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Slant submersions from almost paracontact Riemannian manifolds

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ABSTRACT

In this paper, we introduce slant submersions from almost paracontact Riemannian manifolds onto Riemannian manifolds. We give examples and investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion. We also find necessary and sufficient conditions for a slant submersion to be totally geodesic.

Keywords: Riemannian submersion; almost paracontact Riemannian manifold; slant submersion.

INTRODUCTION

Given a $C^{\infty}-$ submersion π from a Riemannian manifold (M, g) onto a Riemannian manifold (B, g'), there are several kinds of submersions according to the conditions on it: e.g. Riemannian submersion (O'Neill, 1966; Gray, 1967), slant submersion (Ṣahin, 2011; Park, 2012; Gündüzalp, 2013a), almost Hermitian submersion (Watson, 1976), paracontact semi-Riemannian submersion (Gündüzalp & Ṣahin, 2013), anti-invariant semi-Riemannian submersions (Gündüzalp, 2013b), paraquaternionic submersion (Caldarella, 2010), quaternionic submersion (Ianus *et al*., 2008), etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory (Bourguignon & Lawson, 1981; Watson, 1983), Kaluza-Klein theory (Ianus & Visinescu, 1987; Bourguignon & SS Lawson, 1989), supergravity and superstring theories (Ianus & Visinescu, 1991; Mustafa, 2000), etc. On the other hand, slant submanifolds of almost paracontact metric manifolds were studied in (Atçeken, 2010).

Riemannian submersions between almost Hermitian manifolds were studied by Watson in (1976) under the name of holomorphic submersions. One of the main result of this notion is that vertical and horizontal distributions are invariant under almost complex structure. He showed that if the total manifold is a Kähler manifold, then the base manifold is also a Kähler manifold. Recently, (Ṣahin, 2011) has introduced slant submersions from almost Hermitian manifolds to Riemannian manifolds. He showed that the geometry of slant submersions is quite different from holomorphic submersions. Indeed, although every holomorphic submersion is harmonic, slant submersions may not be harmonic. The paper is organized as follows. In the following seedons we recall some notions needed for this paper and we give the definition of slant Riemannian submersions and provide examples. We also investigate the geometry of leaves of the distributions. Finally we give necessary and sufficient conditions for such submersions to be totally geodesic.

PRELIMINARIES

In this section, we define almost paracontact Riemannian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds, and give a brief review of basic facts of Riemannian submersions.

Almost paracontact Riemannian manifolds. Let M be a $(m+1)$-dimensional manifold. If there exist on M a $(1,1)$ type tensor field F, a vector ξ, and 1-form η satisfying

$$F^2 = I - \eta \otimes \xi, \eta(\xi) = 1, \tag{1}$$

then M is said to be an almost paracontact manifold, where $\otimes$, the symbol, denotes the tensor product. In the almost paracontact manifold, the following relations hold good:

$$F\xi = 0,\ \eta \circ F = 0, rank(F) = m. \tag{2}$$

An almost paracontact manifold is said to be an almost paracontact metric manifold if Riemannian metric g on M satisfies

$$g(FX, FY) = g(X,Y) - \eta(X)\eta(Y),\ \eta(X) = g(X,\xi) \tag{3}$$

for all $X, Y \in \Gamma$(TM). From (2) and (3), we can easily derive the relation

$$g(FX, Y) = g(X, FY). \tag{4}$$

An almost paracontact metric manifold is said to be an almost paracontact Riemannian manifold with $(F, g, \xi, \eta)-$ Levi Civita connection if $\nabla F = 0$ and $\nabla \eta = 0$, where ∇ denotes the connection on M. Since $F^2 = I - \eta \otimes \xi$, the vector field ξ is also parallel with respect to ∇ (Ianus et al., 1985).

Example 1. The almost paracontact Riemannian structure (F, g, ξ, η) is defined on R^5 in the following way:

$$F(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}, F(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1}, F(\frac{\partial}{\partial y_1}) = \frac{\partial}{\partial y_2}, F(\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_1},$$

$$F(\frac{\partial}{\partial t}) = 0,\ \xi = \frac{\partial}{\partial t}, \eta = dt.$$

If $Z = a_i(\frac{\partial}{\partial x_i}) + b_i(\frac{\partial}{\partial y_i}) + v(\frac{\partial}{\partial t}) \in T(R^5)$, then we have

$$g(Z,Z) = \sum_{i=1}^{2} a_i^2 + \sum_{i=1}^{2} b_i^2 + v^2.$$

From this definition, it follows that

$$g(Z,\xi) = \eta(Z) = v, g(FZ, FZ) = g(Z,Z) - \eta^2(Z),\ F\xi = 0,\ \eta(\xi) = 1$$

for an arbitrary vector field Z. Thus (R^5, F, g, ξ, η) becomes an almost paracontact Riemannian manifold, where g and $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial t}\}$ denote usual inner product and standard basis of $T(R^5)$, respectively.

Riemannian submersions. Let (M, g) and (B, g') be two Riemannian manifolds. A surjective $C^\infty -$ map $\pi : M \to B$ is a $C^\infty -$ submersion if it has maximal rank at any point of M. Putting $\mathcal{V}_x = ker\pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called horizontal distribution. A $C^\infty -$ submersion $\pi : M \to B$ between two Riemannian manifolds (M, g) and (B, g') is called a Riemannian submersion if, at each point x of M, π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is $\pi -$ related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to B$ determines two $(1,2)$ tensor fields T and A on M, by the formulas:

$$T(E,F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{5}$$

and

$$A(E,F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{6}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (Falcitelli et al., 2004). From (5) and (6), one can obtain

$$\nabla_U W = T_U W + \hat{\nabla}_U W; \tag{7}$$

$$\nabla_U X = T_U X + h(\nabla_U X): \tag{8}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{9}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{10}$$

for any $X,Y \in \Gamma(\ker\pi_*)^{\perp})$, $U,W \in \Gamma(\ker\pi_*)$. Moreover, if X is basic then

$$h(\nabla_U X) = h(\nabla_X U) = A_X U. \tag{11}$$

We note that for $U,V \in \Gamma(\ker\pi_*)$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X,Y \in \Gamma(\ker\pi_*)^{\perp})$. $A_X Y = \frac{1}{2}v[X,Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X,Y \in \Gamma(\ker\pi_*)^{\perp})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U,V \in \Gamma(\ker\pi_*)$.

We now recall the following result which will be useful for later.

Lemma 1. (O'Neill, 1966). If $\pi : M \to B$ is a Riemannian submersion and X,Y basic vector fields on M, $\pi-$related to X' and Y' on B, then we have the following properties

1. $h[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X',Y'] \circ \pi$;
2. $h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;
3. $[E,U] \in \Gamma(\ker\pi_*)$. for any $U \in \Gamma(\ker\pi_*)$ and for any basic vector field E.

We recall the notion of harmonic maps between Riemannian manifolds. Let (M,g_M) and (N,g_N) be Riemannian manifolds and suppose that $\pi : M \to N$ is a smooth map between them. Then the differential π_* of π can be viewed a section of the bundle $Hom\ (TM, \pi^{-1}TN) \to M$, where $\pi^{-1}TN$ is the pullback bundle which has fibres $(\pi^{-1}\mathrm{TN})_p = T_{\pi(p)}N, p \in M$. $Hom\ (TM, \pi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection ∇^π. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X,Y) = \nabla^\pi_X \pi_* Y - \pi_*(\nabla^M_X Y) \tag{12}$$

for $X,Y \in \Gamma(TM)$. Recall that π is said to be harmonic if $trace(\nabla\pi_*) = 0$ and π is called a *totally geodesic* map if $(\nabla\pi_*)(X,Y) = 0$ for $X,Y \in \Gamma(TM)$ (Baird & Wood, 2003). It is known that the second fundamental form is symmetric.

SLANT SUBMERSIONS

In this section, we define slant submersions from an almost paracontact Riemannian manifold onto a Riemannian manifold by using the definition of a slant distribution given in (Atceken, 2010). We give examples, investigate the geometry of leaves of distributions. We also obtain a necessary and sufficient condition for such submersions to be totally geodesic maps.

Definition 1. Let π be a Riemannian submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2). If for any non-zero vector $X \in ker\pi_* - sp\{\xi_p\}$: $p \in M_1$, the angle $\theta(X)$ between FX and the space $(ker\pi_* - sp\{\xi_p\})$, is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $(ker\pi_* - sp\{\xi_p\})$, then we say that π is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion.

It is known that the distribution $(ker\pi_*)$ is integrable for a Riemannian submersion between Riemannian manifolds. In fact, its leaves are $\pi^{-1}(p)$, $p \in M_1$, i.e., fibers. Thus it follows from above definition that the fibers of a slant submersion are slant submanifolds of M_1, for slant submanifolds, (Atceken, 2010). We note that the characteristic vector field ξ is a vertical vector field.

We first give some examples of slant submersions.

Example 2. Every anti-invariant Riemannian submersion from an almost paracontact Riemannian manifold onto a Riemannian manifold is a slant submersion with $\theta = 90$.

Example 3. Consider the following Riemannian submersion given by

$$\pi : R^5 \to R^2$$
$$(x_1, x_2, y_1, y_2, t) \to (\frac{x_1 - y_1}{\sqrt{2}}, x_2).$$

Then π is a slant submersion with slant angle $\theta = 45$.

Example 4. Define a map $\pi : R^5 \to R^2$ by

$$\pi(x_1, x_2, y_1, y_2, t) = (x_2, x_1 \cos\alpha - y_1 \sin\alpha),$$

where $0 < \alpha < 90$. Then the map π is a slant submersion with the slant angle $\theta = \alpha$.

Example 5. Define a map $\pi : R^5 \to R^2$ by

$$\pi(x_1, x_2, y_1, y_2, t) = (x_1 \cos\alpha - y_1 \sin\alpha, x_2 \sin\beta - y_2 \cos\beta).$$

Then the map π is a slant submersion with the slant angle θ with

$$\cos\theta = |\sin(\alpha + \beta)|.$$

Let π be a Riemannian submersion from an almost paracontact Riemannian manifold M_1 with the structure (g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2), Then for $X \in \Gamma(ker\pi_*)$, we write

$$FX = \phi X + \omega X, \tag{13}$$

where ϕX and ωX are vertical and horizontal parts of *FX*. From (4) and (13), one can easily see that

$$g_1(X, \phi Y) = g_1(\phi X, Y). \tag{14}$$

for any $X, Y \in \Gamma(ker\pi_*)$.

Also for $Z \in \Gamma((ker\pi_*)^{\perp})$. we have

$$FZ = BZ + CZ, \tag{15}$$

where *BZ* and *CZ* are vertical and horizontal component of *FZ*. From (4) and (15), one can easily see that

$$g_1(Z_1, CZ_2) = g_1(CZ_1, Z_2) \tag{16}$$

for any $Z_1, Z_2 \in \Gamma(ker\pi_*)^{\perp})$.

Span $\{\xi\}$ defines the vertical vector field distribution. If $X \in ker\pi_*$ is a vertical vector field, which is orthogonal to ξ, then

$$g(FX, FX) = g(X, X) \geq 0,$$

the same is valid for ϕX. For vertical vector fields the Cauchy-Schawrz inequality, $g(X, Y) \leq |X| Y|$, is verified. Therefore the Wirtinger angle, θ, is given by:

$$\frac{g(FX, øX)}{|FX||øX|} = \cos\theta.$$

We define the covariant derivatives of ϕ and ω as follows

$$(\nabla_X \phi)Y = \hat{\nabla}_X \phi Y - \phi \hat{\nabla}_X Y \tag{17}$$

and

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y \tag{18}$$

for $X, Y \in \Gamma(ker\pi_*)$. where $\hat{\nabla}_X Y = v\nabla_X Y$. Then we easily have

Lemma 2. Let (M_1, g_1, F, ξ, η) be an almost paracontact Riemannian manifold and (M_2, g_2) a Riemannian manifold. Let $\pi : (M_1, g_1, F) \to (M_2, g_2)$ be a slant submersion. Then we get

$$\hat{\nabla}_X \phi Y + T_X \omega Y = \phi \hat{\nabla}_X Y + BT_X Y$$

$$T_X \phi Y + h\nabla_X \omega Y = \omega \hat{\nabla}_X Y + CT_X Y$$

for any $X, Y \in \Gamma(ker\pi_*)$.

Let π be a slant submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2) with the slant angle $\theta \in (0,90)$, then we say that ω is parallel with respect to the Levi-Civita connection ∇ on $(ker\pi_*)$ if its covariant derivative with respect to ∇ vanishes, i.e., we have

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y = 0 \tag{19}$$

for $X, Y \in \Gamma(ker\pi_*)$.

Invariant and anti-invariant submanifolds are particular classes of slant submanifolds with slant angles $\theta = 0$ and $\theta = 90$, respectively. A slant submanifold which is neither invariant nor anti-invariant submanifold is called a proper slant submanifold (At c eken, 2010).

Theorem 1. Let π be a Riemannian submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2), Then π is a proper slant submersion if and only if there exists a constant $\lambda \in [0,1]$ such that

$$\phi^2 X = \lambda(X - \eta(X)\xi)$$

for $X \in \Gamma(ker\pi_*)$ If π is a proper slant submersion, then $\lambda = \cos^2\theta$.

Proof: For any nonzero $X \in \Gamma(ker\pi_*)$, we can write

$$\cos\theta(X) = \frac{\| øX \|}{\| FX \|}, \tag{20}$$

where $\theta(X)$ is the slant angle. By using (14), (20) and (1) we get

$$\begin{aligned} g_1(\phi^2 X, X) &= g_1(\phi X, \phi X) \\ &= \cos^2\theta(X) g_1(FX, FX) \\ &= \cos^2\theta(X) g_1(F^2 X, X) \\ &= \cos^2\theta(X) g_1(X - \eta(X)\xi, X) \end{aligned} \tag{21}$$

for all $X \in \Gamma(ker\pi_*)$ Since g_1 is Riemannian metric, from (21) we have

$$\phi^2 X = \cos^2\theta(X)(X - \eta(X)\xi),\ X \in \Gamma(ker\pi_*). \tag{22}$$

Let $\lambda = \cos^2\theta$. Then it is obvious that $\lambda \in [0,1]$ and $\phi^2 = \lambda(I - \eta \otimes \xi)$.

Conversely, let us assume that there exist a constant $\lambda \in [0,1]$ such that $\phi^2 = \lambda(I - \eta \otimes \xi)$ is satisfied. From (13), (14) and (1) we get

$$\cos\theta(X) = \frac{g_1(FX, øX)}{\| FX \| \| øX \|}$$

$$= \frac{\lambda g_1(FX, FX)}{\| FX \| \| øX \|},$$

for all $X \in \Gamma(ker\pi_*)$. Thus we have

$$cos\,\theta(X) = \frac{\lambda \| FX \|}{\| øX \|}.$$

Since $cos\,\theta(X) = \frac{\| øX \|}{\| FX \|}$, then by using the last equation we obtain $cos^2\theta(X) = \lambda$, which implies that $\theta(X)$ is a constant and π is a proper slant submersion.

From Theorem 1, (13) and (3) we have the following result.

Corollary 1. Let π *be a slant submersion from an almost paracontact Riemannian manifold* (M_1, g_1, F, ξ, η) *onto a Riemannian manifold* (M_2, g_2) *with slant angle* $\theta \in (0,90)$. *Then, for any* $X, Y \in \Gamma(ker\pi_*)$. *we have*

$$g_1(\phi X, \phi Y) = \cos^2\theta(g_1(X,Y) - \eta(X)\eta(Y) \tag{23}$$

$$g_1(\omega X, \omega Y) = \sin^2\theta(g_1(X,Y) - \eta(X)\eta(Y)). \tag{24}$$

From (1), (13) and (15) we have the following result.

Corollary 2. Let π be a slant submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0,90)$. Then, we have

$$\phi^2 + B\omega = I - \eta \otimes \xi \tag{25}$$

$$\omega\phi + C\omega = 0. \tag{26}$$

From Theorem 1 and (25) we have the following result.

Corollary 3. Let π be a slant submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0,90)$. Then π is a proper slant submersion if and only if there exists a constant $\mu \in [0,1]$ such that

$$B\omega = \mu(I - \eta \otimes \xi).$$

If π is a proper slant submersion, then $\mu = \sin^2\theta$.

Proposition 1. Let π be a slant submersion from an almost paracontact Riemannian manifold onto a Riemannian manifold with the slant angle $\theta \in (0,90)$. If ω is parallel with respect to ∇ on $(ker\pi_*)$, then we have

$$T_{\phi X}\phi X = \cos^2\theta(T_X X - \eta(X)T_X\xi) \tag{27}$$

for $X \in (ker\pi_*)$.

Proof: If ω is parallel, then from Lemma 2 we have $CT_X Y = T_X\phi Y$ for $X, Y \in (ker\pi_*)$. Interchanging the role of X and Y, we get $CT_Y X = T_Y\phi X$. Thus we have

$$CT_X Y - CT_Y X = T_X\phi Y - T_Y\phi X$$

Since T is symmetric, we derive $T_X\phi Y = T_Y\phi X$. Then substituting Y by ϕX we get $T_X\phi^2 X = T_{\phi X}\phi X$. Finally using Theorem 1 we obtain (27).

We now investigate the geometry of the leaves of the distributions $(ker\pi_*)$ and $(ker\pi_*)^\perp$.

Theorem 2. Let π be a slant submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0,90)$. Then the distribution $(ker\pi_*)$ defines a totally geodesic foliation on M_1 if and only if

$$\eta(Y)g_1(T_X\xi, Z) = g_1(h\nabla_X \omega\phi Y, Z) + g_1(h\nabla_X \omega Y, CZ + g_1(T_X\omega Y, BZ)$$

for $X, Y \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$.

Proof: For $X, Y \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$, from (3) and (13) we have

$$g_1(\nabla_X Y, Z) = g_1(\nabla_X \phi Y, FZ) + g_1(\nabla_X \omega Y, FZ).$$

Using (3),(13) and (15) we get

$$g_1(\nabla_X Y, Z) = g_1(\nabla_X \phi^2 Y, Z) + g_1(\nabla_X \omega\phi Y, Z) + g_1(\nabla_X \omega Y, BZ) + g_1(\nabla_X \omega Y, CZ).$$

Then from (7), (8) and Theorem 1 we obtain

$$g_1(\nabla_X Y, Z) = \cos^2\theta g_1(\nabla_X Y, Z) - \eta(Y)g_1(T_X\xi, Z) + g_1(h\nabla_X \omega\phi Y, Z) + g_1(T_X\omega Y, BZ) + g_1(T_X\omega Y, CZ).$$

Hence we have

$$\sin^2\theta g_1(\nabla_X Y, Z) = -\eta(Y)g_1(T_X\xi, Z) + g_1(h\nabla_X \omega\phi Y, Z) + g_1(T_X\omega Y, BZ) + g_1(h\nabla_X \omega Y, CZ)$$

which proves assertion.

Theorem 3. Let π be a slant submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0,90)$. Then the distribution $(ker\pi_*)^\perp$ defines a totally geodesic foliation on M_1 if and only if

$$\phi(v\nabla_X BY + A_X CY) + B(A_X BY + h\nabla_X CY) = 0$$

for $X, Y \in \Gamma(\ker\pi_*)^{\perp})$.

Proof: For $X, Y \in \Gamma(\ker\pi_*)^{\perp})$, since M is an almost paracontact Riemannian manifold, we have $\nabla_X Y = F\nabla_X FY$. From (9), (10),(13) and (15) we get

$$\begin{aligned}\nabla_X Y &= F(\nabla_X BY + \nabla_X CY) \\ &= F(A_X BY + v\nabla_X BY + A_X CY + h\nabla_X CY) \\ &= BA_X BY + CA_X BY + \text{ø}v\nabla_X BY + \omega v\nabla_X BY + \text{ø}A_X CY + \omega A_X CY \\ &\quad + Bh\nabla_X CY + Ch\nabla_X CY,\end{aligned}$$

which proves the assertion.

Finally we give necessary and sufficient conditions for a slant submersion with slant angle $\theta \in (0,90)$ to be totally geodesic. Recall that a differentiable map π between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla\pi_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$.

Theorem 4. Let π be a slant submersion from an almost paracontact Riemannian manifold (M_1, g_1, F, ξ, η) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0,90)$. Then π is totally geodesic if and only if

$$\cos^2\theta\eta(Y)g_1(T_X\xi, Z) = g_1(h\nabla_X \omega\text{ø}Y, Z) + g_1(h\nabla_X \omega Y, CZ) + g_1(T_X \omega Y, CZ)$$

and

$$g_1(h\nabla_{Z_1}\omega\text{ø}X, Z_2) = g_1(A_{Z_1}BZ_2 + h\nabla_{Z_1}CZ_2, \omega X) + \cos^2\theta\eta(X)g_1(A_{Z_1}\xi, Z_2)$$

for $Z, Z_1, Z_2 \in \Gamma(\ker\pi_*)^{\perp})$ and $X, Y \in \Gamma(\ker\pi_*)$.

Proof: First of all, since π is a Riemannian submersion we have

$$(\nabla\pi_*)(Z_1, Z_2) = 0$$

for $Z_1, Z_2 \in \Gamma(\ker\pi_*)^{\perp})$.

For $X, Y \in \Gamma(\ker\pi_*)$ and $Z, Z_1, Z_2 \in \Gamma(\ker\pi_*)^{\perp})$. from (3), (12) and (13) we have

$$g_2((\nabla\pi_*)(X, Y), \pi_* Z) = -g_1(\nabla_X F\text{ø}Y, Z) - g_1(\nabla_X \omega Y, FZ).$$

Using (13) and (15) we get

$$\begin{aligned}g_2((\nabla\pi_*)(X, Y), \pi_* Z) &= -g_1(\nabla_X \text{ø}^2 Y, Z) - g_1(\nabla_X \omega\text{ø}Y, Z) \\ &\quad - g_1(\nabla_X \omega Y, BZ) - g_1(\nabla_X \omega Y, CZ).\end{aligned}$$

Then Theorem 1, (7) and (8) imply that

$$g_2((\nabla\pi_*)(X,Y),\pi_*Z) = -\cos^2\theta g_1(\nabla_X Y,Z) + \cos^2\theta\eta(Y)g_1(T_X\xi,Z) - g_1(h\nabla_X \omega ø Y,Z)$$
$$-g_1(T_X\omega Y,BZ) - g_1(h\nabla_X\omega Y,CZ).$$

Hence we obtain

$$\sin^2\theta g_2((\nabla\pi_*)(X,Y),\pi_*Z) = \cos^2\theta\eta(Y)g_1(T_X\xi,Z) - g_1(h\nabla_X\omega\varphi Y,Z)$$
$$-g_1(T_X\omega Y,BZ) - g_1(h\nabla_X\omega Y,CZ). \tag{28}$$

Similarly, we get

$$\sin^2\theta g_2((\nabla\pi_*)(X,Y),\pi_*Z) = \cos^2\theta\eta(Y)g_1(T_X\xi,Z) - g_1(h\nabla_X\omega ø Y,Z)$$
$$-g_1(T_X\omega Y,BZ) - g_1(h\nabla_X\omega Y,CZ).$$

$$\sin^2\theta g_2((\nabla\pi_*)(X,Z_1),\pi_*Z_2) = g_1(A_{Z_1}BZ_2 + h\nabla_{Z_1}CZ_2,\omega X) + \cos^2\theta\eta(X)g_1(A_{Z_1}\xi,Z_2)$$
$$-g_1(h\nabla_{Z_1}\omega ø X,Z_2). \tag{29}$$

Then the proof follows from (28) and (29).

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غمور جانبية من منطويات ريمانية مثل – تلامسية تقريباً

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خلاصة

نقدم في هذا البحث غمور جانبية من منطويات ريمانية مثل – تلامسية تقريباً إلى منطويات ريمانية. نعطي بعض الامثلة و ندرس هندسة التوريقات التي تنتج عن هذا الغمر. كما نجد شروط ضرورية و كافية ليكون الغمر الجانبي جيوديزي كلياً.