Minimum total irregularity index of tricyclic graphs

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Abstract

The quantitative characterization of the topological structures of irregular graphs has been demonstrated through several irregularity measures. In the literature, not only different chemical and physical properties can be well comprehended but also quantitative structure-activity relationship (QSPR) and quantitative structure-property relationship (QSAR) are documented through these measures. A simple graph $G = (V, E)$ is a collection of V and E as vertex and edge sets respectively, with no multiple edges or loops. Keeping in view the importance of various irregularity measures, in (Abdo *et al.*, 2014a) the authors defined the total irregularity of a simple graph $G = G(V, E)$ as

$$irr_t(G) = \frac{1}{2}\sum_{u,v\in V} |d_G(u) - d_G(v)|,$$

where $d_G(u)$ indicates the degree of the vertex u, where $u \in V(G)$. In this paper, we have determined the first minimum, second minimum and third minimum total irregularity index of the tricyclic graphs on the n vertices.

Keywords: Irregularity; topological index; total irregularity index; λ-transformation; tricyclic graphs.

1. Introduction

Let $G = (V, E)$ be a graph with edge and vertex sets as denoted by E and V respectively. The number of edges attached on a vertex v of a graph G is the degree $d_G(v)$ of vertex v. If $V = \{v_i\}_{i=1}^{n}$, then sequence $(d_1, d_2, d_3, \ldots d_n)$ is called degree sequence of G (Bondy & Murty, 1976), where d_i is the degree of i^{th} vertex of G. We assume the sequence $(d_G(v_i))_{i=1}^{n}$ is in decreasing order $i.e.$ for $i < z$, $(d(v_z) \leq d(v_i))$. For convenience, we will use $\mathcal{DS}$ as the notation for degree sequence of a graph G.
With recent advances in graph theory in different areas, chemical graph theory is one of the most active area of research. Chemical graph theory or the theory of chemical graphs is a sub-branch of mathematical chemistry that describes non-trivial graph theory applications for solving molecular problems where the chemical structure is transformed into a mathematical structure. A representation of an object only provides information on the number of elements it

comprises, and its connectivity is defined as the graph's topological representation.
A topological index is a numerical value that is used primarily for predicting chemical and physical properties of various compounds and structures. A molecular graph is called a topological representation of a molecule. Significant number of topological indices during the last two decades have been documented. Many existing topological indices based on degrees can be classified as BID index, whose general form is

$$BID(G) = \sum_{uv \in E} f(d_u, d_v), \tag{1}$$

where uv is the edge connecting vertices u and v of the graph. There are numerous indices introduced such as the ABC index, Zagreb index, Randic index, etc. Some information can be found in the articles ((Akbar & Akhlaq, 2016), (Akbar & Akhlaq, 2017), (Hassan *et al.*, 2019) cited therein. Currently, the study of such types of indices has become a very active research area in the theory of chemical graphs. One such area is the quantitative analysis of different topological structures of irregular graphs.
The graph that has the same degree of all its vertices is *regular*, otherwise, it is *irregular*. Several approaches have been proposed which characterize the irregularity of a graph. Albertson in (Albertson, 1997) introduced $|d_G(u) - d_G(v)|$ as an imbalance of an edge $e = uv \in E$ and defined

$$irr(G) = \sum_{uv \in E} |d_G(u) - d_G(v)| \tag{2}$$

as an irregularity of a graph G. More results about the above-mentioned concepts are mentioned in ((Dimitrov & Skrekovski, 2015), (Abdo *et al.*, 2014b), (L.H. You *et al.*, 2014a), (L.H. You *et al.*, 2014b), (Henning & Rautenbach, 2007), (Albertson, 1997), (Hensen & $M\acute{e}lot$, 2005)). Taking inspiration from the structure and significance of Equation 2, a new irregularity measure was introduced by the authors in (Abdo *et al.*, 2014a) termed the total irregularity index, defined as

$$irr_t(G) = \frac{1}{2} \sum_{u,v \in V} |d_G(u) - d_G(v)| \tag{3}$$

Even though both graph invariants compute irregularity, the irregularity is captured by one parameter, i.e. the vertex degree, but in some respects the later is preferable to the old one. For instance, equation (3) has the known characteristic of an irregularity computation that the graphs with identical total irregularity have the same $\mathcal{DS}$, whereas equation (2) does not possess this property. Clearly, equation (3) is an upper bound of equation (2). In (Dimitrov & Skrekovski, 2015), the relationship between $irr(G)$ and $irr_t(G)$ for the connected graph on n vertices have been derived, that is, $irr_t(G) \leq n^2\{\frac{irr(G)}{4}\}$. Furthermore, for any tree, they also computed that $irr_t(T) \leq (n-2)irr(T)$. In (Abdo *et al.*, 2014a) the bounds on $irr_t(G)$ on cycle, path, and the star graph, denoted as C_n, P_n, and S_n, on the n vertices respectively, were computed. They also proved that the graph with maximal total irregularity on n vertices between all the trees is the star graph. Following result is due to (Abdo *et al.*, 2014a).

Theorem 1.1. *Let G be an n-vertex simple and undirected graph. Then*

(i) $irr_t(G) \leq (2n^3 - 3n^2 - 2n + 3)$.

(ii) $irr_t(G) \leq (n-1)(n-2)$ *if G is a tree, with equality iff* $G \cong S_n$.

The authors in (L.H. You *et al.*, 2014a) and (Hensen & $M\acute{e}lot$, 2005) examined the total irregularity of the unicyclic and bicyclic graphs and defined graphs with $n^2 - n - 6$ as maximum total irregularity among all the unicyclic graphs and graphs with $n^2 + n - 16$ as maximum total irregularity among all bicyclic graphs on n vertices respectively. By using the Gini index in (M. Eliasi, 2015), the author obtained the ordering of the total irregularity index for some classes of connected graphs, with the same number of vertices. Recently, the authors in (F. Gao *et al.*, 2021) characterized trees T of order n and triangulation graphs with respect to difference of Mostar index and irregularity of graphs. For more related research, readers are requested to see (Xu & Das, 2016).
In Section 2, we have described an important transformation in the current note to examine the minimum total irregularity of tricyclic graphs. We have also determined first, second and third minimum total irregularity of tricyclic graphs on n vertices in Section 3. Lastly, summary of the note is mentioned in Section 4.

2. λ-Transformation

An important transformation in this section is explained to explore the minimum total irregularity of graphs. Before introduction of transformation, let us define induced subgraph and hanging tree (Yingxue Zhu *et al.*, 2014).
Let G be an n-vertex graph then a subset of the vertices of G having edges incident on the vertices in the subset as endpoints is known as vertex-induced or simply induced subgraph of G. Let T be induced sub-tree of G, if G can be obtained back by connecting T to a vertex of $G \setminus T$. Then T is a hanging tree of G. Now we introduce the λ-Transformation as:
λ-Transformation: Let G be a simple graph with at least two leaves. Let u be a vertex of $d_G(u) \geq 3$ and T be hanging tree of G connecting to u with $|V(T)| \geq 1$, and v be the leaf of G with $v \notin T$. By removing T from u and connecting it to the vertex v and the graph obtained be denoted as G^*. Then this transformation from vertex u to v is a λ-transformation on G (see Figure 1).

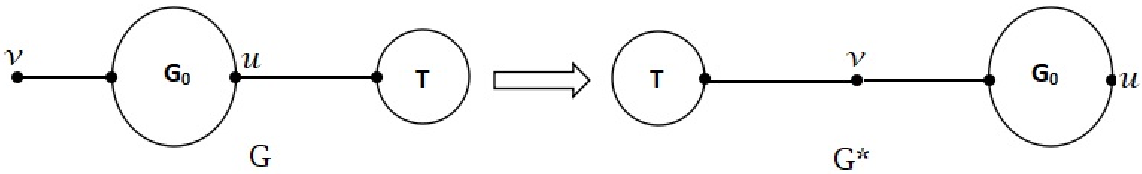

Fig. 1. G and G^*(obtained from λ-Transformation)

The following result is due to (Yingxue Zhu *et al.*, 2014), after λ-Transformation and it will be used in the main results as it will help us to compute total irregularity index of tricyclic graphs.

Lemma 2.1. *(Yingxue Zhu et al., 2014) Let G be an $n-vertex$ graph then $irr_t(G) > irr_t(G^*)$, where G^* is the graph obtained from G, after λ-Transformation from u to v.*

Proof. Let $G = (V, E)$, consider the vertex set $V = V^1 \cup V^2 \cup V^3$ such that

$$V^1 = \{x|d_G(x) \geq d_G(u), x \in V\}$$

$$V^2 = \{x|d_G(x) = 1, x \in V\}$$

$$V^3 = \{x|2 \leq d_G(x) < d_G(u), x \in V\}$$

Clearly, $u \in V^1, v \in V^2$. Let $|V^1| = j$, $|V^2| = k$, $|V^3| = l$, then $j \geq 1, k \geq 2$ and $j+k+l = n$. Note by λ-transformation, the degrees of v and u become $d_{G^*}(v) = d_G(v) + 1 = 2$, $d_{G^*}(u) = d_G(u) - 1$ and $d_{G^*}(w) = d_G(w)$ for any $w \in V \setminus \{u, v\}$. Let $U = V \setminus \{u, v\}$. Then

$$|d_{G^*}(u) - d_{G^*}(v)| - |d_G(u) - d_G(v)| = -2,$$

$$\sum_{w \in U} (|d_{G^*}(u) - d_{G^*}(w)| - |d_G(u) - d_G(w)|) = (j-1) - (l+k-1) = j - l - k,$$

$$\sum_{w \in U} (|d_{G^*}(v) - d_{G^*}(w)| - |d_G(v) - d_G(w)|) = -(j-1) - l + (k-1)$$
$$= -j - l + k.$$

Thus, we have $irr_t(G^*) - irr_t(G) = -2 + (j - l - k) + (-j - l + k) = -2l - 2 < 0$.

Remark. *Let λ-transformation be performed on G from the vertex u to v and G^* be the resulting graph. Then by λ-transformation and Lemma 2.1, we have $d_{G^*}(u) = d_G(u) - 1 \geq 2$ and $d_{G^*}(v) = d_G(v) + 1 = 2$. If $d_{G^*}(u) \geq 3$, G^* has at least two leaves, and there's a hanging tree of G^* connecting to vertex u, we can repeat λ-transformation from vertex u on G^*, till the degree of u equals 2, or the resulting graph consists of just one leaf, or no hanging tree connects to vertex u.*

We can see from the above arguments that λ-transformation can be achieved on G iff three conditions hold mentioned below:

(i) *There exists a vertex u with degree greater or equal to 3;*

(ii) *There is a hanging tree of G, connecting to vertex u;*

(iii) *G has at least two leaves.*

Following trivial result will be useful to establish our main results.

Lemma 2.2. *((Bondy & Murty, 1976)) Let $G = (V, E)$ be a graph and $|E| = m$. Then* $\sum_{v \in V} d_G(v) = 2m$.

■

In the following section, we establish the main results by describing different classes in tricyclic graphs on n vertices.

3. The Total Irregularity of Tricyclic Graphs

A connected (n, m) graph G is said to be a tricyclic graph if $m = n + 2$. Within this section, the extremal graphs are described by computing, the first, second and third minimum total irregularity of n-vertex tricyclic graphs.
Tricyclic graphs can be divided into three types: $\xi - graph$, $\Omega - graph$, and $\vartheta - graph$.
The class of $\xi - graph$, denoted by $\xi(p, q, r, s, i)$ contains three types of tricyclic graphs (see Figure 2). The first one is obtained from three cycles C_p, C_q, and C_r having one common vertex (say u), between C_p and C_q, and one (say v), between C_q and C_r (i.e. having no paths between

the cycles see Figure 2(a)). It is denoted by $\xi_1(p,q,r,s,i) = \xi_1$. The second is obtained having one common vertex u between C_p and C_q a path between C_q and C_r to any vertex $w \in V \setminus u$ (see Figure 2(b). It is denoted by $\xi_2(p,q,r,s,i) = \xi_2$. Lastly, third is obtained by attaching two disjoint paths P_s and P_i between C_p and C_q and one between C_q and C_r respectively (see Figure 2(c)), where $p,q,r \geq 3$. It is denoted by $\xi_3(p,q,r,s,i) = \xi_3$.

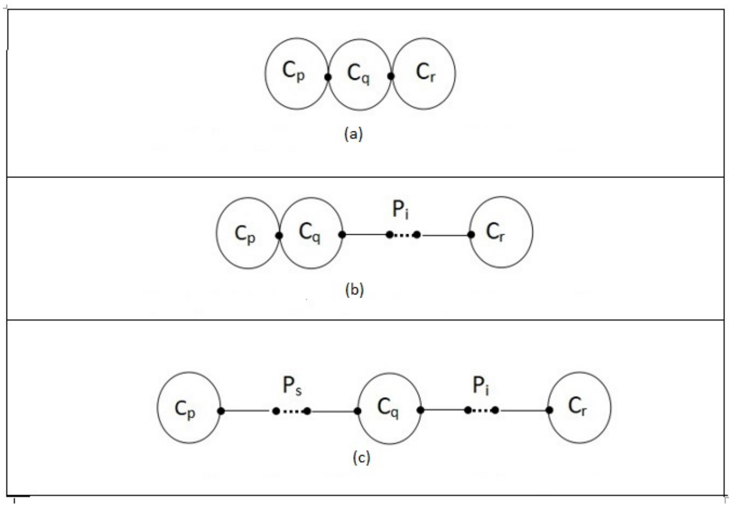

Fig. 2. Tricyclic graphs: (a) $\xi_1(p,q,r,s,i)$; (b) $\xi_2(p,q,r,s,i)$; (c) $\xi_3(p,q,r,s,i)$

An $\Omega - graph$ denoted by $\Omega(p,q,r,s,i,y)$, contains four types of tricyclic graphs (see Figure 3 and 4). The first graph, denoted by $\Omega_1 = \Omega_1(p,q,r,s,i,y)$, with only one common vertex, (say u), attached to C_p, C_q and C_r (see Figure 3(a)). The second graph, denoted by $\Omega_2 = \Omega_2(p,q,r,s,i,y)$ is obtained from Ω_1 by attaching a path P_y of length $y \geq 1$ between vertex u and C_r (see Figure 3(b)). The third graph, denoted by $\Omega_3 = \Omega_3(p,q,r,s,i,y)$, obtained from Ω_2 by attaching a path P_i of length $i \geq 1$ between vertex u and C_q (see Figure 4(a)). Lastly, the fourth graph, denoted by $\Omega_4 = \Omega_4(p,q,r,s,i,y)$ is obtained from Ω_3 by attaching a path P_s of length $s \geq 1$ between vertex u and C_p (see Figure 4(b)), where $p,q,r \geq 3$.

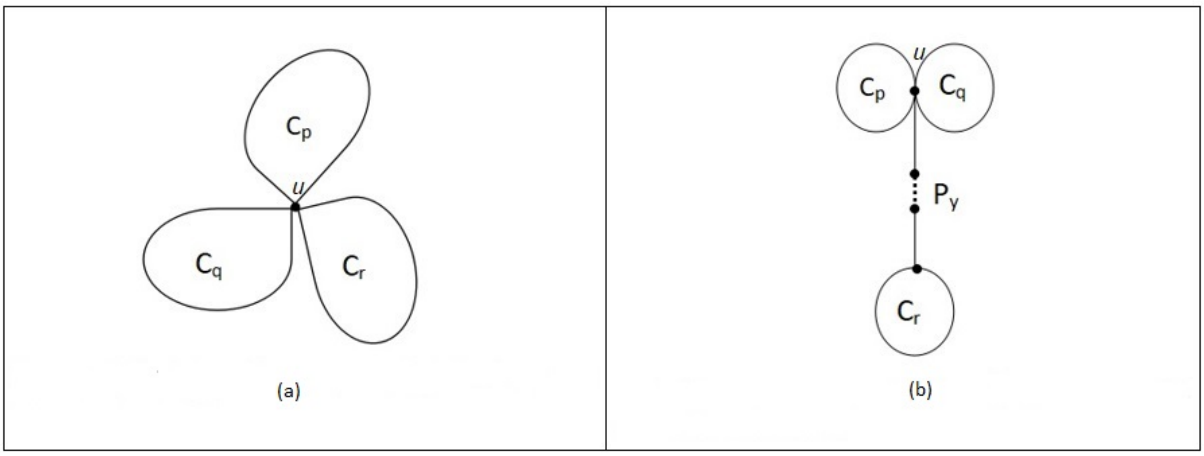

Fig. 3. Tricyclic graphs: (a) Ω_1; (b) Ω_2

A $\vartheta - graph$, denoted by $\vartheta(p,q,r,s,i)$ contains four types of tricyclic graphs (see Figure 5). The first graph, denoted by $\vartheta_1 = \vartheta_1(p,q,r,s,i)$, is a graph with three cycles (namely, C_p, C_q, C_r) on $p+q+r-s-i$ vertices, having $(s+i)$ vertices as common with each other (see Figure 5(a)). In the second case, the graph denoted by $\vartheta_2 = \vartheta_2(p,q,r,s,i)$, is obtained

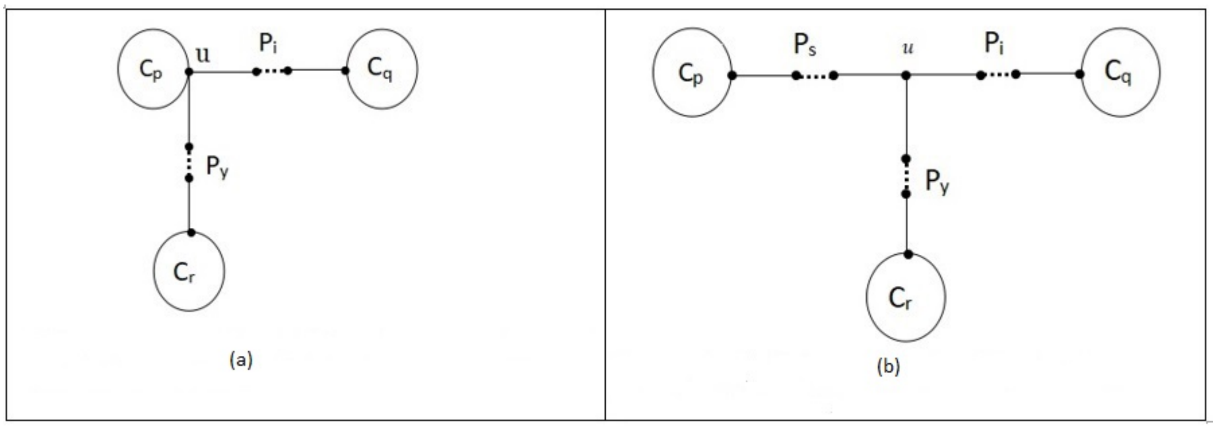

Fig. 4. Tricyclic graphs: (a) Ω_3; (b) Ω_4

from ϑ_1 by removing C_r from C_q and attaching it to one of the end vertices $\{f_1, f_s\}$ (see Figure 5(b)). In the third case, the graph is obtained from ϑ_1 by attaching a path P_{r-i} from one of the end vertices $\{e_1, e_{p-s}, h_1, h_i\}$ with a vertex of disjoint cycle C_r(see Figure 5(c)), let it be denoted by $\vartheta_3 = \vartheta_3(p, q, r, s, i)$. Lastly, the graph denoted by $\vartheta_4 = \vartheta_4(p, q, r, s, i)$ is obtained by attaching a path between the cycle C_r and one of the end vertices $\{f_1, f_s\}$ (see Figure 5(d)), where $p, q, r \geq 3$ and $s, i \geq 2$.

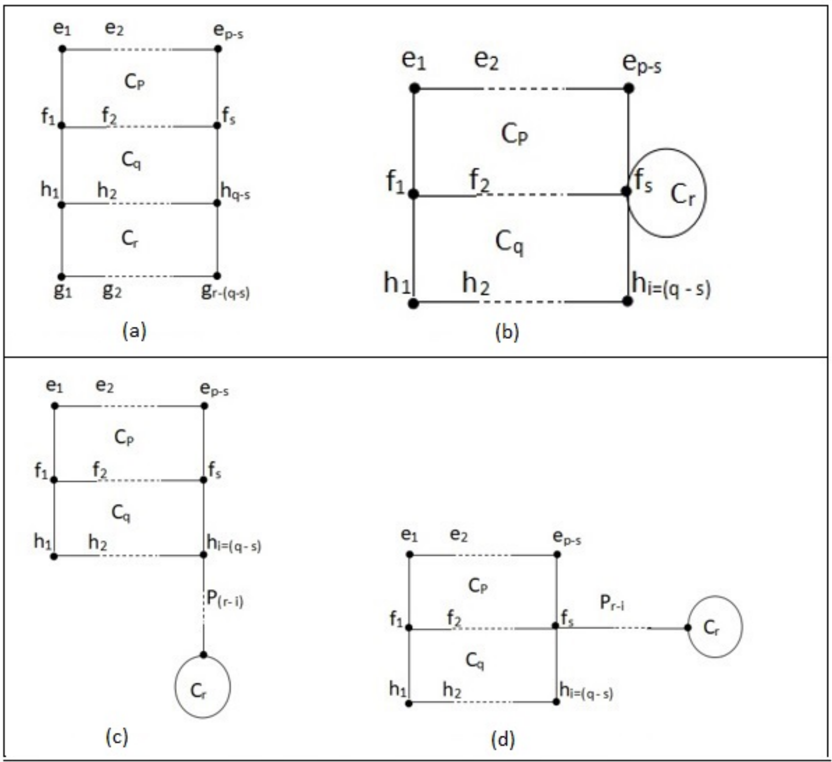

Fig. 5. Tricyclic graphs: (a) ϑ_1; (b) ϑ_2; (c) ϑ_3; (d) ϑ_4;

Let the set of all tricyclic graphs on n vertices be denoted by $\mathcal{T}_n$. As defined above $\mathcal{T}_n$ is based on three types of graphs $\xi - graph$, $\Omega - graphs$, and $\vartheta - graph$.

3.1. Graphs having minimum total irregularity in $\xi(p, q, r, s, i)$

In this section, we determine the minimum total irregularity of tricyclic graphs in $\xi(p, q, r, s, i)$. Let $\xi_1 = \xi_1(p, q, r, s, i)$ having no paths (see Figure 2(a)), $\xi_2 = \xi_2(p, q, r, s, i)$ with a one path P_i with length $i \geq 1$ (see Figure 2(b)) and $\xi_3 = \xi_3(p, q, r, s, i)$ with two paths P_s and P_i with lengths $s, i \geq 1$ respectively (see Figure 2(c)).

Theorem 3.1. *Let* $n \geq 7$, $G \in \xi_1 = \xi_1(p, q, r, s, i)$ *then*

(i) $irr_t(G) \geq 4n - 8$ *and equality holds iff* $(4, 4, 2, 2, \dots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(4, 4, 2, 2, \dots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 14$*, with equality iff the* $\mathcal{DS}$ *of G is* $(4, 4, 3, 2, 2, \dots, 2, 1)$.

Proof. We know that $\sum_{v \in V} d_G(v) = 2(n + 2)$ from Lemma 2.2. Let us divide the vertex set as follows,

$$j = |\{x | d_G(x) \geq 3, x \in V\}|,$$

$$k = |\{x | d_G(x) = 1, x \in V\}|,$$

$$t = |\{x | d_G(x) = \Delta_G, x \in V\}|.$$

Since $G \in \xi_1 = \xi_1(p, q, r, s, i)$, then $j \geq 2, k \geq 0, 1 \leq t \leq j$ and $\Delta_G \geq 4$. Note $G \in \xi_1$ if $j = 2, \Delta_G \geq 5$ or $j \geq 3$ so vertex u with $d_G(u) \geq 3$ exists and hanging tree of G which connects to u exists. We complete the proof by considering following cases:

Case 1. If $j = 2$, then there are three subcases mentioned below:

Subcase (i): If $\Delta_G = 4$, then $k = 0$ and the $\mathcal{DS}$ is $(4, 4, 2, 2, \dots, 2)$ as $2(n+2) = \sum_{v \in V} d_G(v) = 8 + 2(n - 2 - k) + k$, then $irr_t(G) = 4n - 8$.

Subcase (ii): If $\Delta_G = 5$, then $k = 1$ and the $\mathcal{DS}$ is $(5, 4, 2, 2, \dots, 2, 1)$ as $2(n + 2) = \sum_{v \in V} d_G(v) = 5 + 4 + 2(n - 2 - k) + k$, then $irr_t(G) = 6n - 10 > 6n - 14$.

Subcase (iii): If $\Delta_G \geq 6$, then $k \geq \Delta_G - 4 \geq 2$ as $2(n + 2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 4 + 2(n - 2 - k) + k$ and λ-transformation can be done $(k - 1)-$ times on G till the $\mathcal{DS}$ of the graph obtained becomes $(5, 4, 2, 2, \dots, 2, 1)$. Let the graph obtained be denoted as F_1, then $irr_t(G) > irr_t(F_1) = 6n - 10 > 6n - 14$ by Lemma 2.1.

Case 2. Now if $j \geq 3$, then consider following subcases:

Subcase (i): If $j + \Delta_G = 7$, then $j = 3, \Delta_G = 4, 2 \leq t \leq 3$.

If $t = 2$, then $k = 1$ and the $\mathcal{DS}$ is $(4, 4, 3, 2, 2, \dots, 2, 1)$ as $2(n + 2) = \sum_{v \in V} d_G(v) = 4 + 4 + 3 + 2(n - 3 - k) + k = 11 + 2(n - 3 - k) + k$, so $irr_t(G) = 6n - 14$.

If $t = 3$, then $k = 2$ as $2(n + 2) = \sum_{v \in V} d_G(v) = 4t + 2(n - 3 - k) + k$, and λ-transformation can be done once on G so the $\mathcal{DS}$ of obtained graph is $(4, 4, 3, 2, 2, \dots, 2, 1)$. Let the obtained graph be denoted as F_2, then $irr_t(G) > irr_t(F_2) = 6n - 14$ by Lemma 2.1.

Subcase (ii): If $j + \Delta_G \geq 8$, then $k \geq \Delta_G + j - 6 \geq 2$ as $2(n + 2) = \sum_{v \in V} d_G(v) \geq$

$\Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of graph obtained is $(4,4,3,2,2,\ldots,2,1)$. Let the obtained graph be denoted as F_3, then $irr_t(G) > irr_t(F_3) = 6n-14$ by Lemma 2.1.

□

Theorem 3.2. *Let $n \geq 8$, $G \in \xi_2 = \xi_2(p,q,r,s,i)$ then*

(i) $irr_t(G) \geq 4n-10$ and equality holds iff $(4,3,3,2,2,\ldots,2)$ is the $\mathcal{DS}$ of G.

(ii) If $(4,3,3,2,2,\ldots,2)$ is not the $\mathcal{DS}$ of G, then $irr_t(G) \geq 6n-18$, with equality iff the $\mathcal{DS}$ of G is $(4,3,3,3,2,2,\ldots,2,1)$.

Proof. It is easy to see that $\sum_{v\in V} d_G(v) = 2(n+2)$ from Lemma 2.2.
Let us divide the vertex set as,

$$j = |\{x|d_G(x) \geq 3, x \in V\}|,$$

$$k = |\{x|d_G(x) = 1, x \in V\}|,$$

$$t = |\{x|d_G(x) = \Delta_G, x \in V\}|.$$

Since $G \in \xi_2 = \xi_2(p,q,r,s,i)$ then $j \geq 3,\ k \geq 0, 1 \leq t \leq j$ and $\Delta_G \geq 4$.
Note $G \in \xi_2$ if $j = 3,\ \Delta_G \geq 4$ or $j \geq 4$ so there exists a vertex u with $d_G(u) \geq 3$ and there exists a hanging tree of G which connects to u. We complete the proof by considering following cases:

Case 1. If $j = 3$, then consider following subcases:
Subcase (i): If $\Delta_G = 4$, then $k = 0$ and the $\mathcal{DS}$ is $(4,3,3,2,2,\ldots,2)$ as $2(n+2) = \sum_{v\in V} d_G(v) = 4+3+3+2(n-3-k)+k$, then $irr_t(G) = 4n-10$.
Subcase (ii): If $\Delta_G = 5$, then $1 \leq t \leq 3$
If $t = 1$, then $k = 1$ and $k = 2$. For $k = 1$ the $\mathcal{DS}$ is $(5,3,3,2,2,\ldots,2,1)$ as $2(n+2) = \sum_{v\in V} d_G(v) \geq \Delta_G + 3 + 3 + 2(n-3-k) + k$ and $irr_t(G) = 6n-12 > 6n-18$. For $k = 2$ λ-transformation can be done on G once and the $\mathcal{DS}$ of the graph obtained becomes $(5,3,3,2,2,\ldots,2,1)$. Let the obtained graph denoted by F_4, then $irr_t(G) > irr_t(F_4) = 6n-12 > 6n-18$ from Lemma 2.1.
If $t \geq 2$, then $k \geq 3$ as $2(n+2) = \sum_{v\in V} d_G(v) \geq 5+5+3+2(n-3-k)+k$ λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of obtained graph becomes $(5,3,3,2,2,\ldots,2,1)$. Let the obtained graph denoted by F_5, then $irr_t(G) > irr_t(F_5) = 6n-12 > 6n-18$ by Lemma 2.1.
Subcase (iii): If $\Delta_G \geq 6$, then $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v\in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of obtained graph is $(5,4,2,2,\ldots,2,1)$. Let the obtained graph be denoted as F_6, then $irr_t(G) > irr_t(F_6) = 6n-10 > 6n-14$ by Lemma 2.1.
Case 2. If $j \geq 4$, then consider following subcases:
Subcase (i): If $j + \Delta_G = 8$, then $k = 1$, and the $\mathcal{DS}$ of G is $(4,3,3,3,2,2,\ldots,2,1)$ as $2(n+2) = \sum_{v\in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, then $irr_t(G) = 6n-18$.

Subcase (ii): If $j + \Delta_G \geq 9$, then $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of obtained graph is $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$. Let the obtained graph be denoted as F_7, then $irr_t(G) > irr_t(F_7) = 6n - 18$ by Lemma 2.1.

□

Theorem 3.3. *Let $n \geq 9$, $G \in \xi_3 = \xi_3(p, q, r, s, i)$ then*

(i) $irr_t(G) \geq 4n - 16$ and equality holds iff $(3, 3, 3, 3, 2, 2, \ldots, 2)$ is the $\mathcal{DS}$ of G.

(ii) If $(3, 3, 3, 3, 2, 2, \ldots, 2)$ is not the $\mathcal{DS}$ of G, then $irr_t(G) \geq 6n - 26$, with equality iff the $\mathcal{DS}$ of G is $(3, 3, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

Proof. It is easy to see that $\sum_{v \in V} d_G(v) = 2(n+2)$ from Lemma 2.2.
Let us divide vertex set as below,

$$j = |\{x | d_G(x) \geq 3, x \in V\}|,$$

$$k = |\{x | d_G(x) = 1, x \in V\}|,$$

$$t = |\{x | d_G(x) = \Delta_G, x \in V\}|.$$

Since $G \in \xi_3 = \xi_3(p, q, r, s, i)$ then $j \geq 4,\ k \geq 0, 1 \leq t \leq j$ and $\Delta_G \geq 3$.
Note $G \in \xi_3 = \xi_3(p, q, r, s, i)$ if $j = 4,\ \Delta_G \geq 3$ or $j \geq 5$ so there exists a vertex u with $d_G(u) \geq 3$ and there exists hanging tree of G which connects to u. We have completed the proof by considering the following cases:
Case 1. If $j = 4$, then consider following subcases:
Subcase (i): If $\Delta_G = 3$, then $k = 0$ and the $\mathcal{DS}$ is $(3, 3, 3, 3, 2, 2, \ldots, 2)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, then $irr_t(G) = 4n - 16$.
Subcase (ii): If $\Delta_G = 4$, then $1 \leq t \leq 4$.
If $t = 1$, then $k = 1$. For $k = 1$ the $\mathcal{DS}$ is $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and $irr_t(G) = 6n - 18 > 6n - 26$.
If $t \geq 2$, then $k \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of obtained graph becomes $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$. Let the obtained graph denoted by F_8, thus $irr_t(G) > irr_t(F_8) = 6n - 18 > 6n - 26$ by Lemma 2.1.
Subcase (iii): If $\Delta_G \geq 5$,
then $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of obtained graph is $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$. Let the obtained graph be denoted as F_9, thus $irr_t(G) > irr_t(F_9) = 6n - 18 > 6n - 26$ by Lemma 2.1.
Case 2. If $j \geq 5$, then consider the following subcases:
Subcase (i): If $j + \Delta_G = 8$, then $k = 1$, and the $\mathcal{DS}$ of G is $(3, 3, 3, 3, 3, 2, 2, \ldots, 2, 1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, then $irr_t(G) = 6n - 26$.

Subcase (ii): If $j + \Delta_G \geq 9$, then $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of obtained graph is $(3,3,3,3,3,2,2,\ldots,2,1)$. Let the graph obtained be denoted as F_{10}, then $irr_t(G) > irr_t(F_{10}) = 6n - 26$ by Lemma 2.1.

□

3.2. The graphs with minimum total irregularity in $\Omega - graph$

In this section, we determine the first minimum, second minimum, and third minimum total irregularity of tricyclic graphs in $\Omega(p,q,r,s,i,y)$.

Theorem 3.4. *Let $n \geq 7$, $G \in \Omega_1 = \Omega_1(p,q,r,s,i,y)$ then*

- *(i) $irr_t(G) \geq 4n - 4$ and equality holds iff $(6,2,2,\ldots,2)$ is the $\mathcal{DS}$ of G.*
- *(ii) If $(6,2,2,\ldots,2)$ is not the $\mathcal{DS}$ of G, then $irr_t(G) \geq 6n - 8$, with equality iff the $\mathcal{DS}$ of G is $(6,3,2,2,\ldots,2,1)$.*

Proof. It is obvious that $\sum_{v \in V} d_G(v) = 2(n+2)$ from Lemma 2.2.
Let us consider the vertex set as,

$$j = |\{x | d_G(x) \geq 3, x \in V\}|,$$

$$k = |\{x | d_G(x) = 1, x \in V\}|,$$

$$t = |\{x | d_G(x) = \Delta_G, x \in V\}|.$$

Since $G \in \Omega_1 = \Omega_1(p,q,r,s,i,y)$, then $j \geq 1,\ k \geq 0, 1 \leq t \leq j$ and $\Delta_G \geq 6$.
Note $G \in \Omega_1$ if $j = 1,\ \Delta_G \geq 6$ or $j \geq 2$ so there exists a vertex u with $d_G(u) \geq 3$ and there exists hanging tree of G which connects to u. We complete the proof by considering the following cases:
Case 1. If $j = 1$, then consider the following subcases:
Subcase (i): If $\Delta_G = 6$, then $k = 0$ and the $\mathcal{DS}$ is $(6,2,2,\ldots,2)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, thus $irr_t(G) = 4n - 4$.
Subcase (ii): If $\Delta_G = 7$, then $k = 1$. For $k = 1$, $\mathcal{DS}$ is $(7,2,2,\ldots,2,1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and $irr_t(G) = 6n - 6 > 6n - 8$.
Subcase (iii): If $\Delta_G \geq 7$, then $k \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of graph obtained is $(7,2,2,\ldots,2,1)$. Let the graph obtained be denoted as F_{11}, then $irr_t(G) > irr_t(F_{11}) = 6n - 6 > 6n - 8$ by Lemma 2.1.
Case 2. If $j \geq 2$, then consider the following subcases:
Subcase (i): If $\Delta_G = 6$, then $1 \leq t \leq 2$,
If $t = 1$ then $1 \leq k \leq 3$,. For $k = 1$ the $\mathcal{DS}$ of G is $(6,3,2,2,\ldots,2,1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, thus $irr_t(G) = 6n - 8$. For $k \geq 2$ and we can do λ-transformation $(k-1)-$times on G till the $\mathcal{DS}$ of graph obtained is $(6,3,2,2,\ldots,2,1)$. Let the graph obtained be denoted as F_{12}, then $irr_t(G) > irr_t(F_{12}) = 6n - 8$ by Lemma 2.1.

Subcase (ii): If $\Delta_G \geq 7$, then $1 \leq t \leq 2$ and $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of graph obtained is $(6, 3, 2, 2, \ldots, 2, 1)$. Let the graph obtained be denoted as F_{13}, then $irr_t(G) > irr_t(F_{13}) = 6n - 8$ by Lemma 2.1.

□

By following the same pattern as above we get the following results by direct calculations.

Theorem 3.5. *Let $n \geq 8$, $G \in \Omega_2 = \Omega_2(p, q, r, s, i, y)$ then*

(i) $irr_t(G) \geq 4n - 6$ *and equality holds iff* $(5, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(5, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 12$*, with equality iff the* $\mathcal{DS}$ *of G is* $(5, 3, 3, 2, 2, \ldots, 2, 1)$.

□

Theorem 3.6. *Let $n \geq 9$, $G \in \Omega_3 = \Omega_3(p, q, r, s, i, y)$ then*

(i) $irr_t(G) \geq 4n - 10$ *and equality holds iff* $(4, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(4, 3, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 18$*, with equality iff the* $\mathcal{DS}$ *of G is* $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

□

Theorem 3.7. *Let $n \geq 10$, $G \in \Omega_4 = \Omega_4(p, q, r, s, i, y)$*

(i) $irr_t(G) \geq 4n - 16$ *and equality holds in case* $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 12$*, with equality iff the* $\mathcal{DS}$ *of G is* $(3, 3, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

□

3.3. The graphs with minimum total irregularity in $\vartheta - graph$

In this section, we have determined first minimum, second minimum, and third minimum total irregularity of tricyclic graphs in $\vartheta(p, q, r, s, i)$.

Theorem 3.8. *Let $n \geq 5$, $G \in \vartheta_1 = \vartheta_1(p, q, r, s, i)$*

(i) $irr_t(G) \geq 4n - 10$ *and equality holds iff* $(4, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(4, 3, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 18$*, with equality iff the* $\mathcal{DS}$ *of G is* $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

Proof. We know that $\sum_{v \in V} d_G(v) = 2(n+2)$ from Lemma 2.2.
Consider the following distribution of vertex set as,

$$j = |\{x | d_G(x) \geq 3, x \in V\}|,$$

$$k = |\{x | d_G(x) = 1, x \in V\}|,$$

$$t = |\{x|d_G(x) = \Delta_G, x \in V\}|.$$

Since $G \in \vartheta_1 = \vartheta_1(p, q, r, s, i,)$ then $j \geq 3,\ k \geq 0, 1 \leq t \leq j$ and $\Delta_G \geq 4$.
Note $G \in \vartheta_1$ if $j = 3,\ \Delta_G \geq 4$ or $j \geq 4$ so there exists a vertex u with $d_G(u) \geq 3$ and there exists hanging tree of G which connects to u. We prove by considering the following cases:
Case 1. If $j = 3$, then consider the following cases:
Subcase (i): If $\Delta_G = 4$,
then $1 \leq t \leq 3$. If $t = 1$ then $k = 0$ and the $\mathcal{DS}$ is $(4, 3, 3, 2, 2, \ldots, 2)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, then $irr_t(G) = 4n - 10$.
If $t = 2$ then $k = 1$ and the $\mathcal{DS}$ is $(4, 4, 3, 2, 2, \ldots, 2, 1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, thus $irr_t(G) = 6n - 14 > 6n - 18$.
If $t = 3$ then $k = 2$ and λ-transformation can be done once on G s.t. the $\mathcal{DS}$ of graph obtained is $(4, 4, 3, 2, 2, \ldots, 2, 1)$. Let the graph obtained be denoted by F_{14}, thus $irr_t(G) \geq irr_t(F_{14}) = 6n - 14 > 6n - 18$.
Subcase (ii): If $\Delta_G = 5$, then $k = 1$. For $k = 1$ $\mathcal{DS}$ is $(5, 3, 3, 2, 2, \ldots, 2, 1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and $irr_t(G) = 6n - 12 > 6n - 18$.
Subcase (iii): If $\Delta_G \geq 6$, then $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of graph obtained is $(5, 3, 3, 2, 2, \ldots, 2, 1)$. Let the graph obtained be denoted as F_{15}, then $irr_t(G) > irr_t(F_{15}) = 6n - 12 > 6n - 18$ by Lemma 2.1.
Case 2. If $j \geq 4$, then consider the following subcases:
Subcase (i): If $j + \Delta_G = 8$, then $k = 1$. For $k = 1$ the $\mathcal{DS}$ of G is $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$, thus $irr_t(G) = 6n - 18$.
Subcase (ii): If $j + \Delta_G \geq 9$, then $k \geq \Delta_G + j - 7 \geq 2$ as $2(n+2) = \sum_{v \in V} d_G(v) \geq \Delta_G + 3(j-1) + 2(n-j-k) + k$ and λ-transformation can be done $(k-1)-$times on G till the $\mathcal{DS}$ of graph obtained is $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$. Let the graph obtained be denoted as F_{16}, thus $irr_t(G) > irr_t(F_{16}) = 6n - 18$ by Lemma 2.1.

□

Similarly, by direct calculation, we have the following results.

Theorem 3.9. *Let $n \geq 6$, $G \in \vartheta_2 = \vartheta_2(p, q, r, s, i)$ then*

(i) $irr_t(G) \geq 4n - 6$ *and equality holds iff* $(5, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(5, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 12$*, with equality iff the* $\mathcal{DS}$ *of G is* $(5, 3, 3, 2, 2, \ldots, 2, 1)$.

□

Theorem 3.10. *Let $n \geq 7$, $G \in \vartheta_3 = \vartheta_3(p, q, r, s, i)$ then*

(i) $irr_t(G) \geq 4n - 16$ *and equality holds iff* $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 26$*, with equality iff the* $\mathcal{DS}$ *of G is* $(3, 3, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

□

Theorem 3.11. *Let $n \geq 7$, $G \in \vartheta_4 = \vartheta_4(p, q, r, s, i)$ then*

(i) $irr_t(G) \geq 4n - 10$ *and equality holds iff* $(4, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(4, 3, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 18$*, with equality iff the* $\mathcal{DS}$ *of G is* $(4, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

□

4. The graphs with minimum total irregularity in $\mathcal{T}_n$

By section 3 we have determined first minimum, second minimum and the third minimum total irregularity in $\mathcal{T}_n$ immediately.

Theorem 4.1. *Let $n \geq 7$, $G \in \mathcal{T}_n$ then*

(i) $irr_t(G) \geq 4n - 16$ *and equality holds iff* $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G.*

(ii) If $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *is not the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 4n - 10$*, with equality iff the* $\mathcal{DS}$ *of G is* $(4, 3, 3, 3, 2, 2, \ldots, 2)$.

(iii) If neither $(3, 3, 3, 3, 2, 2, \ldots, 2)$ *nor* $(4, 3, 3, 3, 2, 2, \ldots, 2)$ *is the* $\mathcal{DS}$ *of G, then* $irr_t(G) \geq 6n - 26$*, with equality iff the* $\mathcal{DS}$ *of G is* $(3, 3, 3, 3, 2, 2, \ldots, 2, 1)$.

□

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