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Generalized Bour's theorem

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ABSTRACT

We give the classical isometric minimal helicoidal and rotational surfaces using generalized Bour's theorem in Euclidean 3-space. Furthermore, we investigate the minimality and have same Gauss map of the surfaces.

Keywords: Gauss map; gaussian curvature; helicoidal surface; mean curvature; rotational surface.

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INTRODUCTION

Theory of surfaces in three dimensional Euclidean and Minkowskian spaces have been studied for a long time and many examples of such surfaces have been discovered. Many nice books have been published, such as Do Carmo (1976), and Kühnel (2006).

In classical surface geometry in Euclidean space, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. Rotational or surface of revolution) surface which is minimal. If we focus on the ruled (helicoid) and rotational characters, we have Bour's theorem (Bour, 1862).

Ikawa (2000) determine pairs of surfaces by Bour's theorem with the additional condition that they have the same Gauss map in Euclidean 3-space. About helicoidal surfaces in Euclidean 3-space, Do Carmo & Dajczer (1982), prove that, by using a result of Bour (1862), there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface.

Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces in Euclidean 3-space are shown by Güler *et al*. (2010). In addition, they give Bour's theorem on the Gauss map and some special examples.

Ji & Kim (2013) prove that, in Minkowski 3-space, a minimal helicoidal surface with Gauss curvature K has an isometric minimal rotational surface if and only if $K \leq 0$. Moreover, they show that a timelike right helicoid does not have an isometric minimal rotational surface. On another hand, Martinez *et al*. (2013) give a complete classification of the helicoidal flat surfaces in the hyperbolic 3-space in terms of meromorphic data as well as by means of linear harmonic functions.

In this paper, we give the generalized Bour's theorem in Euclidean 3-space. In the following section, we recall some basic notions of the Euclidean geometry and the reader can find a definition of helicoidal surface. Isometric helicoidal surface and rotational surface are obtained, and Gauss'Theorema Egregium is given in section which follows next. In the last section, properties of the isometric surfaces, that have the same Gauss map and the minimality, are investigated.

PRELIMINARIES

In the rest of this paper we shall identify a vector $(a,b,c)^t$ with its transpose. In this section, we will obtain the rotational and helicoidal surfaces in Euclidean 3-space.The reader can find basic elements of differential geometry in Do Carmo (1976).

Now, we define the rotational surface and helicoidal surface in $\mathbb{E}^3$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \to \Pi$ be a curve in a plane Π in $\mathbb{R}^3$, and let ℓ be a straight line in $\prod$. A rotational surface in E^3 is defined as a surface rotating a curve γ around a line ℓ (these are called the profile curve and the axis, respectively). Suppose that, when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the helicoidal surface with axis ℓ and pitch $a \in \mathbb{R} \setminus \{0\}$.

We may suppose that ℓ is the line spanned by the vector. The orthogonal matrix which fixes the above vector is

$$A(v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v \in \mathbb{R}.$$

The matrix A can be found, by solving the following equations simultaneously:

$$A\ell = \ell, A^t A = AA^t = I_3, detA = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the z-axis of $\mathbb{R}^3$. Parametrization of the profile curve is given by

$$\gamma(u) = (f(u), 0, \phi(u)),$$

where $f(u)$, $\phi(u)$: $I \subset \mathbb{R} \to \mathbb{R}$ are differentiable functions for all $u \in I$. A helicoidal surface in 3-Euclidean space which is spanned by the vector with pitch, as follows:

$$H(u,v) = A(v) \cdot \gamma(u) + av(0,0,1).$$

When a=0, the surface is just a rotational surface as follows:

$$R(u,v) = (f(u)cos\ v, f(u)sin\ v, \phi(u)).$$

For a surface $X(u,v)$, the coefficients of the first and second fundamental forms, the Gauss map and the other objects are defined in Do Carmo (1976) and Kühnel (2006).

GENERALIZED BOUR'S THEOREM

In this section, helicoidal and rotational surfaces are going to generalized surfaces, by Bour's theorem in three dimensional Euclidean space.

Theorem 1. (Generalized Bour's Theorem). A helicoidal surface

$$H(u,v) = (f(u)cos\ v, f(u)sin\ v, \phi(u) + av) \tag{1}$$

is locally isometric to a rotational surface

$$R(u,v) = \begin{pmatrix} \sqrt{f^2+a^2}\, cos\left(v + \int \frac{a\phi'}{f^2+a^2}du\right) \\ \sqrt{f^2+a^2}\, sin\left(v + \int \frac{a\phi'}{f^2+a^2}du\right) \\ \int \sqrt{\frac{(af')^2+(f\phi')^2}{f^2+a^2}}du \end{pmatrix}, \tag{2}$$

where f and ϕ are differentiable functions, $0 \le v < 2\pi$ and $u, a \in \mathbb{R}\backslash\{0\}$.

Proof. We assume that the profile curve is

$$\gamma_H(u_H) = (f_H(u_H), 0, \phi_H(u_H)).$$

Since the helicoidal surface is given by rotating the profile curve γ around the axis ℓ= (0,0,1) and simultaneously displacing parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation, we have the following representation of the helicoidal surface

$$H(u_H, v_H) = (f_H(u_H)cosv_H, f_H(u_H)sinv_H, \phi_H(u_H) + av_H),$$

where $u_H, a \in \mathbb{R}\backslash\{0\}$ and $0 \le v_H < 2\pi$. The line element of the helicoidal surface as above is given by

$$ds_H^2 = (f_H'^2 + \phi_H'^2)du_H^2 + 2a\phi_H' du_H dv_H + (f_H^2 + a^2)dv_H^2. \tag{3}$$

Helices in $H(u_H, v_H)$ are curves defined by $u_H = const$, so curves in $H(u_H, v_H)$ that are orthogonal to helices supply the orthogonality condition as follow

$$a\phi_H' du_H + (f_H^2 + a^2)dv_H = 0.$$

Thus, we obtain

$$v_H = -\int \frac{a\phi_H'}{f_H^2 + a^2} du_H + c,$$

where c is constant. Hence if we put

$$\bar{v}_H = v_H + \int \frac{a\phi_H'}{f_H^2 + a^2} du_H,$$

then curves orthogonal to helices are given by v_H= const..Substituting the equation

$$dv_H = d\bar{v}_H - \frac{a\phi_H'}{f_H^2 + a^2} du_H$$

into the line element , we have

$$ds_H^2 = \left(f_H'^2 + \frac{f_H^2 \phi_H'^2}{f_H^2 + a^2}\right) du_H^2 + (f_H^2 + a^2) d\bar{v}_H^2. \tag{4}$$

Setting

$$\bar{u}_H := \int \sqrt{f_H'^2 + \frac{f_H^2 \phi_H'^2}{f_H^2 + a^2}}\, du_H, \quad k_H(\bar{u}_H) := \sqrt{f_H^2 + a^2},$$

(4) becomes

$$ds_H^2 = d\bar{u}_H^2 + k_H^2(\bar{u}_H) d\bar{v}_H^2. \tag{5}$$

On the other hand, the rotational surface

$$R(u_R, v_R) = (f_R(u_R)cosv_R, f_R(u_R)sinv_R, \phi_R(u_R)),$$

has the line element

$$ds_R^2 = (f_R'^2 + \phi_R'^2) du_R^2 + f_R^2 d\bar{v}_R^2. \tag{6}$$

Again, setting

$$\bar{u}_R := \int \sqrt{f_R'^2 + \phi_R'^2}\, du_R, \quad k_R(\bar{u}_R) := f_R, \quad \bar{v}_R := v_R,$$

then (6) becomes

$$ds_R^2 = d\bar{u}_R^2 + k_R^2(\bar{u}_R)d\bar{v}_R^2. \tag{7}$$

Comparing (5) with (7), if we take

$$\bar{u}_H = \bar{u}_R, \quad \bar{v}_H = \bar{v}_R, \quad k_H(\bar{u}_H) = k_R(\bar{u}_R),$$

then we have an isometry between $H(u_H, v_H)$ and $R(u_R, v_R)$. Therefore, it follows that

$$\int \sqrt{f_H'^2 + \frac{f_H^2 \phi_H'^2}{f_H^2 + a^2}}\, du_H = \int \sqrt{f_R'^2 + \phi_R'^2}\, du_R,$$

and we get

$$\int \phi_R' du_R = \int \sqrt{\frac{(af')^2 + (f\phi')^2}{f^2 + a^2}}\, du_H\,.$$

this completes the proof.

Corollary 1. Let $f(u) = u$, $\phi(u)=0$, in generalized Bour's theorem. It is easily seen the results of the isometric surfaces in Ikawa (2000).

Example 1. A helicoidal surface

$$H(u,v) = (u^2 \cos v, u^2 \sin v, u^3 + av)$$

is isometric to the rotational surface

$$R(u,v) = \begin{pmatrix} \sqrt{u^4 + a^2}\cos\left(v + \int \frac{3au^2}{u^4 + a^2} du\right) \\ \sqrt{u^4 + a^2}\sin\left(v + \int \frac{3au^2}{u^4 + a^2} du\right) \\ \int \sqrt{\frac{4a^2u^2 + 9u^8}{u^4 + a^2}}\, du \end{pmatrix}$$

by generalized Bour's theorem, where $u, a \in \mathbb{R}\backslash\{0\}$ and $0 \le v < 2\pi$. Moreover, when $a=0$, these surfaces have the form of the rotational surface as follow

$$(u^2 \cos v, u^2 \sin v, u^3).$$

Following theorem called "Theorema Egregium"(Latin: "Remarkable Theorem") was published by German Mathematician C.F. Gauss in 1827. It is one of the great discoveries of nineteenth-century (Kühnel, 2006).

Theorem 2. (Gauss' Theorema Egregium). The Gaussian curvature K of a 2-dimensional surface element $f{:}\,U \to \mathbb{R}^3$ of class depend only on the first fundamental form (and is consequently an intrinsic quantity of the surface).

The mean curvature **H** does not depend only on the first fundamental form. For example, the cylinder and the plane have the same fundamental form, but have $\mathbf{H} \neq 0$ and $\mathbf{H}=0$, respectively. Gauss 'Theorema Egregium can be stated in the language of isometries:

Theorem 3. If $I{:}\,M \to M^*$ is an isometry (and surfaces are locally isometric), then the Gausssian curvatures at corresponding points are equal. That is,

$$K(p) = K^*(I(p))$$

for all point p in M.

Therefore, we give an example for the truth of Theorem 1 as follows.

Example 2. Let $f(u) = u$ and $\phi(u)=0$ in Theorem 1, then the right helicoid

$$H(u,v) = (u\cos v, u\sin v, av) \tag{8}$$

is isometric to the catenoid

$$R(u,v) = \left(\sqrt{u^2+a^2}\cos v,\ \sqrt{u^2+a^2}\sin v,\ a\log\left(u+\sqrt{u^2+a^2}\right)\right), \tag{9}$$

where, $a \in \mathbb{R}\backslash\{0\}$, $0 \leq v < 2\pi$. The coefficients of the first and second fundamental forms of these surfaces are

$$E_{H(u,v)} = 1 = E_{R(u,v)},$$

$$F_{H(u,v)} = 0 = F_{R(u,v)},$$

$$G_{H(u,v)} = u^2 + a^2 = G_{R(u,v)},$$

$$L_{H(u,v)} = 0, \qquad M_{H(u,v)} = -\frac{a}{\sqrt{u^2+a^2}}, \qquad N_{H(u,v)} = 0,$$

$$L_{R(u,v)} = -\frac{a}{u^2+a^2}, \qquad M_{R(u,v)} = 0, \qquad N_{R(u,v)} = a.$$

Hence, the surfaces have

$$LN - M^2 = -\frac{a^2}{u^2+a^2}.$$

So, the Gaussian curvatures of the right helicoid and the catenoid are

$$K_{H(u,v)} = -\frac{a^2}{(u^2+a^2)^2} = K_{R(u,v)}.$$

Corollary 2. If $I{:}H(u,v)\to R(u,v)$ is an isometry (and surfaces are locally isometric), then the Gausssian curvatures at corresponding points are equal, and

$$K_H(p) = \frac{u^3\phi'\phi'' - a^2}{(u^2+a^2+u^2\phi'^2)^2} = K_R(I(p))$$

for all p point in H.

THE GAUSS MAP

In this section, we prove the relations among the isometric surfaces by generalized Bour's theorem.

Theorem 4. Let a helicoidal and a rotational surface be isometrically related by generalized Bour's theorem. If these two surfaces have the same Gauss map, then a pair of two surfaces is

$$H(u,v) = (f(u)cos\; v, f(u)sin\; v, \phi(u) + av) \tag{10}$$

and

$$R(u,v) = \begin{pmatrix} \sqrt{f^2+a^2}\, cos\left(v + \int \frac{a\phi'}{f^2+a^2}du\right) \\ \sqrt{f^2+a^2}\, sin\left(v + \int \frac{a\phi'}{f^2+a^2}du\right) \\ b\; argcosh\left(\frac{\sqrt{f^2+a^2}}{b}\right) \end{pmatrix}, \tag{11}$$

where

$$\phi(u) = \sqrt{b^2-a^2}\log\left(\sqrt{\frac{\sqrt{f^2-a^2}+\sqrt{f^2+a^2-b^2}}{\sqrt{f^2-a^2}-\sqrt{f^2+a^2-b^2}}}\right)$$

$$-a\; arctan\left(\frac{\sqrt{b^2-a^2}}{a}\sqrt{\frac{f^2+a^2}{f^2+a^2-b^2}}\right) + c,$$

$f = f(u)$ is a differentiable function, $u,a,b \in \mathbb{R}\setminus\{0\}$ and $0 \le v < 2\pi$.

Proof. First we consider the helicoidal surface (10). Differentiating H_u and H_v, we obtain

$$H_{uu} = \begin{pmatrix} f'' \cos v \\ f'' \sin v \\ \phi'' \end{pmatrix}, \quad H_{uv} = \begin{pmatrix} -f' \sin v \\ f' \cos v \\ 0 \end{pmatrix}, \quad H_{vv} = \begin{pmatrix} -f \cos v \\ -f \sin v \\ 0 \end{pmatrix}.$$

By virtue of the first and second fundamental forms

$$E_H = f'^2 + \phi'^2, \quad F_H = a\phi', \quad G_H = f^2 + a^2,$$

$$L_H = \frac{f(-\phi' f'' + f'\phi'')}{\sqrt{(f^2+a^2)f'^2 + f^2\phi'^2}},$$

$$M_H = -\frac{af'^2}{\sqrt{(f^2+a^2)f'^2 + f^2\phi'^2}},$$

$$N_H = -\frac{f^2\phi'}{\sqrt{(f^2+a^2)f'^2 + f^2\phi'^2}},$$

the Gauss map and the mean curvature of the helicoidal surface are

$$e_H = \frac{1}{\sqrt{(f^2+a^2)f'^2 + f^2\phi'^2}} \begin{pmatrix} a\phi' \sin v - f\phi' \cos v \\ -a\phi' \cos v - f\phi' \sin v \\ ff' \end{pmatrix}, \tag{12}$$

$$\mathbf{H}_H = \frac{\mathbf{\Phi}(u)}{[(f^2+a^2)f'^2 + f^2\phi'^2]^{3/2}}, \tag{13}$$

where

$$\mathbf{\Phi}(u) := (f^2 f'^2 - f^3 f'' - a^2 f f'' + 2a^2 f'^2)\phi' + f^2\phi'^3 + (f^3 f' + a^2 f f')\phi''. \tag{14}$$

Next, we calculate the Gauss map e_R and the mean curvature $\mathbf{H}_R$ of the rotational surface (2). Since

$$R_u = \begin{pmatrix} \frac{ff'}{\sqrt{f^2+a^2}} \cos\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) - \frac{a\phi'}{\sqrt{f^2+a^2}} \sin\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) \\ \frac{ff'}{\sqrt{f^2+a^2}} \sin\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) + \frac{a\phi'}{\sqrt{f^2+a^2}} \cos\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) \\ \sqrt{\frac{(af')^2 + (f\phi')^2}{f^2+a^2}} \end{pmatrix},$$

$$R_v = \begin{pmatrix} -\sqrt{f^2+a^2}\, sin\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) \\ \sqrt{f^2+a^2}\, cos\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) \\ 0 \end{pmatrix},$$

the Gauss map of the rotational surface is

$$e_R = \frac{1}{\sqrt{(f^2+a^2)f'^2 + f^2\phi'^2}} \begin{pmatrix} \sqrt{(af')^2+(f\phi')^2}\, cos\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) \\ \sqrt{(af')^2+(f\phi')^2}\, sin\left(v + \int \frac{a\phi'}{f^2+a^2} du\right) \\ ff' \end{pmatrix}. \quad (15)$$

Using the coefficients of the second fundamental form, by the straight calculation, the mean curvature of the rotational surface is

$$\mathbf{H}_R = \frac{f^2\phi'\boldsymbol{\Phi}(u)}{2\,[(f^2+a^2)f'^2 + f^2\phi'^2]^2\sqrt{f^2+a^2}\sqrt{(af')^2+(f\phi')^2}}, \quad (16)$$

where $\Phi\,(u)$ is the function in (14).

Now, suppose that the Gauss map e_H is identically equal to e_R .

If $\phi' = 0$, then the helicoidal surface reduces to right helicoid and the mean curvature of the rotational surface is identically zero. Hence, the rotational surface is the catenoid and the function $\phi_R(u_R)$ of (2) is $\phi_R(u_R)=bargcosh(\frac{u_R}{b})$ in (11) , where b is a constant. Comparing this function and the third element of (2), we have

$$bargcosh\left(\frac{\sqrt{f^2+a^2}}{b}\right) = \int \frac{af'}{\sqrt{f^2+a^2}} du.$$

By differentiating this equation, it follows that

$$\frac{bf}{\sqrt{f^2+a^2-b^2}} = a.$$

Therefore, we have $a = b$. Next, we suppose $\phi' \neq 0$. Comparing e_H and e_R, we have

$$argtanh\left(\frac{af'}{f\phi'}\right) = \int \frac{a\phi'}{f^2+a^2} du.$$

Differentiating this equation, we obtain

$$(f^2f'^2 - f^3f'' - a^2ff'' + 2a^2f'^2)\phi' + f^2\phi'^3 + (f^3f' + a^2ff')\phi'' = 0. \qquad (17)$$

This equation means $\Phi(u)$=0 in (13) and (16). So, the helicoidal surface and the rotational surface are minimal surfaces. Hence, again, the rotational surface reduces to the catenoid. Then, it follows that

$$b\ argcosh\left(\frac{\sqrt{f^2+a^2}}{b}\right) = \int \sqrt{\frac{(af')^2 + (f\phi')^2}{f^2+a^2}}du.$$

Using this eqution, we can find the profile curve ϕ of the helicoidal surface. Then, we have

$$\phi' = \frac{\sqrt{b^2-a^2}\sqrt{f^2+a^2}f'}{f\sqrt{f^2+a^2-b^2}}.$$

To solve this this differential equation, we put

$$t := \sqrt{\frac{f^2+a^2}{f^2+a^2-b^2}}, \quad r := \sqrt{b^2-a^2}.$$

Then, it folows that

$$\phi = rb^2 \int \frac{t^2}{(r^2t^2+a^2)(1-t^2)}dt.$$

This completes the proof.

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خلاصة

نقدم في هذا البحث السطوح الكلاسيكية المتقايسة الأصغرية اللولبية والدورانية، وذلك باستخدام تعميم لمبرهنة بور في الفضاء الإقليدي الثلاثي الأبعاد. زيادة على ذلك، نقوم بدراسة خاصة الأصغرية ونحصل على نفس تطبيق غاوس للسطوح.