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On conjugacy classes of the homomorphic images of a certain Bianchi group

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Abstract

In this paper, we classify the conjugacy classes of the action of $PSL_2(O_2)$ on the projective line over finite fields, $PL(\mathrm{F}_p)$ where p is the $M-S$ prime, by using the method of parameterization and investigate the behavior of coset diagrams of these actions. We prove that the action of $PSL_2(O_2)$ on $PL(\mathrm{F}_p)$ is transitive for all conjugacy classes except for the conjugacy class in which 2 is a perfect square in F_p. We also prove that the homomorphic images of $PSL_2(O_2)$ represented by these coset diagrams are isomorphic to the rank one Chevalley groups, $\mathrm{L}_2(p)$ for all $p \geq 11$. We also study the behavior of the coset diagram of the homomorphic images of $PSL_2(O_2)$ for the conjugacy class in which 2 is a perfact square in F_p and prove that these coset diagrams admit symmetry about the vertical line of axis in two dimensional space. We also prove that these coset diagrams depict intransitive action of $PSL_2(O_2)$ on $PL(\mathrm{F}_p)$ in this case. This algebraic fact leads us to develop a formula to count the number of orbits occurring in each coset diagram of this particular class.

Key words: Conjugacy class; coset diagrams; finite simple groups; parameterization; the group $PSL_2(O_2)$.

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1. Introduction

Let d be a positive square free integer. Then O_d be the ring of algebraic integers over the imaginary quadratic number field Q($\sqrt{(-d)}$). A Bianchi group denoted by $PSL_2(O_d)$ (or Γ_d) is defined as

$$PSL_2(O_d) = \left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} : w, x, y, z \in O_d, \atop \mathrm{wz} - \mathrm{xy} = 1 \right\}.$$

The study of this class of groups was initiated in the 1890's by Bianchi as a natural extension of the study of the Modular group. Bianchi was able to find generators for many members of this class.

He proved that each Γ_d acts discontinuously on hyperbolic 3-space H^3, Fine (1989). He further developed a technique for determining fundamental domains for Γ_d in H^3 which allows one to compute presentations for Γ_d. Bianchi groups, as is well-known, are the discrete groups and have applications in hyperbolic geometry, topology and number theory. The Bianchi groups are classified into three classes $\{\Gamma_1\}$, $\{\Gamma_3\}$ and $\{\Gamma_2, \Gamma_7, \Gamma_{11}\}$ based on their relative amalgam structures.

In particular the groups $PSL_2(O_2)$, $PSL_2(O_7)$ and $PSL_2(O_{11})$ can be decomposed as free product with amalgamation. Moreover, these groups can also be studied as HNN groups. The study of these groups play an important role in the fields of hyperbolic geometry, number theory and automorphic function theory. A finite presentation of the group $PSL_2(O_2)$ is given by $PSL_2(O_2) = \langle a, t, u : a^2 = (at)^3 = (u^{-1}aua)^2 = [t,u] = 1\rangle$ where $a : z \to (-1)/z$, $t : z \to z+1$ and $u : z \to z+\sqrt{-2}$ are the linear fractional transformations. The matrix representation corresponding to each respective linear fractional transformation is given as

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{bmatrix}.$$

By taking $s = at$, $m = u^{-1}au$, $v = u^{-1}su$ and the application of Tietz transformations to the above stated presentation of $PSL_2(O_2)$ yield the following new presentation as

$$\left\langle \begin{array}{c} a,s,m,v,u:\ a^2 = s^3 = m^2 = v^3 \\ = (am)^2 = (s\,v^{-1})^2 = 1, m = \\ u^{-1}au,\ v = u^{-1}su,\ \ am = s\,v^{-1} \end{array} \right\rangle$$

Where $a: z \to \frac{-1}{z}$, $s: z \to \frac{-1}{z+1}$,

$m: z \to \frac{-\sqrt{-2}\,z+1}{2z+2\sqrt{-2}}$,

$v: z \to \frac{-\sqrt{-2}\,z+(1-\sqrt{-2})}{z+(1+\sqrt{-2})}$ and

$u: z \to z+\sqrt{-2}$ are the respective linear fractional transformations.

It is well known that $PSL_2(O_2)$ can be decomposed as a free product of $G_1$and G_2 with amalgamated subgroup H written as $\Gamma_2 = G_1 *_H G_2,$ where G_1 and G_2 are HNN groups of Klein-4 group D_2 and the alternating group A_4 and $\Gamma_2 = Z * Z_2$. To discuss more on the amalgam structure of Bianchi groups and HNN extensions, we refer to Sengun (2011) and Wilson (1998).

In the subsequent section, we study the coset diagram of the action of $PSL_2(O_2)$ on $PL(F_p)$ and the method of parameterization. We can find a Conjugacy class corresponding to each perfect square in F_p by using this method. The Conjugacy classes of these actions are represented graphically by coset diagrams. Since these conjugacy classes are represented by coset diagrams, we can establish a correspondence between the elements θ of finite field F_p and these coset diagrams. In section three, we classify these conjugacy classes and investigate the behavior of the coset diagram for each conjugacy class. In section four, We prove that the action of $PSL_2(O_2)$ on $PL(F_p)$ is transitive for all conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$.

We also prove that the permutation subgroup of $PSL_2(O_2)$ represented by these coset diagrams are isomorphic to the rank one Chevalley groups, $L_2(p)$, for all $p \geq 11$. In the last section of this paper, we investigate the behavior of the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ for the conjugacy class corresponding to the element $\theta = 2$ and prove that these coset diagrams admit symmetry about the vertical line of axis in two dimensional space. We also prove that these coset diagrams depict intransitive action of $PSL_2(O_2)$ on $PL(F_p)$. This algebraic fact leads us to establish a formula to count the number of orbits occurring in each coset diagram of the action of $PSL_2(O_2)$ on $PL(F_p)$ for this particular conjugacy class.

2. Coset diagram of the action of $PSL_2(O_2)$ on $PL(F_p)$

Every odd prime of the sequence in which -2 is a perfect square modulo p can be expressed as either $4n+1$ if n is even or $4n-1$ if n is odd. Such primes along with the solo even prime are called the M-S primes. The group $PSL_2(O_2)$ acts on projective line over the finite field, $PL(F_p)$ only if -2 is a perfect square in F_p.

We use special graphs to investigate behavior of the group $PSL_2(O_2)$ in this paper. Let G be a permutation group defined on a set X, and that G be generated by the elements $x_1, x_2, x_3, \dots, x_k$. Then the elements of X may be represented by the vertices of a diagram, with the directed edge of colour i diagram from vertex u to vertex v whenever $ux_i = v$. The diagram is connected if for any two vertices in X can be joined by a word w , where w is the product of generators and power of the generators. In this case, the vertices are right cosets of the stabilizer H of any element of X in G, so that the colour i joins the coset Hg to the coset Hgx_i, for each g in G.

The idea of the coset diagrams for the modular group was propounded by Professor Graham Higman about 40 years ago. Later, Mushtaq laid the foundation of these diagrams. To see more on coset diagrams we refer to Ashiq (2006), Everitt (1997), Higman *et al.* (1983), Mushtaq (1992) and Torstensson (2010).

The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$, where p is a $M-S$ prime, are made of four generators $\bar{a}$, $\bar{s}$, $\bar{m}$, $\bar{v}$. We denote these generators graphically as follows. The three cycles of the permutations $\bar{s}$ is represented by triangles having solid lines whereas $\bar{v}$ is represented by triangles having edges consisting of bold solid lines. The involution $\bar{a}$ is denoted by broken edges and $\bar{m}$ is denoted by dotted edges. Fixed points are represented by heavy dots if they exist. Each diagram represents finite, non-Abelian and simple subgroups of A_{p+1}, for all $p \geq 11$. Where as these coset diagrams represent the permutation subgroups isomorphic to symmetric group of degree three and the alternating group of degree four for $p = 2$ and $p = 3$, respectively.

There are two connectors namely C_1 and C_2 of the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$. The connectors C_1 and C_2 graphically represent the behavior of linear fractional transformations $a(z)$ and $m(z)$, respectively in these coset diagrams. These connectors join each vertex of a fragment to the other vertex in a unique way. Initially, the coset diagram contains different orbits. Each orbit represents the alternating group A_4. When we start joining these orbits through the connectors C_1 and C_2, the diagram starts becoming connected. Once all these orbits are completely joined by C_1 and C_2, we obtain connected coset diagrams. This connection is in fact due to the amalgam structure of $PSL_2(O_2)$ in which different fragments of A_4 and D_2 are joined together. To see more on understanding for the study of these particular coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ and finite simple groups, we refer to Moghaddamfar (2008), Mushtaq *et al.* (2013) and Shen *et al.* (2016).

For instance, consider the action of $PSL_2(O_2)$ on $PL(F_{19})$ which is depicted by the following coset diagram.

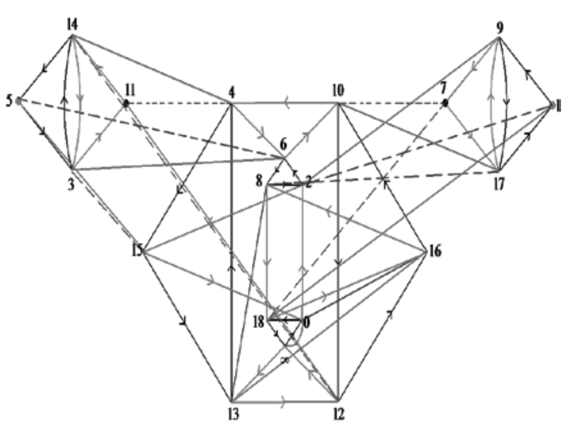

Fig. 1. Coset diagtam of the action of$PSL_2(O_2)$ on $PL(F_{19})$.

The following Remark Mushtaq *et al.* (2013) is about the conditions of existence of the fixed points occurring in the action of $PSL_2(O_2)$ on $PL(F_p)$ for the linear fractional transformations a, m, ,s, v in each $M - S$ prime under the action of $PSL_2(O_2)$ on $PL(F_p)$.

Remark 1. Under the action of $PSL_2(O_2)$ on $PL(F_p)$.

i- Fixed points of transformations a and m exist if -1 is a perfect square modulo p.

ii- Fixed points of transformations s and v exist if -3 is a perfect square modulo p.

A homomorphism $\delta : PSL(2, Z[\sqrt{-2}]) \rightarrow \mathrm{PSL}(2, \mathrm{p})$ which maps $a\delta = \overline{a}$, $s\delta = \overline{s}$, $m\delta = \overline{m}$ and $v\delta = \overline{v}$ in $PSL_2(O_2)$ such that

$$\overline{a}^2 = \overline{m}^2 = \overline{s}^3 = \overline{v}^3 = (\overline{am})^2 = ((\overline{sv})^{-1})^2 = 1$$

defines action of $PSL(2, Z[\sqrt{-2}])$ on $PL(F_p)$. In other words, the action will yield subgroups of the alternating group of degree $P + 1$ for $P \geq 3$. Each action is depicted by a coset diagram. The parameter for δ or of the conjugacy class containing δ, is the parameter of $\overline{as}$.

In the following result, we parameterize the actions of $PSL_2(O_2)$ on $PL(F_p)$ where p is the $\mathrm{M - S}$ prime, that is, we establish a link between the elements $\theta \in F_p$ and a conjugacy class of linear fractional transformations a, m, s and v such that

$$a^2 = m^2 = s^3 = v^3 = (am)^2 = ((sv)^{-1})^2 = 1.$$

That is, corresponding to each perfect square in F_p, there is a conjugacy class of such actions defined by the non-degenerate homomorphism δ.

Theorem 1. Corresponding to each perfect square θ in F_p, where p is the $\mathrm{M - S}$ prime, we can find a conjugacy class of the action of $PSL_2(O_2)$ on $PL(F_p)$ represented by the coset diagram $D(\theta, p)$.

Proof. If the mapping $\mathrm{PSL}(2, \mathrm{p}) \rightarrow \mathrm{GL}(2, \mathrm{p})$ maps an element g of $\mathrm{PSL}(2, \mathrm{p})$ to a matrix N of $\mathrm{GL}(2, \mathrm{p})$ (where p is the $M - S$prime) then $\theta = \frac{(\mathrm{trace\ N})^2}{\det(N)}$ is the invariant of the conjugacy class of g. We refer θ as a parameter of g or of the conjugacy class containing g.

Let $A = \begin{bmatrix} b & c \\ d & t \end{bmatrix}$, $S = \begin{bmatrix} i & j \\ k & l \end{bmatrix}$,

$M = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, $V = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be the elements of $\mathrm{GL}(2, \mathrm{p})$ which yield the elements $\overline{a}$, $\overline{s}$, $\overline{m}$ and $\overline{v}$ of $\mathrm{PSL}(2, \mathrm{p})$.

Since $a^2 = 1$, $m^2 = 1$, $s^3 = 1$, and $v^3 = 1$, therefore, A^2, M^2, S^3 and V^3 are scaler matrices and hence the determinants of these matrices are square in F_p. Thus replacing these matrices by suitable scalar matrices, we assume that the determinants of these matrices are equal to one.

Since $a^2 = 1$ implies that $Tr(A) = 0$ implies that $t = -b$. Since $\det(A) = 1$, therefore,

$$b^2 + cd = -1 \ (1)$$

This means that the matrix A becomes

$$A = \begin{bmatrix} b & c \\ d & -b \end{bmatrix}.$$

Since $m^2 = 1$ implies that $Tr\,(M) = 0$ implies that $h = -e$. Since $\det(M) = 1$, therefore

$$e^2 + fg = -1 \ (2)$$

This means that the matrix M becomes

$$M = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}.$$

Now as $s^3 = 1$ implies that

$$[Tr(S)]^2 = \det(S) \qquad \text{(i)}$$

Suppose that $Tr(S) = -1$, then $l = i - 1$.

Since $\det(S) = 1$, therefore, $il - jk - 1 = 0$ Substituting the values of $Tr(S)$ and $\det(S)$ in (i), and after simplification, we obtain

$$i^2 + i + jk + 1 = 0 \qquad (3)$$

Where $S = \begin{bmatrix} i & j \\ h & -i-1 \end{bmatrix}$.

Now as $v^3 = 1$ implies that

$$[Tr(V)]^2 = \det(V) \qquad \text{(ii)}$$

Suppose that $Tr(V) = -1$, then

$z = -w - 1$. Since $\det(V) = 1$, therefore, $wz - xy - 1 = 0$. Substituting the values of $Tr(V)$ and $\det(V)$ in (ii), and after simplification, we obtain

$$w^2 + w + xy + 1 = 0 \qquad (4)$$

Where $V = \begin{bmatrix} w & x \\ y & -w-1 \end{bmatrix}$. Consider

$$AM = \begin{bmatrix} b & c \\ d & -b \end{bmatrix}\begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \begin{bmatrix} be+cg & bf-ce \\ de-tg & df+be \end{bmatrix}.$$

Since $(am)^2 = 1$ implies that

$Tr(AM) = 0$ implies that $be + cg + df + be = 0$ implies that

$$2be + cg + df = 0 \ (5)$$

Consider

$$SV^{-1} = \begin{bmatrix} i & j \\ k & -i-1 \end{bmatrix}\begin{bmatrix} -w-1 & -x \\ -y & w \end{bmatrix} = \begin{bmatrix} -i(w+1) & -ix+jw \\ -k(w+1)-yj & -w(i+1)-xk \end{bmatrix}.$$

Since $(SV^{-1})^2 = 1$ implies that $Tr(SV^{-1}) = 0$, therefore $-\mathrm{i}(\mathrm{w}+1) - \mathrm{yj} - \mathrm{w}(\mathrm{i}+1) - \mathrm{xk} = 0$ implies that

$$2iw + yj + xk + i + w = 0 \quad (6)$$

Let r be the trace and δ be the determinant of AS. Consider

$$AM = \begin{bmatrix} b & c \\ d & -b \end{bmatrix}\begin{bmatrix} i & j \\ k & -i-1 \end{bmatrix} = \begin{bmatrix} bi+ck & bj-c(i+1) \\ id-bk & dj+b(i+1) \end{bmatrix}.$$

Now $Tr(AS) = bi + ck + dj + b(i+1) = r$ implies that

$$r = b(2i+1) + ck + dj \ (7)$$

Now $\det(AS) = \det(A)\det(S) = 1$ shows that $\Delta = 1$.

Now $\theta = \frac{r^2}{\Delta}$, implies that

$$\theta = r^2 \qquad (8).$$

Thus we can find a conjugacy class corresponding to each element which is a perfect square in F_p.

3. Classification of homomorphic images of $PSL_2(O_2)$

In this section, we classify the actions of $PSL_2(O_2)$ on $PL(F_p)$ where p is the $M-S$ prime by using the method of parameterization and study the behavior of coset diagrams of these actions in each class. We can subdivide the sequence of $M-S$ primes into four subsequences. This subdivision is based on weather -1 and -3 are perfect squares in F_p or not where -1 and -3 are the conditions of existence of the fixed points of the linear fractional transformations a, m, s and v under the action of $PSL_2(O_2)$ on $PL(F_p)$. The subsequences are given as follow;

(i) Neither -1 nor -3 are perfect squares in F_p is $\pi_1 = \{11,\ 59,\ 83, \dots\}$.

(ii) Only -1 is a perfect square in F_p is

$$\pi_2 = \{17, 41, 89, \dots\}.$$

(iii) Only -3 is a perfect square in F_p is

$$\pi_3 = \{19, 43, 67, \dots\}.$$

(iv) Both -1 and -3 are perfect squares in F_p is $\pi_4 = \{73, 97,\ , \dots\}$.

Case I: In this case, we study actions of $PSL_2(O_2)$ on $PL(F_p)$ corresponding to the subsequence π_1 in which neither -1 nor -3 are perfect squares in F_p. The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ yield Chevalley groups of rank one, $L_2(p)$, for the elements θ in F_p corresponding to each prime p in the subsequence π_1. Each diagram is connected. This shows that the action is transitive. Moreover, each diagram consists of $\frac{P+1}{12}$ number of components which are joint together by the transformations a and m. These transformations are graphically represented by orange solid and purple doted edges in each diagram, respectively. If we remove these edges, we obtain the disconnected diagrams in which each of the fragment represents alternating group of degree four.

Case II: In this case, we study actions of $PSL_2(O_2)$ on $PL(F_p)$ corresponding to the subsequence π_2 in which only -1 is a perfectsquare in F_p. The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ yield Chevalley groups of rank one, $L_2(p)$ for the elements θ in F_p corresponding to each prime p in the subsequence π_2. Each diagram is connected. This shows that the action is transitive. Moreover, each diagram consists of $\frac{P-5}{12} + 1$ number of components which are joined together by the transformations a and m. These transformations are graphically represented by orange solid and purple doted edges in each diagram, respectively. If we remove these edges, we obtain the disconnected diagrams in which each of the component represents alternating group of degree four.

Case III: In this case, we study actions of $PSL_2(O_2)$ on $PL(F_p)$ corresponding to the subsequence π_3 in which only -3 is a perfect square in F_p. The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ yield Chevalley groups of rank one, $L_2(p)$ for the elements θ in F_p corresponding to each prime p in the subsequence π_3. Each diagram is connected. This shows that the action is transitive. Moreover, each diagram consists of $\frac{P+5}{12} + 1$ number of fragments which are joined together by the transformations a and m. These transformations are graphically represented by orange solid and purple doted edges in each diagram, respectively. If we remove these edges, we obtain the disconnected diagrams in which each of the fragment represents alternating group of degree four.

Case IV: In this case, we study actions of $PSL_2(O_2)$ on $PL(F_p)$ corresponding to the subsequence π_4 in which both -1 and -3 are perfect squares in F_p. The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ yield Chevalley groups of rank one, $L_2(p)$ for the elements θ in F_p corresponding to each prime p in the subsequence π_4. Each diagram is connected. This shows that the action is transitive. Moreover, each diagram consists of $\frac{P-1}{12} + 2$ number of fragments which are joined together by the transformations a and m. These transformations are graphically represented by orange solid and purple doted edges in each diagram, respectively. If we remove these edges, we obtain the disconnected diagrams in which each of the component represents alternating group of degree four.

4. The rank-one lie type and simple permutation subgroup of $PSL_2(O_2)$

In this section, our focus is to study algebraic characteristics of the homomorphic images of $PSL_2(O_2)$. We study the conjugacy classes of the action of $PSL_2(O_2)$ on $PL(F_p)$ and draw the coset diagrams for these conjugacy classes in the previous section. These connected diagrams provide information that the action of $PSL_2(O_2)$ on $PL(F_p)$ is transitive except for the conjugacy class $\theta = 2$. We prove that these homomorphic images are isomorphic to the rank one Chevalley groups, $L_2(p)$ for all $p \geq 11$ except for the conjugacy class $\theta = 2$.

The Chevalley groups contains four families of linear simple groups.

a. The projective special linear groups, PSL(n, q).
b. The projective special unitary groups, PSL(n, q).
c. The projective special symmetric groups, PSL(n, q).
d. The twisted groups, P ∩ ϵ(n, q).

This family of Chevalley groups is obtained from the special linear groups SL(n + 1, q) and then factoring out by the center. These are all simple linear groups except for $A_1(2)$, and $A_1(3)$. The group $A_1(2)$ is non-simple and is isomorphic to S_3 and $A_1(3)$ is non-simple and is isomorphic to A_4.

There exists blocks of the permutation subgroups of $PSL_2(O_2)$ for all the conjugacy classes of the non-degenerate homomorphisms from $PSL_2(O_2)$ into PSL(2, p). These blocks can graphically be visualized by the orbits of the coset diagrams of these actions. Notice that all these coset diagrams are connected. This shows that the action of $PSL_2(O_2)$ on $PL(F_p)$ is transitive for all the conjugacy classes except for $\theta = 2$. In the following result, we prove the above stated fact.

Theorem 2. $PSL_2(O_2)$ acts transitively on $PL(F_p)$ for all the conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

Proof. Let X be a non-empty subset of $PL(F_p)$. Then the image set X^g for all g in $\overline{\Gamma_2}$, has either non empty intersection with X or it does not coincide with X under the action of $PSL_2(O_2)$ on $PL(F_p)$. This means that $\overline{\Gamma_2}$ does not preserve any non-trivial partition of $PL(F_p)$. We can easily conclude that $\overline{\Gamma_2}$ is a transitive permutation subgroup of $PSL_2(O_2)$ because $\overline{\Gamma_2}$ has only trivial blocks. In other words, either X is the singleton set or the whole set $PL(F_p)$ and we obtain only one orbit of $\overline{\Gamma_2}$ as an image set X^g for all g in $\overline{\Gamma_2}$ of X in both cases. Hence action of $PSL_2(O_2)$ on $PL(F_p)$ is transitive for all the conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

The following remark provides information about the order of these subgroups.

Remark 2. The order of each permutation subgroup resulting from these coset diagrams can be expressed as $\frac{p(p^2-1)}{2}$.

We now show that these permutation subgroups resulting from these coset diagrams are isomorphic to PSL(2, p).

We need the following result to prove the above stated fact.

Theorem 3. Cameron (2000) If G is be a simple group and $|G| = \frac{p(p^2-1)}{2}$, then G is isomorphic to PSL(2, p).

Theorem 4. Every permutation subgroup of the action of $PSL_2(O_2)$ on $PL(F_p)$ is isomorphic to PSL(2, p) , for $p \geq 11$, for all the conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

Proof. We know that the permutation subgroups depicting from the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ are simple for all $p \geq 11$. So by using Theorem 3 and Remark 2, we conclude that every permutation subgroup of the action of $PSL_2(O_2)$ on $PL(F_p)$ is isomorphic to PSL(2, p), for $p \geq 11$ for all the conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

We need the following result to prove the next Theorem.

Proposition1. Carter (1972) $A_1(k)$ is isomorphic to $PSL_2(k)$.

Theorem 5. The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ represent the rank one Chevalley groups $L_2(p)$ for $p \geq 11$, for all the conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

Proof. We know that the permutation groups corresponding to these coset diagrams are isomorphic to PSL(2, p) for all $p \geq 11$ and by using the above Theorem, we note that PSL(2, p) is isomorphic to $L_2(p)$. Thus we conclude that the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ represent the rank one Chevalley groups $L_2(p)$ for $p \geq 11$ for all the conjugacy classes except for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

5. Coset diagram of the action of $PSL_2(O_2)$ on $PL(F_p)$ for the conjugacy class $\theta = 2$

In this section, we consider the action of $PSL_2(O_2)$ on $PL(F_p)$ for the conjugacy class corresponding to the element $\theta = 2$ in F_p. Each such action is depicted by a coset diagram. The subsequence of the sequence of $M - S$ primes in which 2 is a perfect square is given by $\nu = \{2, 17, 41, 73, 89, \ldots\}$. All the primes of the sequence ν are infect the Pythagorean primes except 2. The action of $PSL_2(O_2)$ on $PL(F_2)$ yields the symmetric group of degree 3 whereas we obtain the symmetric group of degree 4 as a homomorphic image of $PSL_2(O_2)$ for all the Pythagorean primes of the sequence ν.

5.1 Action of $PSL_2(O_2)$ on $PL(F_{17})$ for $\theta = 2$. In view of Theorem1. The permutation representation of $PSL_2(O_2)$ on $PL(F_{17})$ for $\theta = 2$ is given by

$\overline{a} = (0,\infty)\,(1,16)\,(2,8)\,(3,11)\,(4)\,(5,16)\,(6,14)\,(7,12)\,(9,15)\,(13)$

$\overline{m} = (0,12)\ (1,10)\ (2,14)\ (3)\ (4,13)$

$(5,16)\ (6,8)\ (7,\infty)\ (9,15)(11)$

$\overline{s} = (0,9,\infty)(1,8,13)(2,14,11)(3,5,16)(4,6,10)(7,15,12)$

$\overline{v} = (0,15,\infty)(1,10,11)(2,16,13)(3,6,8)(4,5,14)(7,9,12).$

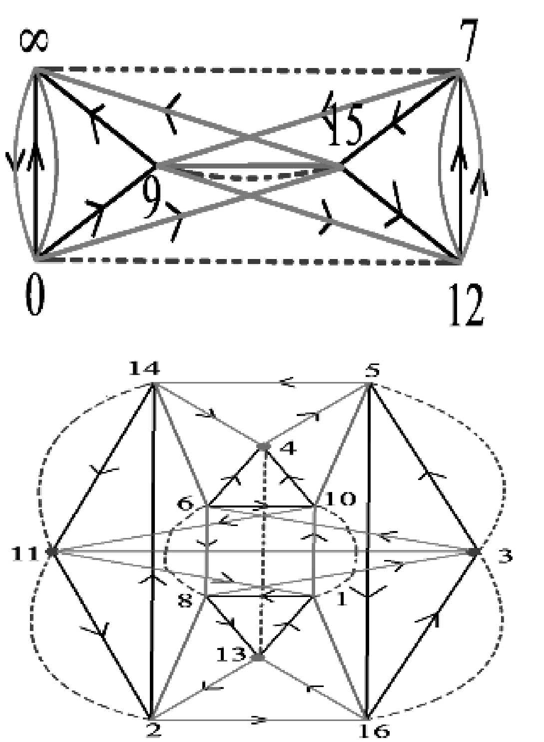

Fig. 2-(b)

Fig. 2. Coset diagram of the action of $PSL_2(O_2)$ on $PL(F_{17})$ for $\theta = 2$.

This coset diagram represents S_4 as a homomorphic image of $PSL_2(O_2)$ on $PL(F_{17})$ for $\theta = 2$. This non-simple group of size 24 has

$$\overline{v}^3 = \overline{s}^3 = m^2 = \overline{a}^2 = (\overline{am})^2 = 1,$$

$$\overline{s} = [\overline{s},\overline{m}],\ \overline{m} = [\overline{a},\overline{s}],\ \overline{ma} = [\overline{m},\overline{s}] = [\overline{a},\overline{v}] = [\overline{m},\overline{v}]$$

as its possible defining relations where $\overline{a}$, $\overline{m}$, $\overline{s}$ and $\overline{v}$ are its generators. This coset diagram admits symmetry about the vertical line of axis. It has two orbits, namely, $\mu_1 = \{0,7,9,12,15,\infty\}$ and $\mu_2 = \{1,2,3,4,5,6,8,10,11,13,14,16\}$. Consequently, this diagram depicts an intransitive action of $PSL_2(O_2)$ on $PL(F_{17})$ for $\theta = 2$.

5.2. Action of $PSL_2(O_2)$ on $PL(F_{41})$ for $\theta = 2$. In view of Theorem 1, the permutation representation of $PSL_2(O_2)$ on $PL(F_{41})$ for $\theta = 2$ is given by

$\overline{a} = (0,\infty)\,(1,40)\,(2,20)\,(3,27)\,(4,10)\,(5,8)\,(6,34)\,(7,35)\,(9)\,(11,26)\,(12,17)\,(13,22)\,(14,38)\,(15,30)\,(16,23)\,(18,25)\,(19,28)\,(21,39)\,(24,29)\,(31,37)\,(33,36)$

$\overline{m} = (0,26)\,(1,7)\,(2)\,(3,6)\,(4,17)\,(5,18)\,(8,25)\,(9,32)\,(10,12)\,(11,\infty)\,(13,31)\,(14,38)\,(15,21)\,(16,19)\,(22,37)\,(23,28)\,(24,33)\,(27,34)\,(29,36)\,(30,39)\,(35,40)$

$\overline{s} = (0,38,\infty)\,(1,8,29)\,(2,31,13)\,(3,19,35)\,(4,28,5)\,(6,40,16)\,(7,36,35)\,(9,30,37)\,(10,34,33)\,(11,14,26)\,(12,24,27)\,(15,20,21)\,(17,18,23)\,(22,39,32)$

$\overline{v} = (0,14,\infty)\,(1,34,23)\,(2,30,39)\,(3,29,17)$

$(4,36,6)\,(5,40,24)\,(7,28,27)\,(8,19,10)\,(9,31,15)\,(11,38,26)\,(12,16,25)\,(13,32,21)\,(18,33,35)\,(21,22,37).$

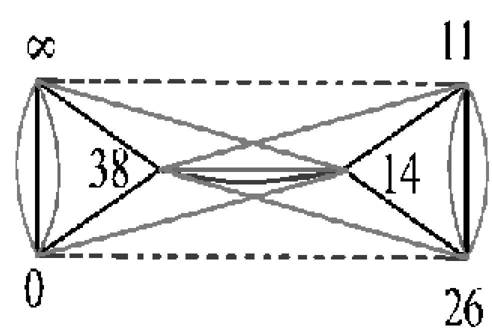

Fig. 3-(a)

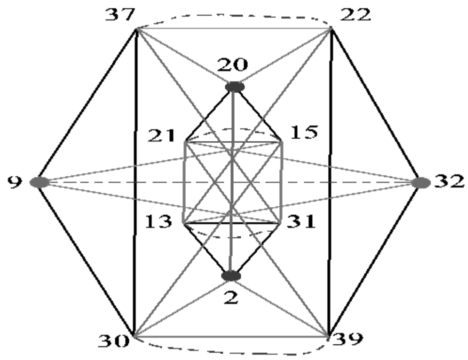

Fig. 3-(b)

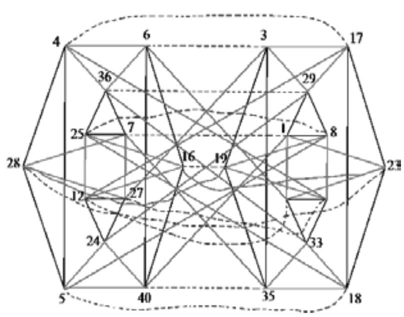

Fig. 3-(c)

Fig. 3. Coset diagram of the action of $PSL_2(O_2)$ on $PL(F_{41})$ for $\theta = 2$.

This coset diagram represents S_4 as a homomorphic image of $PSL_2(O_2)$ on $PL(F_{41})$ for $\theta = 2$. The diagram represents a non-simple group of size 24 which has

$$\overline{v}^3 = \overline{s}^3 = m^2 = \overline{a}^2 = (\overline{am})^2 = 1,\ \overline{s} = [\overline{s}\ \ \overline{v}],\ \overline{m} = [\overline{a}\ \ \overline{s}],$$

$$\overline{a} = [\overline{m}\ \ \overline{v}],\ \overline{ma} = [\overline{m}\ \ \overline{s}]$$

as its possible defining relations where $\overline{a}, \overline{m}, \overline{s}$ and $\overline{v}$ serve as its generators. The coset diagram admits symmetry about the vertical line of axis. It has three orbits, namely,

$\mu_1 = \{14,38\}$, $\mu_2 = \{4,28,6,36,25,7,27,12,24,40,16,$

$19\ ,3,17,35,18,33,8,23\}$ and

$\mu_3 = \{9,30,37,22,20,21,15,13,31,2,39.32,22\}$.

Consequently, this diagram depicts an intransitive action of $PSL_2(O_2)$ on $PL(F_{41})$ for $\theta = 2$.

5.3. Action of $PSL_2(O_2)$ on $PL(F_{73})$ for $\theta = 2$. In view of Theorem 1, The permutation representation of $PSL_2(O_2)$ on $PL(F_{73})$ for $\theta = 2$ is given by

$\overline{a} = (0,\infty)$ (1,72) (2,36) (3,24) (4,18) (5,29) (6,12) (7,52) (8,9) (10,51) (11,53) (13,28) (14,26) (15,34) (16,41) (17,30) (19,23) (20,62) (21,66) (22,63) (25,35) (27) (31,40) (32,57) (33,42) (37,71) (38,48) (39,58) (43,56) (44,68) (45,60) (46) (47,59) (49,70) (50,54) (55,69) (61,67) (64,65)

$\overline{m}$= (0,6) (1,32) (2,34) (3,4) (5,33) (7,56) (8,67) (9,61) (10,49) (11,13) (12,∞) (14,48) (15,36) (16,30) (17,41) (18,24) (19,64) (20,21) (22,63) (23,65) (25,40) (26,38) (27,46) (28,53) (29,42) (31,35) (37,47) (39) (43,52) (44,69) (45,54) (50,60) (51,70) (55,68) (57,72) (58) (59,71) (62,66)

$\overline{s}$= (0,63,∞) (1,44,36) (2,16,13) (3,26,54) (4,45,38) (5,42,43) (6,12,22)(7) (8,35,14) (9,37,60) (10,68,53) (11,30,34) (15,69,32) (17,72,70) (18,59,25) (19,62,27) (20,21,58) (23,39,65) (24,40,71) (28,55,49) (29,33,52) (31,67,48) (41,51,57) (46,66,64) (47,61,50) (56)

$\overline{v}$=(0,22,∞) (1,30,49) (2,68,72) (3,37,31) (4,33,47) (5,7,42) (6,12,63) (8,45,71) (9,26,25) (10,16,32) (11,44,51) (13,70,69) (14,24,50) (15,17,53) (18,48,60) (19,64,58) (20,23,27) (21,46,65) (28,41,36) (29,56,33) (34,57,55) (38,61,40) (39,66,62) (43) (52) (54,67,59).

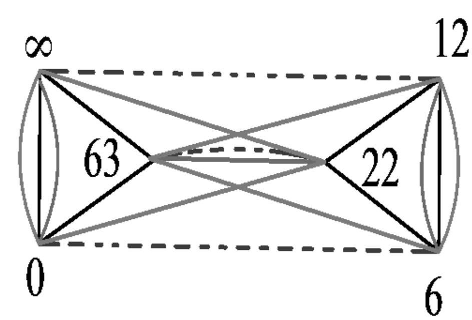

Fig. 4-(a)

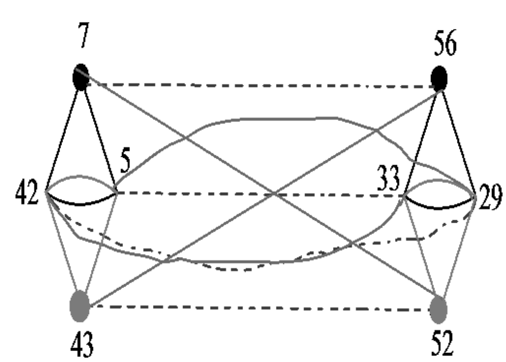

Fig. 4-(b)

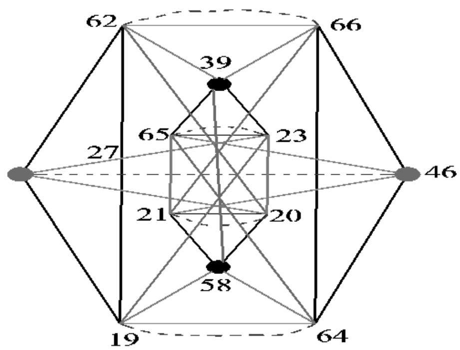

Fig. 4-(c)

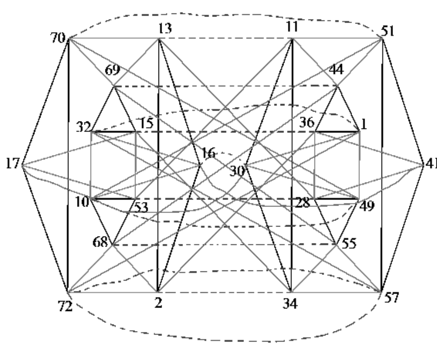

Fig. 4-(d)

Fig. 4. Coset diagram of the action of $PSL_2(O_2)$ on $PL(F_{73})$ for $\theta = 2$.

This coset diagram represents S_4 as a homomorphic image of $PSL_2(O_2)$ on $PL(F_{73})$ for $\theta = 2$. This non-simple group of size 24 has

$$\overline{v}^3 = \overline{s}^3 = m^2 = \overline{a}^2 = (\overline{am})^2 =$$

$$(\overline{av})^2 = 1, \overline{s} = [\overline{s} \;\; \overline{v}] \;, m = [\overline{a} \;\; \overline{s}]$$

$$, \overline{a} = [\overline{m} \;\; \overline{v}], \overline{ma} = [\overline{m} \;\; \overline{s}]$$

as its possible defining relations where $\overline{a}$, $\overline{m}$, $\overline{s}$ and $\overline{v}$ serve as its generators. This coset diagram admits symmetry about the vertical line of axis. It has three orbits, namely

$$\mu_1 = \{5,7,29,33,42,43,56,52\},$$

$$\mu_2 = \{0,6,8,12,22,63\},$$

$$\mu_3 = \left\{\begin{matrix}19,20,21,23,27,39,46, \\ 62,64,65,66\end{matrix}\right\},$$

$$\mu_4 = \{1,2,10,11,13,15,16,17,28,30,32,34,6,41,44,49,51,53,55,57,68,69,70,72\},$$

$$\mu_4 = \{3,4,8,9,14,18,24,25,26,31,35,37,38,40,45,47,48,50,54,59,60,61,67,71\}.$$

Consequently, this diagram depicts an intransitive action of $PSL_2(O_2)$ on $PL(F_{73})$ for $\theta = 2$.

In the following result we prove that each vertex of the coset diagram of the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$ is fixed by $(as)^4$.

Theorem 6. Under the action of $PSL_2(O_2)$ on $PL(F_p)$ there exists a coset diagram $D(\theta, p)$ such that each vertex of the diagram is fixed by $(as)^4$ for the conjugacy class in which 2 is a perfect square in F_P.

Proof. Let A and S be $2*2$ non-singular matrices corresponding to a and s with $det(AS) = 1$ and $trace\,(AS) = r$. Then (AS) satisfies the characteristic equation.

$$(AS)^2 - r(AS) + \mathrm{I} = 0 \quad (1)$$

Multiply (1) by AS, and after simplification we obtain the following relation

$$(\mathrm{AS})^3 = (\mathrm{AS})(r^2 - I) - \mathrm{rI} \qquad (2)$$

Multiplying equation (2) by AS and after simplification we get

$$(\mathrm{AS})^4 = (r^3 - 2r)(\mathrm{AS}) - r^2 - I.$$

To find fourth root of unity $(r^3 - 2r) = 0$ by putting $\theta = r^2$, we get $\theta = 2$.

In view of Theorem 1, we get the relations with parameters b, c, d, e, f, g, i, j, k, w, x

and y with $\sqrt{-1}$ and $\sqrt{-2}$. Since $-1 \equiv p - 1 \,(mod\, p)$ and $-2 \equiv p - 2\,(mod\, p)$ map elements of $PL(F_P)$ onto the elements of $PL(F_p)$ if $p-1$ and $p-2$ are perfect squares in F_p, therefore, as^4 is also a relater of $PSL_2(O_2)$ for $\theta = 2$. Hence we obtain a coset diagram $D(\theta,\ p)$ in which each vertex is fixed by as^4.

It is interesting to note that each coset diagram of the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$ admits symmetry. The following result is about the existence of symmetry in the coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$.

Theorem 7. Under the action of $PSL_2(O_2)$ on $PL(F_p)$ coset diagram admits symmetry for the conjugacy class in which 2 is a perfect square in F_p.

Proof. The subsequence of M − S primes (where p > 2) in which 2 is a perfect square modulo p is $\{17,41,73,89,\ldots\}$. To prove the existence of symmetry of the diagrams about the vertical line of axis, we show that the transformation

$m: \to \frac{-\sqrt{-2}z+1}{z+\sqrt{-2}}$ inverts the transformations a, s and v that is $m^2 = (sm)^2 = (vm)^2 = 1$.

Since the value of $\sqrt{-2}$ is not fixed and it changes as the prime p changes, therefore, the value of the transformation m is different in each p.

In the action of $PSL_2(O_2)$ on $PL(F_{17})$ the values of the transformations a, m, s and v are as follows;

$a: z \to \frac{-1}{z}$, $m: z \to \frac{7z+1}{z-7}$, $s: z \to \frac{8}{2z-1}$,

$a: z \to \frac{2}{8z-1}$.

Let A, M, S and V be the corresponding matrix representations of the transformations a, m, s and v respectively, that is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 7 & 1 \\ 1 & -7 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 8 \\ 2 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 2 \\ 8 & -1 \end{bmatrix}$$

Then $M^2 = -50I$, that is $M^2 = I$ Also $(AM)^2 = I$ and $(SM)^2 = 16I$, and $(VM)^2 = 16I$. This shows that the transformation m inverts a, s and v.

The following table indicates the above stated fact for first few prime numbers in which 2 is a perfect square modulo p.

Table 1. A list of first few prime numbers in which 2 is a perfect square in F_p.

Prime Number	Value of m
17	$\frac{7z+1}{z-7}$
41	$\frac{11z+1}{z-11}$
73	$\frac{12z+1}{z-12}$
89	$\frac{40z+1}{z-40}$

The coset diagrams of the action of $PSL_2(O_2)$ on $PL(F_p)$ admit symmetry about the vertical line of axis for the conjugacy class in which 2 is a perfect square in F_p.

5.4. Intransitivity of the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$ In this section, we study the intransitivity of the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$. We also obtain a formula to count the number of orbits of the permutation subgroup obtained from the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$.

In the following result we prove that the blocks of the permutation group of the action of $PSL_2(O_2)$ on $PL(F_p)$ for $\theta = 2$ serve as the orbit of this group.

Theorem 8. Blocks of the permutation group of the action of $PSL_2(O_2)$ on $PL(F_p)$ for conjugacy class corresponding to the element for $\theta = 2$ are the orbits of this group.

Proof. Let $X \subseteq PL(F_p)$ be a block of the permutation group $\overline{\Gamma_2}$ of the action of $PSL_2(O_2)$ on $PL(F_p)$ for conjugacy class corresponding to the element $\theta = 2$. Then either $X = X^g$, for all g in $\overline{\Gamma_2}$ or $X \cap X^g = \emptyset$.

Case 1: If $X = X^g$, since X^g is an orbit of $\overline{\Gamma_2}$ and $X = X^g$, therefore, X itself becomes an orbit of $\overline{\Gamma_2}$. Since $PL(F_p)$ is finite and we know that orbits partition the set, therefore, there exists another block, say , Y of $\overline{\Gamma_2}$ such that $Y = Y^g$ for all g in $\overline{\Gamma_2}$. This means that Y is also an orbit of $\overline{\Gamma_2}$ because the image set Y^g of Y is an orbit of $\overline{\Gamma_2}$. Continuation of the above process leads us to conclude that blocks of $\overline{\Gamma_2}$ are infect the orbits of $\overline{\Gamma_2}$ for conjugacy class corresponding to the element $x_i = 2$.

Case 2: If $X \cap X^g = \emptyset$, this means that the image set X^g of X will coincide with some other subset Z of $PL(F_p)$ such that $Z = X^g$. This implies that Z is a block of $\overline{\Gamma_2}$. Thus Z is an orbit of $\overline{\Gamma_2}$ as X^g is an orbit of $\overline{\Gamma_2}$.

Hence in either case the blocks of $\overline{\Gamma_2}$ are infect the orbits of this group.

The following result leads to the point that the action of $PSL_2(O_2)$ on $PL(F_p)$ is intransitive for the conjugacy class corresponding to the element $\theta = 2$ in F_p.

Theorem 9. Action of $PSL_2(O_2)$ on $PL(F_p)$ is intransitive for the conjugacy class corresponding to the element $\theta = 2$.

Proof. By the above Theorem, $\overline{\Gamma_2}$ contains more then one block as its orbits. This algebraic fact leads to conclude that the action of $PSL_2(O_2)$ on $PL(F_p)$ is intransitive for the conjugacy class corresponding to the element $\theta = 2$.

The following result leads us to note the above stated algebraic fact.

Theorem 10. Under the action of $PSL_2(O_2)$ on $PL(F_p)$ for the conjugacy class in which 2 is a perfect square modulo p.

i- If -1 is a perfect square modulo p, the number of orbits in a coset diagram are $\frac{p+7}{24} + 1$.

ii- If both -1 and -3 are perfect squares modulo p, then the number of orbits in a coset diagram are $\frac{p-1}{24} + 2$.

Proof. The subsequence of $\mathrm{M-S}$ primes in which 2 is a perfect square modulo $p > 2$ is as follows

$\nu = \{17,41,73,89,97,113,\dots\}$.

We further subdivide the sequence ν into two subsequences, namely, ν_1 and ν_2. The subsequence ν_1 consists of those $\mathrm{M-S}$ primes in which -1 is a perfect square modulo p that is $\nu_1 = \{17,41,89,113,\dots\}$. The subsequence ν_2 consists of those $\mathrm{M-S}$ primes in which both -1 and -3 are perfect square modulo p that is $\nu_2 = \{73,97,\dots\}$.

We discuss ν_1 and ν_2 in the following two separate cases.

Case 1: Consider the sequence $\nu_1 = \{17,41,89,113\dots\}$. For prime $p = 17$, the coset diagram of the action of $PSL_2(O_2)$ on $PL(F_{17})$ has two orbits as shown in fig. 2 that is

$$2 = \frac{17+7}{24} + 1 = \frac{p+7}{24} + 1, \qquad p = 17$$

For the prime $p = 41$ the coset diagram has three orbits, as shown in fig.3 that is

$$3 = \frac{41+7}{24} + 1 = \frac{p+7}{24} + 1, \qquad p = 41$$

Case 2: Consider the sequence $\nu_2 = \{73,97,\dots\}$. For prime $p = 73$, the coset diagram has five orbits, as shown in fig.4 that is

$$5 = \frac{73-1}{24} + 2 = \frac{p-1}{24} + 2, \qquad p = 73$$

The continuation of the above process leads us to note the number of orbits occurring in a coset diagram are $\frac{p-1}{24} + 2$ for the subsequence ν_2. This completes the proof.

Conclusion

We have developed the mechanism to count the number of orbits, occurring in each coset diagram of homomorphic images of $PSL(2, O_2)$ on $PL(F_p)$ for conjugacy class in which 2 is a perfect square in F_p. This work can also be extended to the real and imaginary quadratic fields. We are working on it and will share some interesting results in the future.

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المجموعات المقرنة من الصور المتماثلة لمجموعة بيانكي (Bianchi) معينة

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خلاصة

في هذا البحث، نصنف المجموعات المقرنة للإجراء $PSL_2\,(O_2)$ على الخط الاسقاطي على حقول منتهية، $PL(F_p)$ حيث p عدد أولي باستخدام أسلوب المعلمات وبحث سلوك مجموعة من الرسوم البيانية المصاحبة لهذه الإجراءات. سوف نثبت أن الإجراء $M-S$ PSL_2 (O_2) على $PL(F_p)$ انتقالي لجميع المجموعات المقرنة باستثناء المجموعة المقرنة التي يكون فيها 2 هو مربع كامل في F_p. وسوف نثبت أيضا أن الصور المتماثلة ل $PSL_2\,(O_2)$ والتي يتم تمثيلها من خلال مجموعة الرسوم البيانية هذه تكون متماثلة مع مجموعات شيفالي (Chevalley) ذات الرتبة الواحدة، $L_2\,(p)$ لكل $p \geq 11$. سوف ندرس أيضاً سلوك الرسم البياني الخاص بالصور المتماثلة ل PSL_2 (O_2) على المجموعات المقرنة التي يكون فيها 2 هو مربع كامل في F_p وإثبات أن هذه الرسوم البيانية تتمتع بالتماثل حول الخط العمودي للمحور في فضاء ثنائي الأبعاد. ونثبت أيضاً أن مجموعة الرسوم البيانية هذه تصف تأثير غير انتقالي ل $PSL_2\,(O_2)$ على $PL(F_p)$ في هذه الحالة. تقودنا هذه الحقيقة الجبرية إلى وضع صيغة لحساب عدد المدارات التي تحدث في كل رسم بياني في هذه الفئة المحددة.