Gamma distribution, it is interesting to compare the moments of the Gamma (i.e., the case $\gamma = 0$) and Kummer distributions. This can be done with the help of the following statement describing the monotonicity of moments of $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ with respect to parameter γ. This is a generalization of the result given by Ng & Kotz (1995), who proved it for γ being a negative integer and $p = 1$.

Theorem 5. *For all* $p > 0$ *and* $-\infty < \gamma_1 \leq \gamma_2 < +\infty$, *one has:*

$$\int_0^{+\infty} x^k f_p(x|\alpha,\beta,\gamma_1)dx$$
$$\geq \int_0^{+\infty} x^k f_p(x|\alpha,\beta,\gamma_2)dx, \quad k \in \mathbb{N}_0. \quad (11)$$

Proof. Taking $\gamma_1 \leq \gamma_2$, consider the double integral

$$I_p := \int_0^\infty\int_0^\infty (xy)^{\frac{\alpha}{p}-1}e^{-\beta\left(x^{\frac{1}{p}}+y^{\frac{1}{p}}\right)} \times$$

$$(1+x^{\frac{1}{p}})^{-\gamma_1}(1+y^{\frac{1}{p}})^{-\gamma_2}\ (x^k - y^k)dxdy$$

$$= \iint_{A_1} + \iint_{A_2}, \quad (12)$$

where $A_1 := \{(x,y): x \geq y \geq 0\}$ and $A_2 := \{(x,y): y \geq x \geq 0\}$. It can be noticed - by interchanging x and y - that

$$\iint_{A_2} = \iint_{A_1} (xy)^{\frac{\alpha}{p}-1}e^{-\beta\left(x^{\frac{1}{p}}+y^{\frac{1}{p}}\right)} \times$$
$$(1+x^{\frac{1}{p}})^{-\gamma_2}(1+y^{\frac{1}{p}})^{-\gamma_1}(y^k - x^k)dxdy,$$

whence

$$I_p = \iint_{A_1} (xy)^{\frac{\alpha}{p}-1}e^{-\beta\left(x^{\frac{1}{p}}+y^{\frac{1}{p}}\right)} \times$$

$$\left[\left(1+x^{\frac{1}{p}}\right)^{-\gamma_1}\left(1+y^{\frac{1}{p}}\right)^{-\gamma_2}\right.$$
$$\left.-\left(1+x^{\frac{1}{p}}\right)^{-\gamma_2}\left(1+y^{\frac{1}{p}}\right)^{-\gamma_1}\right](x^k - y^k)dxdy, \quad (13)$$

Since $x \geq y$ in A_1, and $\gamma_2 \geq \gamma_1$, it follows that

$$(1+x^{1/p})^{\gamma_2-\gamma_1} \geq (1+y^{1/p})^{\gamma_2-\gamma_1}$$

for all $(x,y) \in A_1$, or, equivalently,

$$(1+x^{\frac{1}{p}})^{-\gamma_1}(1+y^{\frac{1}{p}})^{-\gamma_2} \geq (1+x^{1/p})^{-\gamma_2}(1+y^{1/p})^{-\gamma_1}$$

for all $(x,y) \in A_1$.

This implies that the integrand in (13) is non-negative, and hence, $I_p \geq 0$. By virtue of (12), the latter leads to

$$\int_0^\infty x^{k+\frac{\alpha}{p}-1} e^{-\beta x^{\frac{1}{p}}}(1+x^{\frac{1}{p}})^{-\gamma_1}dx \times \int_0^\infty y^{\frac{\alpha}{p}-1} e^{-\beta y^{\frac{1}{p}}}(1+y^{\frac{1}{p}})^{-\gamma_2}dy$$

$$\geq \int_0^\infty y^{k+\frac{\alpha}{p}-1} e^{-\beta y^{\frac{1}{p}}}(1+y^{\frac{1}{p}})^{-\gamma_2}dy \times \int_0^\infty x^{\alpha/p-1} e^{-\beta x^{1/p}}(1+x^{1/p})^{-\gamma_1}dx.$$

This can be rewritten in the form:

$$\frac{1}{C_p(\alpha,\beta,\gamma_2)}\int_0^\infty x^k \frac{f_p(x|\alpha,\beta,\gamma_1)}{C_p(\alpha,\beta,\gamma_1)}\,dx \geq \frac{1}{C_p(\alpha,\beta,\gamma_1)}\int_0^\infty y^k \frac{f_p(y|\alpha,\beta,\gamma_2)}{C_p(\alpha,\beta,\gamma_2)}\,dy,$$

which proves the theorem.

Corollary 6. *Let M_k, $k \in \mathbb{N}_0$, denote the moments of $\Gamma(\alpha,\beta)$-distribution. If $X \sim \mathrm{Kum}(\alpha,\beta,\gamma)$ and $m_k = \mathbf{E}[X^k]$, then $\gamma(M_k - m_k) \geq 0$ for all $\gamma \in \mathbb{R}$.*

4. M-determinacy of $\mathbf{Kum}_p(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$

It can be observed that, as p increases, the tails of the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ become 'heavier'. This affects M-determinacy of the distribution and the exact result is stated in the next theorem.

Theorem 7. *For $0 < p \leq 2$, the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ is moment-determinate, and, for $p > 2$, it is moment-indeterminate.*

Proof. (i) Given $X \sim \mathrm{Kum}_p(\alpha,\beta,\gamma)$, one has:

$$m_k = \mathbf{E}[X^k] = \int_0^\infty x^{k+\frac{\alpha}{p}-1} e^{-\beta x^{\frac{1}{p}}}\left(1+x^{\frac{1}{p}}\right)^{-\gamma} dx = \int_0^\infty p\, s^{pk+\alpha-1}(1+s)^{-\gamma}e^{-\beta s}ds. \quad (14)$$

When $\gamma \geq 0$, equality (14) implies, by virtue of Stirling's formula for the gamma function, that,

$$m_k \leq \int_0^\infty p\, s^{pk+\alpha-1}e^{-\beta s}ds = p\beta^{-pk-\alpha}\Gamma(pk+\alpha) \sim p\beta^{-pk-\alpha}\sqrt{\frac{2\pi}{pk+\alpha}}\left(\frac{pk+\alpha}{e}\right)^{pk+\alpha},$$

for some $a \geq 0$. See, Stoyanov (2000) and Stoyanov (2013), Section 11, p.101 formula (2b) for details. From (3), it can be derived that

$$\frac{-\ln f_p(x^2)}{1+x^2} \sim \frac{\beta x^{2/p}}{1+x^2} \quad \text{as } x \to \infty.$$

Therefore,

$$\int_a^\infty \frac{-\ln f_p(x^2)}{1+x^2}\,dx < \infty$$

as $k \to \infty$. Thus, with some constant $C_1 > 0$, one obtains

$$m_k \leq C_1(e\beta)^{-pk}(pk+\alpha)^{pk+\alpha-1/2}. \quad (15)$$

In the case $\gamma < 0$, it follows from (14) that

$$m_k = \int_0^1 p\, s^{pk+\alpha-1}e^{-\beta s}(1+s)^{-\gamma}ds + \int_1^\infty p\, s^{pk+\alpha-1}e^{-\beta s}(1+s)^{-\gamma}ds$$

$$\leq \int_0^1 p\, 2^{-\gamma}e^{-\beta s}ds + \int_1^\infty p\, s^{pk+\alpha-1}e^{-\beta s}(2s)^{-\gamma}ds$$

$$\leq C + p2^{-\gamma}\int_0^\infty e^{-\beta s}\, s^{pk+\alpha-\gamma-1}ds$$

$$= C + p2^{-\gamma}\beta^{\gamma-pk-\alpha}\Gamma(pk+\alpha-\gamma)$$

$$\sim p2^{-\gamma}\beta^{\gamma-pk-\alpha}\sqrt{\frac{2\pi}{pk+\alpha-\gamma}}\left(\frac{pk+\alpha-\gamma}{e}\right)^{pk+\alpha-\gamma},$$

as $k \to \infty$. Hence,

$$m_k \leq C_2(e\beta)^{-pk}(pk+\alpha-\gamma)^{pk+\alpha-\gamma-1/2} \quad (16)$$

for some $C_2 > 0$. As a result, inequalities (15) and (16) yield

$$m_k^{-1/(2k)} \leq C_3 k^{-p/2}, \qquad k \to \infty,$$

where C_3 is a positive constant. That is, when $0 < p < 2$, the series $\sum_{k=0}^\infty m_k^{-1/2k}$ diverges to ∞ for any $\gamma \in \mathbb{R}$. Using the Carleman condition for the Stieltjes moment problem (see Stoyanov, 2013, p.100), one may conclude that the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ is moment-determinate, whenever $p \in (0,2]$.

(ii) To prove the moment-indeterminacy of $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ with $p > 2$, we use the Krein condition pertinent to the Stieltjes moment problem. This condition is applicable to absolutely continuous distributions having support $[0,\infty)$ and possessing moments of all orders. Such a distribution is moment-indeterminate if

$$\int_a^\infty \frac{-\ln f(x^2)}{1+x^2}\,dx < \infty,$$

for all $a > 0$, $p > 2$. Thus, the distribution $\mathrm{Kum}_p(\alpha,\beta,\gamma)$ with $p > 2$ is M-indeterminate.

Dealing with applications of an M-indeterminate distribution, it is now desirable to exhibit explicitly other distributions with the same moments of all orders. As it has already been mentioned, absolutely continuous distributions make it possible to achieve this goal with the help of the Stieltjes classes. The next theorem provides examples of perturbation functions related to density (3) in the case $p > 2$.

Theorem 8. *Let* $p > 2$ *and*

$$\tilde{h}(x) := (1 + x^{\frac{1}{p}})^{\gamma} \exp\left\{-(b-\beta)x^{\frac{1}{p}}\right\} \times \sin\left[b \tan\left(\frac{\pi}{p}\right)x^{1/p} - \frac{\alpha\pi}{p}\right] \quad (17)$$

where $b > \beta$. *Then*

$$h(x) = \frac{\tilde{h}(x)}{\|\tilde{h}(x)\|_{[0,\infty)}} \quad (18)$$

can serve as a perturbation function for density (3).

Proof. It has been shown in Ostrovska & Stoyanov (2005) that, for all $k \in \mathbb{N}_0$,

$$\int_0^{\infty} x^{k+\frac{\alpha}{p}-1} e^{-bx^{\frac{1}{p}}} \sin\left[b \tan\left(\frac{\pi}{p}\right)x^{1/p} - \frac{\alpha\pi}{p}\right] dx = 0$$

whenever $\alpha > 0,\ b > 0,\ p > 2$.

Equivalently,

$$\int_0^{\infty} x^k f_p(x)\tilde{h}(x)dx = 0, \quad (19)$$

where $\tilde{h}(x)$ is given by (17). Clearly, as long as $b > \beta$, function $\tilde{h}(x)$ is bounded on $[0,\infty)$ whatever $\gamma \in \mathbb{R}$ is. As a result, $h(x)$ defined by (18) satisfies all the conditions of being a perturbation function for f_p.

Remark 1. If $\gamma \leq 0$, then one may take $b = \beta$ and obtain $\tilde{h}(x)$ in a simpler form:

$$\tilde{h}(x) = (1 + x^{\frac{1}{p}})^{\gamma} \sin\left[\beta \tan\left(\frac{\pi}{p}\right)x^{1/p} - \frac{\alpha\pi}{p}\right].$$

Corollary 5. *The set*

$$S := \{f_p(x)[1 + \varepsilon h(x)] : \varepsilon \in [-1,1]\}$$

is a Stieltjes class for f_p.

5. Conclusion

The results of this work provide new properties of the Kummer distribution and its positive powers. It has been established that for powers $p > 1$, the distribution is decreasing failure rate for all admissible values of the parameters when x is large enough, while, for $0 < p < 1$, the situation is opposite. Since the Kummer distribution includes as special cases some important probability distributions used in reliability theory and actuarial sciences, this result is important for applications. The case $p = 1$ is more complicated and the relations between parameters which guarantee the distribution to be increasing/decreasing failure rate have been provided.

New results concerning the moment-(in)determinacy of powers of the Kummer distribution have been obtained. The following important conclusion has been reached: For $0 < p \leq 2$ powers of the Kummer distribution are moment-determinate, whereas for $p > 2$ they are moment-indeterminate. In the latter case, the Stieltjes classes have been written explicitly.

The results of this work are important for applications in such areas as probability theory, financial mathematics, reliability theory, actuarial science, and others, due to the fact that they provide easily checkable conditions for some properties of the Kummer distribution and its powers.

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حول قدرات توزيع كومير (Kummer)

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خلاصة

توزيع كومير (kummer) عبارة عن توزيع احتمالي، والذي يُعبر عن كثافته بالمعادلة:

$$f(x) = Cx^{\alpha-1}(1+\delta x)^{-\gamma}e^{-\beta x}, \quad x > 0,$$

حيث $\alpha, \beta, \delta > 0, \gamma \in \mathbb{R}$, و C عبارة عن ثابت المعايرة. في هذا البحث، تم الوضع في الاعتبار توزيعات المتغير العشوائي X^p، $P > 0$، حيث X لديه توزيع كومير (kummer) مع الشروط اللازمة لضمان كون التوزيع IFR/DFR، تعتمد بعض خصائص العزوم على المعلمات وعلى كون العزوم مُحددة أو غير مُحددة للتوزيع، في حالة كون العزوم مُحددة للتوزيع، تم بناء توزيعات ستيلتجز (Stieltjes) مثالية.