Introduction to $k-$ Horadam hybrid numbers

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Abstract

In this paper, we consider the $k-$ Horadam hybrid numbers and investigate some of their properties. We also give some applications related to the $k-$ Horadam hybrid numbers in matrices.

Keywords: Complex numbers; fibonacci and lucas numbers; generating functions; hyperbolic numbers; recurrences.

1. Introduction

Let k be any positive real number and $h(k)$ and $g(k)$ are scaler value polynomials. For $m \geq 0$ and $h^2(k) + 4g(k) > 0$ the generalized $k-$ Horadam sequence $\{H_{k,m}\}_{m\in\mathbb{N}}$ is described by

$$H_{k,m+2} = h(k)H_{k,m+1} + g(k)H_{k,m} \tag{1}$$

with initial conditions $H_{k,0} = w$, $H_{k,1} = z$.

The solutions of the equation $y^2 - h(k)y - g(k) = 0$ associated with the recurrence relation (1) are

$$\begin{aligned} \lambda =& \frac{h(k) + \sqrt{h^2(k) + 4g(k)}}{2} \quad and \\ \gamma =& \frac{h(k) - \sqrt{h^2(k) + 4g(k)}}{2}. \end{aligned} \tag{2}$$

Note that:

$$\lambda + \gamma = h(k), \quad \lambda - \gamma = \sqrt{h^2(k) + 4g(k)}, \quad \lambda\gamma = -g(k). \tag{3}$$

So the Binet formula for the $k-$ Horadam sequence is given by

$$H_{k,m} = \frac{N\lambda^m - K\gamma^m}{\lambda - \gamma}, \tag{4}$$
$$where \quad N = z - w\gamma, \quad K = z - w\lambda.$$

$k-$ Horadam sequence is a generalization of some sequences such as the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, $k-$ Fibonacci and $k-$ Lucas sequences. These sequences have applications in algebra, number theory and geometry. Hence, these sequences have been studied by a number of researchers; see for examples: (Horadam, 1965; Koshy, 2001; Koshy, 2018; Kilic, 2019a; Kilic, 2019b; Kilic, 2019c; Akkus & Kizilaslan, 2019; Yazlik & Taskara, 2012).

Remark 1.1. *Some particular cases of (1) are ;*
• *If $h(k) = k$, $g(k) = 1$, $w = 0$ and $z = 1$ then we get the $k-$ Fibonacci sequence;*

$$F_{k,m+2} = kF_{k,m+1} + F_{k,m}, \quad F_{k,0} = 0, F_{k,1} = 1.$$

• *If $h(k) = k$, $g(k) = 1$, $w = 2$ and $z = k$ then we have the $k-$ Lucas sequence;*

$$L_{k,m+2} = kL_{k,m+1} + L_{k,m}, \quad L_{k,0} = 2 \quad L_{k,1} = k.$$

• *If $h(k) = 2$, $g(k) = k$, $w = 0$ and $z = 1$ then we obtain the $k-$ Pell sequence;*

$$P_{k,m+2} = 2P_{k,m+1} + kP_{k,m}, \quad P_{k,0} = 0 \quad P_{k,1} = 1.$$

• *If $h(k) = p$, $g(k) = q$, then we find the Horadam sequence;*

$$H_{m+2} = pH_{m+1} + qH_m, \quad H_0 = w \quad H_1 = z.$$

• *If $h(k) = 1$, $g(k) = 1$, $w = 0$ and $z = 1$ then we get the Fibonacci sequence;*

$$F_{m+2} = F_{m+1} + F_m, \quad F_0 = 0 \quad F_1 = 1.$$

• *If $h(k) = 2$, $g(k) = 1$, $w = 1$ and $z = 1$ then we have the Pell-Lucas sequence;*

$$PL_{m+2} = 2PL_{m+1} + PL_m, \quad PL_0 = 1 \quad PL_1 = 1.$$

The set $\mathbb{T}$, defined below, represents the set of hybrid numbers;

$$\mathbb{T} = \{t_1 + t_2 i + t_3\varepsilon + t_4\mathbf{h} | t_1, t_2, t_3, t_4 \in \mathbb{R}; i^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, i\mathbf{h} = -\mathbf{h}i = \varepsilon + i\}. \quad (5)$$

Addition and subtraction of hybrid numbers are done by adding and subtracting corresponding terms. The product of any two hybrid numbers, using the expression (5), can be defined as in Szynal-Liana (2018) (see Table 1).

The conjugate of a hybrid number $P = t_1 + t_2 i + t_3\varepsilon + t_4\mathbf{h}$ is defined by

$$\overline{P} = t_1 - t_2 i - t_3\varepsilon - t_4\mathbf{h}. \quad (6)$$

The nth Horadam hybrid numbers $\widetilde{h}H_n$ is described as

$$\widetilde{h}H_n = H_n + iH_{n+1} + \varepsilon H_{n+2} + H_{n+3}\mathbf{h}. \quad (7)$$

where H_n is the nth Horadam number and $i, \varepsilon, \mathbf{h}$ denote hybrid units.

Hybrid numbers and hybrid polynomials have been studied by several researchers; (Szynal-Liana, 2018) introduced Horadam hybrid numbers and found some results about them. In 2019, Szynal-Liana & Włoch, studied Fibonacci and Lucas hybrinomials. The authors (Liana *et al.*, 2019) introduced Pell hybrid numbers and investigated some of their properties. Kizilates (2020) studied the $q-$ Fibonacci hybrid numbers. Kilic, (2019a) considered split $k-$ Jacobsthal and $k-$ Jacobsthal Lucas quaternions and obtained some properties of them. Also, Kilic studied Horadam octonions and dual Horadam octonions (Kilic, 2019b, 2019c).

Now, we introduce the $k-$Horadam hybrid numbers and investigate some of their properties. We also give some applications related to the $k-$ Horadam hybrid numbers in matrices.

2. $k-$ Horadam hybrid numbers

For $m > 0$, the $k-$ Horadam hybrid numbers are defined by

$$\widetilde{h}H_{k,m} = H_{k,m} + iH_{k,m+1} + \varepsilon H_{k,m+2} + H_{k,m+3}\mathbf{h} \tag{8}$$

where $H_{k,m}$ is the mth $k-$ Horadam number and $i, \varepsilon, \mathbf{h}$ are hybrid units.

Theorem 2.1. *Let $s \geq 2$ be an integer. Then*

$$\widetilde{h}H_{k,s} = h(k)\widetilde{h}H_{k,s-1} + g(k)\widetilde{h}H_{k,s-2} \tag{9}$$

with $\widetilde{h}H_{k,0} = w + iz + \varepsilon[h(k)z + g(k)w] + \mathbf{h}[h^2(k)z + h(k)g(k)w + g(k)z]$ *and* $\widetilde{h}H_{k,1} = z + i[h(k)z + g(k)w] + \varepsilon[h^2(k)z + h(k)g(k)w + g(k)z] + \mathbf{h}[h^3(k)z + h^2(k)g(k)w + 2h(k)g(k)z + g^2(k)w]$.

Proof. If $s = 2$ then, we get

$$\begin{aligned}
&h(k)\widetilde{h}H_{k,1} + g(k)\widetilde{h}H_{k,0} \\
&= h(k)\{z + i[h(k)z + g(k)w] + \varepsilon[h^2(k)z + h(k)g(k)w + g(k)z] + \mathbf{h}[h^3(k)z + h^2(k)g(k)w \\
&+ 2h(k)g(k)z + g^2(k)w]\} + g(k)\{w + iz + \varepsilon[h(k)z + g(k)w] + \mathbf{h}[h^2(k)z + h(k)g(k)w \\
&+ g(k)z]\} \\
&= h(k)z + g(k)w + i[h^2(k)z + h(k)g(k)w + g(k)z] + \varepsilon[h^3(k)z + h^2(k)g(k)w + 2h(k)g(k)z \\
&+ g^2(k)w] + \mathbf{h}[h^4(k)z + h^3(k)g(k)w + 3h^2(k)g(k)z + 2h(k)g^2(k)w + g^2(k)z] \\
&= H_{k,2} + iH_{k,3} + \varepsilon H_{k,4} + \mathbf{h}H_{k,5} \\
&= \widetilde{h}H_{k,2}.
\end{aligned}$$

If $s \geq 3$ then using (1), we obtain

$$\begin{aligned}
&\widetilde{h}H_{k,s} \\
&= H_{k,s} + iH_{k,s+1} + \varepsilon H_{k,s+2} + \mathbf{h}H_{k,s+3} \\
&= h(k)H_{k,s-1} + g(k)H_{k,s-2} + i[h(k)H_{k,s} + g(k)H_{k,s-1}] + \varepsilon[h(k)H_{k,s+1} + g(k)H_{k,s}] \\
&+ \mathbf{h}[h(k)H_{k,s+2} + g(k)H_{k,s+1}] \\
&= h(k)[H_{k,s-1} + iH_{k,s} + \varepsilon H_{k,s+1} + \mathbf{h}H_{k,s+2}] + g(k)[H_{k,s-2} + iH_{k,s-1} + \varepsilon H_{k,s} + \mathbf{h}H_{k,s+1}] \\
&= h(k)\widetilde{h}H_{k,s-1} + g(k)\widetilde{h}H_{k,s-2}
\end{aligned}$$

which completes the proof.

Theorem 2.2. *(Binet Formula) For $m \geq 0$, the Binet formula for $k-$Horadam hybrid numbers is*

$$\widetilde{h}H_{k,m} = \frac{N\lambda^m\hat{\lambda} - K\gamma^m\hat{\gamma}}{\lambda - \gamma} \tag{10}$$

where $N = z - w\gamma, \quad K = z - w\lambda, \quad \hat{\lambda} = 1 + i\lambda + \varepsilon\lambda^2 + \mathbf{h}\lambda^3$ *and* $\hat{\gamma} = 1 + i\gamma + \varepsilon\gamma^2 + \mathbf{h}\gamma^3$.

Proof. Using (8) and (4), we can write the following expression

$$\begin{aligned}
&\widetilde{h}H_{k,m} \\
&= H_{k,m} + iH_{k,m+1} + \varepsilon H_{k,m+2} + \mathbf{h}H_{k,m+3} \\
&= [\frac{N\lambda^m - K\gamma^m}{\lambda - \gamma}] + i[\frac{N\lambda^{m+1} - K\gamma^{m+1}}{\lambda - \gamma}] + \varepsilon[\frac{N\lambda^{m+2} - K\gamma^{m+2}}{\lambda - \gamma}] + \mathbf{h}[\frac{N\lambda^{m+3} - K\gamma^{m+3}}{\lambda - \gamma}] \\
&= \frac{N\lambda^m}{\lambda - \gamma}[1 + i\lambda + \varepsilon\lambda^2 + \mathbf{h}\lambda^3] - \frac{K\gamma^m}{\lambda - \gamma}[1 + i\gamma + \varepsilon\gamma^2 + \mathbf{h}\gamma^3] \\
&= \frac{N\lambda^m\hat{\lambda} - K\gamma^m\hat{\gamma}}{\lambda - \gamma}.
\end{aligned}$$

Remark 2.3. *For $h(k)=k$, $g(k)=1$, $w=0$ and $z=1$, we obtain $\lambda=\frac{k+\sqrt{k^2+4}}{2}$, $\gamma=\frac{k-\sqrt{k^2+4}}{2}$, $N=1$, $K=1$ and the Binet formula for the $k-$ Fibonacci hybrid number $FH_{k,m}$ has the form*

$$\begin{aligned}&FH_{k,m}\\&=\frac{1}{\sqrt{k^2+4}}\Bigg\{\left(\frac{k+\sqrt{k^2+4}}{2}\right)^m\left[1+(\frac{k+\sqrt{k^2+4}}{2})i+(\frac{k+\sqrt{k^2+4}}{2})^2\varepsilon+(\frac{k+\sqrt{k^2+4}}{2})^3\mathbf{h}\right]\\&-\left(\frac{k-\sqrt{k^2+4}}{2}\right)^m\left[1+(\frac{k-\sqrt{k^2+4}}{2})i+(\frac{k-\sqrt{k^2+4}}{2})^2\varepsilon+(\frac{k-\sqrt{k^2+4}}{2})^3\mathbf{h}\right]\Bigg\}.\end{aligned}$$

Remark 2.4. *For $h(k)=k$, $g(k)=1$, $w=2$ and $z=k$, we obtain $\lambda=\frac{k+\sqrt{k^2+4}}{2}$, $\gamma=\frac{k-\sqrt{k^2+4}}{2}$, $N=\sqrt{k^2+4}$, $K=-\sqrt{k^2+4}$ and the Binet formula for the $k-$ Lucas hybrid number $LH_{k,m}$ has the form*

$$\begin{aligned}&LH_{k,m}\\&=\left(\frac{k+\sqrt{k^2+4}}{2}\right)^m\left[1+(\frac{k+\sqrt{k^2+4}}{2})i+(\frac{k+\sqrt{k^2+4}}{2})^2\varepsilon+(\frac{k+\sqrt{k^2+4}}{2})^3\mathbf{h}\right]\\&+\left(\frac{k-\sqrt{k^2+4}}{2}\right)^m\left[1+(\frac{k-\sqrt{k^2+4}}{2})i+(\frac{k-\sqrt{k^2+4}}{2})^2\varepsilon+(\frac{k-\sqrt{k^2+4}}{2})^3\mathbf{h}\right].\end{aligned}$$

Lemma 2.5. *Let $m\geq 1$ be an integer. Then*

$$\begin{aligned}\widetilde{h}H_{k,m}\overline{\widetilde{h}H_{k,m}}&=\frac{1}{h^2(k)+4g(k)}\Big\{N^2\lambda^{2m}[1+\lambda^2-2\lambda^3-\lambda^6]-2NK(-g(k))^m[1-g(k)\\&+g(k)h(k)+g^3(k)]+K^2\gamma^{2m}[1+\gamma^2-2\gamma^3-\gamma^6]\Big\}.\end{aligned}$$

Proof. From (10), we have

$$\widetilde{h}H_{k,m}\overline{\widetilde{h}H_{k,m}}=\left[\frac{N\lambda^m\hat{\lambda}-K\gamma^m\hat{\gamma}}{\lambda-\gamma}\right]\overline{\left[\frac{N\lambda^m\hat{\lambda}-K\gamma^m\hat{\gamma}}{\lambda-\gamma}\right]}$$

By Eq. (3) and some elementary calculations we have

$$\begin{aligned}\widetilde{h}H_{k,m}\overline{\widetilde{h}H_{k,m}}&=\frac{1}{h^2(k)+4g(k)}\Big\{N^2\lambda^{2m}[1+\lambda^2-2\lambda^3-\lambda^6]-2NK(-g(k))^m[1-g(k)\\&+g(k)h(k)+g^3(k)]+K^2\gamma^{2m}[1+\gamma^2-2\gamma^3-\gamma^6]\Big\}.\end{aligned}$$

Theorem 2.6. *The generating function for the $k-$ Horadam hybrid number sequence $\{\widetilde{h}H_{k,m}\}$ is*

$$\sum_{m=0}^{\infty}\widetilde{h}H_{k,m}t^m=\frac{\widetilde{h}H_{k,0}+t[\widetilde{h}H_{k,1}-h(k)\widetilde{h}H_{k,0}]}{1-h(k)t-g(k)t^2}. \tag{11}$$

Proof. Let $A(t)=\sum\limits_{m=0}^{\infty}\widetilde{h}H_{k,m}t^m$. Then

$$A(t)=\widetilde{h}H_{k,0}+\widetilde{h}H_{k,1}t+\widetilde{h}H_{k,2}t^2+... \tag{12}$$

Multiply (12) on both sides by $-h(k)t$ and then $-g(k)t^2$ we have

$$-h(k)tA(t)=-h(k)t\widetilde{h}H_{k,0}-h(k)t^2\widetilde{h}H_{k,1}-h(k)t^3\widetilde{h}H_{k,2}-... \tag{13}$$

$$-g(k)t^2A(t) = -g(k)t^2\widetilde{h}H_{k,0} - g(k)t^3\widetilde{h}H_{k,1} - g(k)t^4\widetilde{h}H_{k,2} - ... \quad (14)$$

By adding (12), (13) and (14), we have

$$A(t) = \frac{\widetilde{h}H_{k,0} + t[\widetilde{h}H_{k,1} - h(k)\widetilde{h}H_{k,0}]}{1 - h(k)t - g(k)t^2}.$$

Theorem 2.7. *Let $m \geq 1$ be an integer. Then*

$$\sum_{l=1}^{m} \widetilde{h}H_{k,l} = \frac{1}{1 - h(k) - g(k)}[\widetilde{h}H_{k,1} + g(k)\widetilde{h}H_{k,0} - \widetilde{h}H_{k,m+1} - g(k)\widetilde{h}H_{k,m}].$$

Proof. By using the Binet formula of the $k-$ Horadam hybrid numbers, we find that

$$\begin{aligned}\sum_{l=1}^{m} \widetilde{h}H_{k,l} &= \sum_{l=1}^{m} \frac{N\lambda^l\widehat{\lambda} - K\gamma^l\widehat{\gamma}}{\lambda - \gamma} \\ &= \frac{N\widehat{\lambda}}{\lambda - \gamma}[\frac{\lambda - \lambda^{m+1}}{1 - \lambda}] - \frac{K\widehat{\gamma}}{\lambda - \gamma}[\frac{\gamma - \gamma^{m+1}}{1 - \gamma}]\end{aligned}$$

By Eqs. (3), (10) and some elementary calculation, we obtain

$$\sum_{l=1}^{m} \widetilde{h}H_{k,l} = \frac{1}{1 - h(k) - g(k)}[\widetilde{h}H_{k,1} + g(k)\widetilde{h}H_{k,0} - \widetilde{h}H_{k,m+1} - g(k)\widetilde{h}H_{k,m}].$$

The results in following theorem can be obtained by Theorem 2.2, Eq.(3) and convenient routine operations.

Theorem 2.8. *For $m \geq 1$, the following identities hold:*
(i) $\sum\limits_{i=1}^{m} \widetilde{h}H_{k,2i} = \frac{\widetilde{h}H_{k,2} - g^2(k)\widetilde{h}H_{k,0} - \widetilde{h}H_{k,2m+2} + g^2(k)\widetilde{h}H_{k,2m}}{1 - h^2(k) - 2g(k) + g^2(k)}$,
(ii) $\sum\limits_{i=1}^{m} \widetilde{h}H_{k,2i+1} = \frac{\widetilde{h}H_{k,3} - g^2(k)\widetilde{h}H_{k,1} - \widetilde{h}H_{k,2m+3} + g^2(k)\widetilde{h}H_{k,2m+1}}{1 - h^2(k) - 2g(k) + g^2(k)}$.

Theorem 2.9. *The exponential generating function for the $k-$ Horadam hybrid number $\widetilde{h}H_{k,l}$ is*

$$\sum_{l=0}^{\infty} \widetilde{h}H_{k,l}\frac{t^l}{l!} = \frac{N\widehat{\lambda}e^{\lambda t} - K\widehat{\gamma}e^{\gamma t}}{\lambda - \gamma}.$$

Proof. By considering Theorem (2.2) we have

$$\begin{aligned}\sum_{l=0}^{\infty} \widetilde{h}H_{k,l}\frac{t^l}{l!} &= \frac{N\widehat{\lambda}}{\lambda - \gamma}\sum_{l=0}^{\infty}\frac{\lambda^l t^l}{l!} - \frac{K\widehat{\gamma}}{\lambda - \gamma}\sum_{l=0}^{\infty}\frac{\gamma^l t^l}{l!} \\ &= \frac{N\widehat{\lambda}e^{\lambda t} - K\widehat{\gamma}e^{\gamma t}}{\lambda - \gamma}.\end{aligned}$$

Corollary 2.10. *The Poisson generating functions for the $k-$ Horadam hybrid numbers are*

$$\sum_{i=0}^{\infty} \frac{\widetilde{h}H_i t^i e^{-t}}{i!} = \frac{N\hat{\lambda}e^{\lambda t} - K\hat{\gamma}e^{\gamma t}}{e^t(\lambda - \gamma)}.$$

Proof. By Theorem 2.2, we have

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{\widetilde{h}H_i t^i e^{-t}}{i!} &= \sum_{i=0}^{\infty}(\frac{N\lambda^i\hat{\lambda}-K\gamma^i\hat{\gamma}}{\lambda-\gamma})\frac{t^i e^{-t}}{i!}\\ &= \frac{N\hat{\lambda}}{\lambda-\gamma}e^{-t}\sum_{i=0}^{\infty}\frac{\lambda^i t^i}{i!}-\frac{K\hat{\gamma}}{\lambda-\gamma}e^{-t}\sum_{i=0}^{\infty}\frac{\gamma^i t^i}{i!}\\ &= \frac{N\hat{\lambda}e^{\lambda t}-K\hat{\gamma}e^{\gamma t}}{e^t(\lambda-\gamma)}.\end{aligned}$$

Theorem 2.11. *For $q > p \geq 0$, we have*

$$\sum_{l=0}^{m}\widetilde{h}H_{k,pl+q} = \frac{(-g(k))^p[\widetilde{h}H_{k,mp+q}-\widetilde{h}H_{k,q-p}]}{(-g(k))^p-\lambda^p-\gamma^p+1}+\frac{-\widetilde{h}H_{k,mp+p+q}+\widetilde{h}H_{k,q}}{(-g(k))^p-\lambda^p-\gamma^p+1}.$$

Proof. From Theorem 2.2, we have

$$\begin{aligned}&\sum_{l=0}^{m}\widetilde{h}H_{k,pl+q} = \widetilde{h}H_{k,q}+\widetilde{h}H_{k,q+p}+...+\widetilde{h}H_{k,q+mp}\\ &= \frac{N\lambda^q\widehat{\lambda}-K\gamma^q\widehat{\gamma}}{\lambda-\gamma}+\frac{N\lambda^{p+q}\widehat{\lambda}-K\gamma^{p+q}\widehat{\gamma}}{\lambda-\gamma}+...+\frac{N\lambda^{mp+q}\widehat{\lambda}-K\gamma^{mp+q}\widehat{\gamma}}{\lambda-\gamma}\\ &= \frac{1}{\lambda-\gamma}\left\{\frac{N\widehat{\lambda}\lambda^q[\lambda^{mp+p}\gamma^p-\lambda^{mp+p}-\gamma^p+1]}{\lambda^p\gamma^p-\lambda^p-\gamma^p+1}-\frac{K\widehat{\gamma}\gamma^q[\gamma^{mp+p}\lambda^p-\gamma^{mp+p}-\lambda^p+1]}{\lambda^p\gamma^p-\lambda^p-\gamma^p+1}\right\}\\ &= \frac{1}{(-g(k))^p-\lambda^p-\gamma^p+1}\left\{[\frac{N\widehat{\lambda}\lambda^{mp+p+q}\gamma^p-K\widehat{\gamma}\gamma^{mp+p+q}\lambda^p}{\lambda-\gamma}]-[\frac{N\widehat{\lambda}\lambda^{mp+p+q}-K\widehat{\gamma}\gamma^{mp+p+q}}{\lambda-\gamma}]\right.\\ &\left.-[\frac{N\widehat{\lambda}\lambda^q\gamma^p-K\widehat{\gamma}\gamma^q\lambda^p}{\lambda-\gamma}]+[\frac{N\widehat{\lambda}\lambda^q-K\widehat{\gamma}\gamma^q}{\lambda-\gamma}]\right\}\\ &= \frac{(-g(k))^p[\widetilde{h}H_{k,mp+q}-\widetilde{h}H_{k,q-p}]}{(-g(k))^p-\lambda^p-\gamma^p+1}+\frac{-\widetilde{h}H_{k,mp+p+q}+\widetilde{h}H_{k,q}}{(-g(k))^p-\lambda^p-\gamma^p+1}.\end{aligned}$$

Theorem 2.12. *(Catalan Identity) For $m, r \in \mathbb{Z}^+$ such that $m \geq r$. Then*

$$\widetilde{h}H_{k,m+r}\widetilde{h}H_{k,m-r}-(\widetilde{h}H_{k,m})^2 = \frac{NK(-g(k))^m}{h^2(k)+4g(k)}\left\{\hat{\lambda}\hat{\gamma}[1-\frac{\lambda^r}{\gamma^r}]+\hat{\gamma}\hat{\lambda}[1-\frac{\gamma^r}{\lambda^r}]\right\}. \tag{15}$$

Proof. By using Theorem 2.2, we obtain;

$$\begin{aligned}&\widetilde{h}H_{k,m+r}\widetilde{h}H_{k,m-r}-(\widetilde{h}H_{k,m})^2\\ &= [\frac{N\lambda^{m+r}\hat{\lambda}-K\gamma^{m+r}\hat{\gamma}}{\lambda-\gamma}][\frac{N\lambda^{m-r}\hat{\lambda}-K\gamma^{m-r}\hat{\gamma}}{\lambda-\gamma}]-[\frac{N\lambda^m\hat{\lambda}-K\gamma^m\hat{\gamma}}{\lambda-\gamma}]^2\\ &= \frac{NK}{(\lambda-\gamma)^2}[\widehat{\lambda}\widehat{\gamma}(\lambda^m\gamma^m-\lambda^{r+m}\gamma^{m-r})+\hat{\gamma}\hat{\lambda}(\lambda^m\gamma^m-\lambda^{m-r}\gamma^{m+r})].\end{aligned}$$

After some elementary calculations, we obtain

$$\widetilde{h}H_{k,m+r}\widetilde{h}H_{k,m-r}-(\widetilde{h}H_{k,m})^2 = \frac{NK(-g(k))^m}{h^2(k)+4g(k)}\left\{\hat{\lambda}\hat{\gamma}[1-\frac{\lambda^r}{\gamma^r}]+\hat{\gamma}\hat{\lambda}[1-\frac{\gamma^r}{\lambda^r}]\right\}.$$

For $r=1$ in the Catalan identity, we get the Cassini identity for $k-$ Horadam hybrid numbers in the next Theorem.

Theorem 2.13. *(Cassini Identity) Let $m \geq 1$ be an integer. Then*

$$\widetilde{h}H_{k,m+1}\widetilde{h}H_{k,m-1}-(\widetilde{h}H_{k,m})^2=\frac{NK(-g(k))^m}{h^2(k)+4g(k)}\Big\{\hat{\lambda}\hat{\gamma}[1-\frac{\lambda}{\gamma}]+\hat{\gamma}\hat{\lambda}[1-\frac{\gamma}{\lambda}]\Big\}.$$

Theorem 2.14. *(d'Ocagne Identity) For $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}$ such that $m > n+1$. Then*

$$\widetilde{h}H_{k,m}\widetilde{h}H_{k,n+1}-\widetilde{h}H_{k,m+1}\widetilde{h}H_{k,n}=\frac{NK(-g(k))^n}{\sqrt{h^2(k)+4g(k)}}[\lambda^{m-n}\hat{\lambda}\hat{\gamma}-\gamma^{m-n}\hat{\gamma}\hat{\lambda}].$$

Proof. From Theorem 2.2 and Eq.(3), we have

$$\begin{aligned}
&\widetilde{h}H_{k,m}\widetilde{h}H_{k,n+1}-\widetilde{h}H_{k,m+1}\widetilde{h}H_{k,n}\\
&=[\frac{N\lambda^m\hat{\lambda}-K\gamma^m\hat{\gamma}}{\lambda-\gamma}][\frac{N\lambda^{n+1}\hat{\lambda}-K\gamma^{n+1}\hat{\gamma}}{\lambda-\gamma}]-[\frac{N\lambda^{m+1}\hat{\lambda}-K\gamma^{m+1}\hat{\gamma}}{\lambda-\gamma}][\frac{N\lambda^n\hat{\lambda}-K\gamma^n\hat{\gamma}}{\lambda-\gamma}]\\
&=\frac{NK}{(\lambda-\gamma)^2}\{\hat{\lambda}\hat{\gamma}[-\lambda^m\gamma^{n+1}+\lambda^{m+1}\gamma^n]+\hat{\gamma}\hat{\lambda}[\lambda^n\gamma^{m+1}-\lambda^{n+1}\gamma^m]\}\\
&=\frac{NK(-g(k))^n}{\sqrt{h^2(k)+4g(k)}}[\lambda^{m-n}\hat{\lambda}\hat{\gamma}-\gamma^{m-n}\hat{\gamma}\hat{\lambda}].
\end{aligned}$$

3. Matrix representations of k-Horadam hybrid numbers

Now, we will give the matrix representation of $k-$ Horadam hybrid numbers. Also we obtain a formula for $k-$ Horadam hybrid numbers $\widetilde{h}H_{k,m}$, in terms of tridiagonal determinant, by using the same kind of approach that was used in (Catarino, 2016; Kizilates et al., 2019).

Theorem 3.1. *Let $u \geq 0$ be an integer. Then*

$$\begin{bmatrix}\widetilde{h}H_{k,u+2} & \widetilde{h}H_{k,u+1}\\ \widetilde{h}H_{k,u+1} & \widetilde{h}H_{k,u}\end{bmatrix}=\begin{bmatrix}\widetilde{h}H_{k,2} & \widetilde{h}H_{k,1}\\ \widetilde{h}H_{k,1} & \widetilde{h}H_{k,0}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}^u \tag{16}$$

Proof. We use the induction method to prove this theorem. If $u=0$, then the result is obvious. Assume the expression is satisfied for $u \geq 0$

$$\begin{bmatrix}\widetilde{h}H_{k,u+2} & \widetilde{h}H_{k,u+1}\\ \widetilde{h}H_{k,u+1} & \widetilde{h}H_{k,u}\end{bmatrix}=\begin{bmatrix}\widetilde{h}H_{k,2} & \widetilde{h}H_{k,1}\\ \widetilde{h}H_{k,1} & \widetilde{h}H_{k,0}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}^u.$$

We next prove that

$$\begin{bmatrix}\widetilde{h}H_{k,u+3} & \widetilde{h}H_{k,u+2}\\ \widetilde{h}H_{k,u+2} & \widetilde{h}H_{k,u+1}\end{bmatrix}=\begin{bmatrix}\widetilde{h}H_{k,2} & \widetilde{h}H_{k,1}\\ \widetilde{h}H_{k,1} & \widetilde{h}H_{k,0}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}^{u+1} \tag{17}$$

We consider the right-hand side of (17)

$$\begin{bmatrix}\widetilde{h}H_{k,2} & \widetilde{h}H_{k,1}\\ \widetilde{h}H_{k,1} & \widetilde{h}H_{k,0}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}^{u+1}=\left\{\begin{bmatrix}\widetilde{h}H_{k,2} & \widetilde{h}H_{k,1}\\ \widetilde{h}H_{k,1} & \widetilde{h}H_{k,0}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}^u\right\}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}$$

From Eq.(9) and induction hypothesis, we have

$$\left\{\begin{bmatrix}\widetilde{h}H_{k,2} & \widetilde{h}H_{k,1}\\ \widetilde{h}H_{k,1} & \widetilde{h}H_{k,0}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}^{u}\right\}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}$$
$$=\begin{bmatrix}\widetilde{h}H_{k,u+2} & \widetilde{h}H_{k,u+1}\\ \widetilde{h}H_{k,u+1} & \widetilde{h}H_{k,u}\end{bmatrix}\times\begin{bmatrix}h(k) & 1\\ g(k) & 0\end{bmatrix}$$
$$=\begin{bmatrix}\widetilde{h}H_{k,u+3} & \widetilde{h}H_{k,u+2}\\ \widetilde{h}H_{k,u+2} & \widetilde{h}H_{k,u+1}\end{bmatrix}$$

Thus the proof is completed.

Using Theorem 2.1 and Eq. (16), the following remarks can be given;

Remark 3.2. *For $h(k) = k$, $g(k) = 1$, $w = 0$ and $z = 1$ in Eq.(16) and Theorem 2.1 , we have the matrix representation of $k-$ Fibonacci hybrid numbers as follows;*

$$\begin{bmatrix}\widetilde{h}FH_{k,u+2} & \widetilde{h}FH_{k,u+1}\\ \widetilde{h}FH_{k,u+1} & \widetilde{h}FH_{k,u}\end{bmatrix}=\begin{bmatrix}\widetilde{h}FH_{k,2} & \widetilde{h}FH_{k,1}\\ \widetilde{h}FH_{k,1} & \widetilde{h}FH_{k,0}\end{bmatrix}\times\begin{bmatrix}k & 1\\ 1 & 0\end{bmatrix}^{u}$$

Remark 3.3. *For $h(k) = k$, $g(k) = 1$, $w = 2$ and $z = k$ in Eq.(16) and Theorem 2.1 , we have the matrix representation of $k-$ Lucas hybrid numbers as follows;*

$$\begin{bmatrix}\widetilde{h}LH_{k,u+2} & \widetilde{h}LH_{k,u+1}\\ \widetilde{h}LH_{k,u+1} & \widetilde{h}LH_{k,u}\end{bmatrix}=\begin{bmatrix}\widetilde{h}LH_{k,2} & \widetilde{h}LH_{k,1}\\ \widetilde{h}LH_{k,1} & \widetilde{h}LH_{k,0}\end{bmatrix}\times\begin{bmatrix}k & 1\\ 1 & 0\end{bmatrix}^{u}$$

The nth terms of $k-$ Horadam hybrid number can be easily obtained by calculating the determinant of the tridiagonal matrix A_{n-1}.

Using Eq.(9) the following propositions can be easily proved.

Proposition 3.4. *The $n \times n$ tridiagonal matrices*

$$A_n=\begin{pmatrix}a & b & & & & & \\ c & d & 1 & & & & \\ & c & d & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & c & d & 1\\ & & & & & c & d\end{pmatrix}$$

satisfy $detA_n = \widetilde{h}H_{n+1}$, where $a = \widetilde{h}H_{k,2}, b = \widetilde{h}H_{k,1}, c = -g(k)$ and $d = h(k)$.

$k-$ Horadam hybrid number can be obtained using another tridiagonal matrix.

Proposition 3.5. *For $m \geq 1$, we have*

$$\widetilde{h}H_{k,m}=\begin{vmatrix}b & a & 0 & 0 & \dots & 0 & 0\\ -1 & 0 & c & 0 & \dots & 0 & 0\\ 0 & -1 & d & c & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & d & c\\ 0 & 0 & 0 & 0 & \dots & -1 & d\end{vmatrix}_{m\times m}$$

where $a = \widetilde{h}H_{k,2}, b = \widetilde{h}H_{k,1}, c = g(k)$ and $d = h(k)$.

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